DYNAMICAL SYSTEMS
AND MATHEMATICAL CONTROL THEORY

(8cfu or 5 cfu, every two years, starting with the first semester 2026/27)

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Aim of the course:

To provide the main tools in dynamical systems and mathematical control theory, enabling to address control problems of interest in various applied sciences.

Both classical problems of control theory governed by ordinary differential equations (ODEs) and examples of control problems involving partial differential equations (PDEs) will be studied.

Motivation:

This course explores the fundamental mathematical principles and methods used to analyze and design control systems which naturally appear in applied sciences, technology, industry, or services.

Control theory analyzes properties of controlled systems, i.e., dynamic systems which are governed by means of a control (or command).
The aim is to transform the system from a given initial state to a particular end-state, respecting certain criteria: for instance, the target could be to stabilize the system in order to make it resistant to certain disruptions or to determine optimal solutions according to a particular criterion (optimal control).

Students of this course can take advantage of their training both in academy and in modern industry, in specialized fields such as aeronautics, aerospace, robotics, in life sciences. The knowledge of the mathematical tools employed in control helps to better understand the existing models and to improve them.

Due to the wide range of applications of control systems, both models involving ODEs and involving PDEs are needed.

The main part of the course deals with dynamical systems of ODEs. Control theory for PDEs is an extremely wide field of study and research. Students will be introduced in this framework by means of some specific examples, such as diffusion problems and vibrating phenomena.

Applications encountered during the course include:

Fuel optimal landing of a space vehicle, minimum drag nose shape problem, boat guidance in a river, reproductive strategies in social insects (in the framework of controlled ODEs), optimal lockdown strategy under pandemic, heating or cooling a building in an optimal way, stabilization and controllability of a string (in the PDEs framework).

Controlled systems:

The evolution of a system, whose configurations can be described by the variables $x = (x_1, \ldots, x_n)$, can be modeled by an ODE

$$\dot{x} = g(t, x).$$

The ingredients of the differential equations are the quantities that describe the phenomenon; the relationships among these quantities arise from intuitive considerations about the properties of the phenomenon, from physical laws based on experimental evidence.

In a controlled system

$$\dot{x} = f(t, x, \alpha).$$

it is introduced a dependence on external parameters called control variables: $\alpha = (\alpha_1, \ldots, \alpha_m)$. α can be interpreted as a controller capable of modifying the system's evolution in order to achieve predetermined objectives.

Optimal control problems:

The goal is to determine an optimal control α which, when applied to our system, optimizes certain performance criteria.

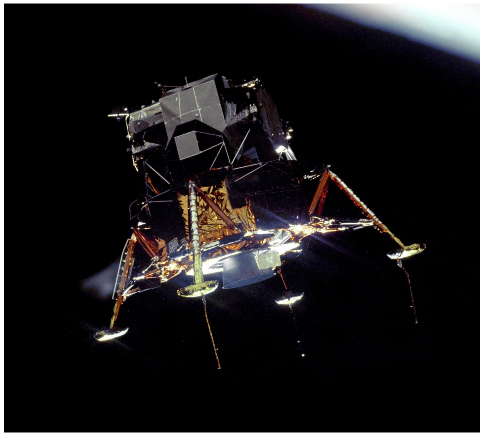
In mathematical terms, we seek an optimal control α that minimizes (or maximizes) a cost functional of the form

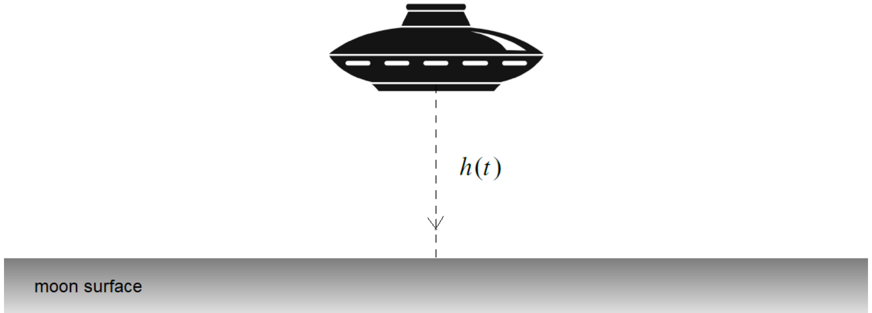
$$P(\alpha) := \int_0^T L(t, x(t), \alpha(t)), dt,$$

where x is a solution of the controlled system

$$\dot{x} = f(t, x, \alpha).$$

An example from the aerospace: the moon lander problem





Problem 1. How should a pilot land safely a spacecraft on the moon surface so as to use the least possible amount of fuel?

Problem 2. How should a pilot drive the spacecraft on the moon surface in minimum time?

An example from the aerospace: the moon lander problem

The variables:

$h(t)$ = height of the spacecraft at time t;

$v(t) = \dot{h}(t)$ = velocity of the spacecraft

$m(t)$ = mass of the spacecraft;

$\alpha(t)$ = thrust at time t.

From Newton's second law the motion of the spacecraft is:

$$m(t)\ddot{h}(t) = -gm(t) + \alpha(t) \qquad \text{and} \qquad \dot{m}(t) = -k|\alpha(t)| \tag{1}$$

Problem 1. How should a pilot land safely a spacecraft on the moon surface so as to use the least possible amount of fuel?

We minimize the total applied thrust before landing

$$P_1(\alpha) := \int_0^\tau \alpha(t)dt$$

among the solutions of (1).

Problem 2. How should a pilot drive the spacecraft on the moon surface in minimum time?

We minimize the time to land safely

$$P_2(\alpha) := \tau = \int_0^\tau dt$$

among the solutions of (1).

An example in the PDEs framework: controlling a string

We would like to control the behavior of a string.
The governing state equation describing the displacement of a string under vibration is the wave equation:

$$y_{tt} - y_{xx} = f, \qquad (t,x) \in (0,T) \times (0,1).$$

$y(t,x)$ represents the displacement of the point x on the string at time t

$f(t,x)$ is the control function, an external force acting on x at time t.

Basic control problems for such a system:

⋆ the stabilization:
to find a control f so that the displacement $y(t,x)$ and its velocity $y_t(t,x)$ go to zero as t goes to infinity;

⋆ the exact null controllability:
to find a control f so that $y(T,x) = y_t(T,x) = 0$, $\forall\, x \in (0,1)$;

⋆ optimal control problems:
to stabilize the vibration with as little energy as possible, we minimize a cost functional as:

$$J(f) = \int_0^{\infty}\int_0^1 \left\{ |y(t,x)|^2 + |y_t(t,x)|^2 + |f(t,x)|^2 \right\} dx dt$$

to find a vibration close to a desired one (in an instrument like a violin), we minimize a cost functional as:

$$J(f) = \int_0^{T}\int_0^1 \left\{ |y(t,x) - \phi(t,x)|^2 + |y_t(t,x) - \psi(t,x)|^2 \right\} dx dt\,.$$

Program:

1. Dynamical systems
Linear systems of ODEs: general results and structure, exponential matrix.
Asymptotic behavior of dynamical systems, stability (linearization, Lyapunov method), phase portraits.

2. Control Systems and asymptotic stabilization
Examples, reachable sets, linear systems, local controllability of nonlinear systems, bang-bang theorem.
Stabilization of linear and nonlinear control systems.

3. Optimal control
Existence of Optimal Controls for Mayer problems and the problem of Bolza.
Necessary conditions for the optimality: the Pontryagin Maximum Principle.
Sufficient Conditions. Dynamic Programming. The linear quadratic case.

4. An introduction to control for partial differential equations
Controllability and stabilization of the wave and the heat equations.
Optimal control for PDE's, an introduction.

5. Further topics
Recent advances in mathematical control theory, dealing e.g., with control problems for multiagent systems, traffic models, moving sets.

The 5cfu version of the course consists in the points **1, 2, 3** of the above program.

Pre-requisites:

First level courses concerning Mathematical Analysis and Geometry.

Some references:

⋆ Bressan, Piccoli: Introduction to Mathematical Theory of Control, AIMS Book Series.

⋆ Cannarsa, Gazzola: Dynamic Optimization for Beginners, EMS Press.

⋆ Evans: An introduction to mathematical control theory, Berkley, Lecture notes.

⋆ Lenhart, Workman: Optimal Control Applied to Biological Models, Chapman and Hall/CRC.

⋆ Li, Yong: Optimal Control Theory for Infinite-Dimensional Systems, Birkhäuser, Boston.

⋆ Lectures notes provided by the teachers.