Global L^p estimates for degenerate Ornstein-Uhlenbeck operators with variable coefficients*

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Abstract

We consider a class of degenerate Ornstein-Uhlenbeck operators in $\mathbb{R}^N$, of the kind

$$\mathcal{A} \equiv \sum_{i,j=1}^{p_0} a_{ij}(x)\,\partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j}$$

where (a_{ij}) is symmetric uniformly positive definite on $\mathbb{R}^{p_0}$ ($p_0 \leq N$), with uniformly continuous and bounded entries, and (b_{ij}) is a constant matrix such that the frozen operator $\mathcal{A}_{x_0}$ corresponding to $a_{ij}(x_0)$ is hypoelliptic. For this class of operators we prove global L^p estimates ($1 < p < \infty$) of the kind:

$$\left\|\partial^2_{x_i x_j} u\right\|_{L^p(\mathbb{R}^N)} \leq c\left\{\|\mathcal{A}u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)}\right\} \text{ for } i,j = 1,2,...,p_0.$$

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We obtain the previous estimates as a byproduct of the following one, which is of interest in its own:

$$\left\|\partial^2_{x_ix_j}u\right\|_{L^p(S_T)} \leq c\left\{\|Lu\|_{L^p(S_T)} + \|u\|_{L^p(S_T)}\right\}$$

for any $u \in C_0^\infty(S_T)$, where S_T is the strip $\mathbb{R}^N \times [-T, T]$, T small, and L is the Kolmogorov-Fokker-Planck operator

$$L \equiv \sum_{i,j=1}^{p_0} a_{ij}(x,t)\,\partial^2_{x_ix_j} + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j} - \partial_t$$

with uniformly continuous and bounded a_{ij}'s.

1 Introduction

Let us consider the following kind of Ornstein-Uhlenbeck operators:

$$\mathcal{A} = \sum_{i,j=1}^{p_0} a_{ij}(x)\,\partial^2_{x_ix_j} + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j}, \tag{1.1}$$

where:

$A_0 = (a_{ij}(x))_{i,j=1}^{p_0}$ is a $p_0 \times p_0$ ($p_0 \leq N$) symmetric, bounded and uniformly positive definite matrix:

$$\frac{1}{\Lambda}|\xi|^2 \leq \sum_{i,j=1}^{p_0} a_{ij}(x)\,\xi_i\xi_j \leq \Lambda|\xi|^2 \tag{1.2}$$

for all $\xi \in \mathbb{R}^{p_0}$, $x \in \mathbb{R}^N$ and for some constant $\Lambda \geq 1$;

the entries a_{ij} are supposed to be uniformly continuous functions on $\mathbb{R}^N$, with a modulus of continuity

$$\omega(r) = \max_{i,j=1,\ldots,p_0} \sup_{\substack{x,y\in\mathbb{R}^N \\ |x-y|\leq r}} |a_{ij}(x) - a_{ij}(y)|\,; \tag{1.3}$$

the constant matrix $B = (b_{ij})_{i,j=1}^N$ has the following structure:

$$B = \begin{bmatrix} * & B_1 & 0 & \ldots & 0 \\ * & * & B_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \ldots & B_r \\ * & * & * & \ldots & * \end{bmatrix} \tag{1.4}$$

where B_j is a $p_{j-1} \times p_j$ block with rank p_j, $j = 1, 2, ..., r$, $p_0 \geq p_1 \geq ... \geq p_r \geq 1$, $p_0 + p_1 + ... + p_r = N$ and the symbols $*$ denote completely arbitrary blocks.

If the a_{ij}'s are constant, the above assumptions imply that the operator $\mathcal{A}$ is hypoelliptic (although degenerate, as soon as $p_0 < N$), see [12]. If the a_{ij}'s are just uniformly continuous, $\mathcal{A}$ is a nonvariational degenerate elliptic operator with continuous coefficients, structured on a hypoelliptic operator. For this class of operators, we shall prove the following global L^p estimates:

Theorem 1.1 *For every $p \in (1, \infty)$ there exists a constant $c > 0$, depending on p, N, p_0, the matrix B, the number Λ in (1.2) and the modulus ω in (1.3) such that for every $u \in C_0^\infty(\mathbb{R}^N)$ one has:*

$$\sum_{i,j=1}^{p_0} \left\| \partial^2_{x_i x_j} u \right\|_{L^p(\mathbb{R}^N)} \leq c \left\{ \| \mathcal{A}u \|_{L^p(\mathbb{R}^N)} + \| u \|_{L^p(\mathbb{R}^N)} \right\}, \tag{1.5}$$

$$\left\| \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u \right\|_{L^p(\mathbb{R}^N)} \leq c \left\{ \| \mathcal{A}u \|_{L^p(\mathbb{R}^N)} + \| u \|_{L^p(\mathbb{R}^N)} \right\}. \tag{1.6}$$

In [4] we have proved this result in the case of constant coefficients a_{ij}. Here we show that exploiting results and techniques contained in [4], together with a careful inspection of the quantitative dependence of some bounds proved in [12] and [9], we can get Theorem 1.1. The striking feature of our result is twofold. On the one side, the merely uniform continuity of the coefficients $a_{ij}(x)$; on the other side the lack of a Lie group structure making translation invariant the frozen operator

$$\mathcal{A}_{x_0} = \sum_{i,j=1}^{p_0} a_{ij}(x_0) \partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j}, \qquad x_0 \in \mathbb{R}^N.$$

As in [4], we overcome this last difficulty by considering the operator $\mathcal{A}$ as the stationary counterpart of the corresponding evolution operator $\mathcal{A} - \partial_t$ and looking for the estimates (1.5) and (1.6) as a consequence of analogous estimates for $\mathcal{A} - \partial_t$ on a suitable strip $S_T = \mathbb{R}^N \times [-T, T]$.

There exists a quite extensive literature related to global L^p estimates for non-degenerate elliptic and parabolic equations on the whole space with unbounded lower order coefficients and variable coefficients a_{ij}. The considered L^p-spaces are defined with respect to Lebesgue measure or with respect to an invariant measure which has also a probabilistic interpretation (see, for instance, [6], [7], [8], [10], [11], [14], [16] and the references therein).

On the other hand, to the best of our knowledge, only the papers [2], [3] and [5] deal with L^p estimates for classes of *degenerate* operators with both unbounded first order coefficients and bounded variable coefficients a_{ij}. However, we want to stress that the estimates there proved are only of *local* type.

We also mention that global L^p estimates like (1.5) are crucial in establishing weak uniqueness theorems for associated stochastic differential equations, see [15] and the references therein. Finally, a priori estimates in non-isotropic Hölder spaces for operators like (1.1) with Hölder continuous a_{ij} were proved by A. Lunardi in [13].

2 Notations and preliminary results

The operator L

Let us introduce the evolution operator

$$
\begin{aligned}
Lu(z) &= \sum_{i,j=1}^{p_0} a_{ij}\left(z\right)\partial^2_{x_i x_j}u(z) + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j}u(z) - \partial_t u(z) \\
&= \sum_{i,j=1}^{p_0} a_{ij}\left(z\right)\partial^2_{x_i x_j}u(z) + \langle x, B\nabla u(z)\rangle - \partial_t u(z)
\end{aligned}
\tag{2.1}
$$

with $z = (x,t)$ in $\mathbb{R}^{N+1}$, where now the coefficients a_{ij} possibly depend also on t. When the a_{ij}'s are time independent, we get $L = \mathcal{A} - \partial_t$. Let

$$
A(z) = \begin{bmatrix} A_0(z) & 0 \\ 0 & 0 \end{bmatrix}
$$

be an $N\times N$ matrix where $A_0(z) = \left(a_{ij}\left(z\right)\right)_{i,j=1}^{p_0}$ is a $p_0 \times p_0$ ($p_0 \le N$) symmetric and uniformly positive definite matrix for all z, satisfying

$$
\frac{1}{\Lambda}\left|\xi\right|^2 \le \sum_{i,j=1}^{p_0} a_{ij}\left(z\right)\xi_i\xi_j \le \Lambda\left|\xi\right|^2 \tag{2.2}
$$

for all $\xi \in \mathbb{R}^{p_0}$ and for some constant $\Lambda \ge 1$.

Moreover, we assume that the functions a_{ij} are uniformly continuous in $\mathbb{R}^{N+1}$ with modulus of continuity

$$
\omega\left(r\right) = \max_{i,j=1,\ldots,p_0} \sup_{\substack{z_1,z_2\in\mathbb{R}^{N+1} \\ |z_1 - z_2| \le r}} \left|a_{ij}\left(z_1\right) - a_{ij}\left(z_2\right)\right|. \tag{2.3}
$$

The operator L_{z_0}

For a fixed $z_0 \in \mathbb{R}^{N+1}$ we consider the operator L_{z_0} that differs from L only for the coefficients a_{ij}'s, that now are constant coefficients:

$$
L_{z_0}u(z) = \sum_{i,j=1}^{p_0} a_{ij}\left(z_0\right)\partial^2_{x_i x_j}u(z) + \langle x, B\nabla u(z)\rangle - \partial_t u(z),
$$

where, as above, $z = (x,t)$.

This operator is hypoelliptic; actually it can be proved (see [12]) that this fact is equivalent to the validity of the condition $C(z_0;t) > 0$ for every $t > 0$, where

$$
C\left(z_0;t\right) = \int_0^t E\left(s\right)A\left(z_0\right)E^T\left(s\right)ds, \text{ where } E\left(s\right) = \exp\left(-sB^T\right).
$$

Moreover, it is proved in [12] that L_{z_0} is left-invariant with respect to the composition law

$$(x,t)\circ(\xi,\tau)=(\xi+E(\tau)\,x,t+\tau)\,.$$

Note that

$$(\xi,\tau)^{-1}=(-E(-\tau)\,\xi,-\tau)\,.$$

We explicitly note that such a composition law is independent of z_0, since only the matrix B is involved.

The operator L_{z_0} has a fundamental solution $\Gamma(z_0;\cdot,\cdot)$,

$$\Gamma(z_0;z,\zeta)=\gamma\left(z_0;\zeta^{-1}\circ z\right)\ \text{ for } z,\zeta\in\mathbb{R}^{N+1},$$

with

$$\gamma(z_0;z)=\begin{cases}0 & \text{for } t\le 0\\ \frac{(4\pi)^{-N/2}}{\sqrt{\det C(z_0;t)}}\exp\left(-\frac{1}{4}\left\langle C^{-1}(z_0;t)\,x,x\right\rangle-t\mathrm{Tr}B\right) & \text{for } t>0\end{cases}$$

where $z=(x,t)$.

The operator K_{z_0}

By *principal part* of L_{z_0} we mean the operator

$$K_{z_0}=\sum_{i,j=1}^{p_0}a_{ij}(z_0)\,\partial^2_{x_ix_j}+\langle x,B_0\nabla\rangle-\partial_t,$$

where the matrix in the drift term is now B_0, obtained by annihilating every $*$ block in (1.4):

$$B_0=\begin{bmatrix}0 & B_1 & 0 & \dots & 0\\ 0 & 0 & B_2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & B_r\\ 0 & 0 & 0 & \dots & 0\end{bmatrix}.$$

The fundamental solution of the principal part operator K_{z_0} is $\Gamma_0(z_0;z,\zeta)=\gamma_0\left(z_0;\zeta^{-1}\circ z\right)$; namely, for $t>0$

$$\gamma_0(z_0;z)=\frac{(4\pi)^{-N/2}}{\sqrt{\det C_0(z_0;t)}}\exp\left(-\frac{1}{4}\left\langle C_0^{-1}(z_0;t)\,x,x\right\rangle\right)$$

with

$$C_0(z_0;t)=\int_0^t E_0(s)\,A(z_0)E_0^T(s)\,ds,\ \text{where } E_0(s)=\exp\left(-sB_0^T\right). \qquad (2.4)$$

Homogeneous dimension, norm and distance

For every $\lambda > 0$, let us define the matrix of *dilations on* $\mathbb{R}^N$,

$$D(\lambda) = \operatorname{diag}\left(\lambda I_{p_0}, \lambda^3 I_{p_1}, ..., \lambda^{2r+1} I_{p_r}\right)$$

where I_{p_j} denotes the $p_j \times p_j$ identity matrix, and the matrix of *dilations* on $\mathbb{R}^{N+1}$,

$$\delta(\lambda) = \left(D(\lambda), \lambda^2\right) = \operatorname{diag}\left(\lambda I_{p_0}, \lambda^3 I_{p_1}, ..., \lambda^{2r+1} I_{p_r}, \lambda^2\right).$$

Note that

$$\det(D(\lambda)) = \lambda^Q, \qquad \det(\delta(\lambda)) = \lambda^{Q+2}$$

with $Q = p_0 + 3p_1 + ... + (2r+1)\, p_r$; Q and $Q+2$ are called the *homogeneous dimension* of $\mathbb{R}^N$ and $\mathbb{R}^{N+1}$, respectively. The operator K_{z_0} is homogeneous of degree two with respect to these dilations.

There is a natural *homogeneous norm* in $\mathbb{R}^{N+1}$, induced by these dilations:

$$\|(x,t)\| = \sum_{j=1}^{N} |x_j|^{1/q_j} + |t|^{1/2},$$

where q_j are positive integers such that $D(\lambda) = \operatorname{diag}(\lambda^{q_1}, ..., \lambda^{q_N})$. Clearly, we have

$$\|\delta(\lambda)\, z\| = \lambda\, \|z\| \quad \text{for every } \lambda > 0, z \in \mathbb{R}^{N+1}.$$

A key geometrical object is the *local quasisymmetric quasidistance* d. Namely,

$$d(z,\zeta) = \left\|\zeta^{-1} \circ z\right\|.$$

Note that the homogeneous norm involved in the definition of d is related to the principal part operator K_{z_0}, while the group law $\circ$ is related to the original operator L_{z_0}. Hence this function d is not a usual quasidistance on a homogeneous group. The function $d(z,\zeta)$ satisfies the quasisymmetric and quasitriangle inequalities only for $d(z,\zeta)$ bounded (see Lemma 2.1 in [9]); this happens for instance on a fixed d-ball $B_\rho(z)$, where

$$B_\rho(z) = \left\{\zeta \in \mathbb{R}^{N+1} : d(z,\zeta) < \rho\right\}.$$

3 Estimates on a strip for evolution operators

Let S_T be the strip $\mathbb{R}^N \times [-T, T]$. We use c to denote constants that may vary from line to line.

Our main result in this section is the following:

Theorem 3.1 *Let L be as in (2.1), with the matrix B satisfying (1.4) and with uniformly continuous coefficients a_{ij} satisfying (2.2).*

For every $p \in (1,\infty)$ there exist constants $c, T > 0$ depending on p, N, p_0, the matrix B, the number Λ in (2.2), c also depending on the modulus of continuity ω in (2.3) such that

$$\sum_{i,j=1}^{p_0} \left\| \partial^2_{x_i x_j} u \right\|_{L^p(S_T)} \leq c \left\{ \|Lu\|_{L^p(S_T)} + \|u\|_{L^p(S_T)} \right\} \tag{3.1}$$

for every $u \in C_0^\infty(S_T)$.

¿From Theorem 3.1 one obtains Theorem 1.1 proceeding as follows.

Proof of Theorem 1.1. If $u : \mathbb{R}^N \to \mathbb{R}$ is a C_0^∞ function, we define

$$U(x,t) = u(x)\,\psi(t),$$

where

$$\psi \in C_0^\infty(\mathbb{R})$$

is a cutoff function with $\operatorname{sprt}\psi \subset [-T,T]$, $\int_{-T}^{T} \psi(t)\,dt > 0$. Then (3.1) applied to U gives (1.5) for u. Moreover, inequality (1.6) immediately follows by difference. ■

The crucial step toward the proof of Theorem 3.1 is a local estimate contained in the following:

Proposition 3.2 *There exist constants c, r_0 such that for every $z_0 \in S_T$, $r \leq r_0$, $u \in C_0^\infty(B_r(z_0))$, we have*

$$\sum_{i,j=1}^{p_0} \left\| \partial^2_{x_i x_j} u \right\|_{L^p(B_r(z_0))} \leq c\,\|Lu\|_{L^p(B_r(z_0))}. \tag{3.2}$$

Proof. Let $z_0 \in S_T$ and $\rho_0 \in (0,T]$ be fixed and choose a cutoff function $\eta \in C_0^\infty\left(\mathbb{R}^{N+1}\right)$ such that

$$\begin{aligned} &\eta(z) = 1 \text{ for } \|z\| \leq \rho_0/2; \\ &\eta(z) = 0 \text{ for } \|z\| \geq \rho_0. \end{aligned}$$

Then, by [9, Proposition 2.11] and (25) in [4], we have, for every $u \in C_0^\infty(B_r(z_0))$,

$$\begin{aligned} \partial^2_{x_i x_j} u = &- PV\left(L_{z_0} u * \left(\eta \partial^2_{x_i x_j} \gamma(z_0;\cdot)\right)\right) - L_{z_0} u * \left((1-\eta)\,\partial^2_{x_i x_j} \gamma(z_0;\cdot)\right) \\ &+ c_{ij}(z_0)\,L_{z_0} u \\ \equiv &- PV\left(L_{z_0} u * k_0(z_0;\cdot)\right) - L_{z_0} u * k_\infty(z_0;\cdot) + c_{ij}(z_0)\,L_{z_0} u \end{aligned} \tag{3.3}$$

having set:

$$\begin{aligned} k_0(z_0;\cdot) &= \eta \partial^2_{x_i x_j} \gamma(z_0;\cdot) \\ k_\infty(z_0;\cdot) &= (1-\eta)\,\partial^2_{x_i x_j} \gamma(z_0;\cdot) \end{aligned} \tag{3.4}$$

and

$$c_{ij}(z_0) = -\int_{\|\zeta\|=1} \partial_{x_i}\gamma_0(z_0;\zeta)\nu_j(\zeta)\, d\sigma(\zeta),$$

where ν_j denotes the j-th component of the exterior normal ν to the boundary of $\{\|\zeta\| < 1\}$. In (3.3) $*$ denotes the convolution with respect to the composition law $\circ$.

Writing

$$\begin{aligned} L_{z_0} u\,(z) &= (L_{z_0} - L)\, u\,(z) + Lu\,(z) \\ &= \sum_{i,j=1}^{p_0} (a_{ij}\,(z_0) - a_{ij}\,(z))\, \partial^2_{x_i x_j} u\,(z) + Lu\,(z) \end{aligned}$$

we get, by (3.3),

$$\begin{aligned} \partial^2_{x_i x_j} u = &- PV\,(Lu * k_0\,(z_0;\cdot)) - PV \left(\sum_{h,k=1}^{p_0} (a_{hk}\,(z_0) - a_{hk}\,(\cdot))\, \partial^2_{x_h x_k} u * k_0\,(z_0;\cdot) \right) \\ &- Lu * k_\infty\,(z_0;\cdot) - \sum_{h,k=1}^{p_0} (a_{hk}\,(z_0) - a_{hk}\,(\cdot))\, \partial^2_{x_h x_k} u * k_\infty\,(z_0;\cdot) \\ &+ c_{ij}\,(z_0)\, Lu + c_{ij}\,(z_0) \sum_{h,k=1}^{p_0} (a_{hk}\,(z_0) - a_{hk}\,(\cdot))\, \partial^2_{x_h x_k} u \\ =& I_1 + I_2 + J_1 + J_2 + A_1 + A_2. \end{aligned} \tag{3.5}$$

We now split the remaining part of the proof into several steps.

Step 1. *L^p-estimate of A_1 and A_2.*
We obviously have

$$\|A_1\|_{L^p(B_r(z_0))} \leq |c_{ij}(z_0)| \|Lu\|_{L^p(B_r(z_0))}.$$

On the other hand, by Theorem 4.1 and Remark 4.2 in Appendix, there exists an absolute constant c such that

$$|c_{ij}(z_0)| \leq \int_{\|\zeta\|=1} |\partial_{x_i}\gamma_0(z_0;\zeta)|\; d\sigma(\zeta) \leq c \int_{\|\zeta\|=1} \frac{1}{\|\zeta\|^{Q+1}}\, d\sigma(\zeta).$$

Therefore

$$\|A_1\|_{L^p(B_r(z_0))} \leq c\|Lu\|_{L^p(B_r(z_0))}. \tag{3.6}$$

Analogously, using the uniform continuity of the coefficients a_{ij}'s, we get

$$\|A_2\|_{L^p(B_r(z_0))} \leq c\,\omega(r) \sum_{h,k=1}^{p_0} \|\partial^2_{x_h x_k} u\|_{L^p(B_r(z_0))}. \tag{3.7}$$

Step 2. *L^p-estimate of J_1 and J_2.*

Without loss of generality we can assume $B_r(z_0) \subseteq S_{2T}$ for every $z_0 \in S_T$. Then

$$\|J_1\|_{L^p(B_r(z_0))} \leq c \int_{S_{2T}} |k_\infty(z_0;\zeta)|\, d\zeta\, \|Lu\|_{L^p(B_r(z_0))},$$

where the presence of the constant c depends on the fact that our group is not unimodular. On the other hand, just proceeding as in [4], pages 799-800, and using the estimates in Appendix (see Proposition 4.6) we get

$$\int_{S_{2T}} |k_\infty(z_0;\zeta)|\, d\zeta \leq c,$$

where c is independent of $z_0 \in S_T$. Therefore

$$\|J_1\|_{L^p(B_r(z_0))} \leq c\|Lu\|_{L^p(B_r(z_0))}. \tag{3.8}$$

Analogously, using the uniform continuity of the a_{ij}'s, we get

$$\|J_2\|_{L^p(B_r(z_0))} \leq c\omega(r) \sum_{h,k=1}^{p_0} \|\partial^2_{x_h x_k} u\|_{L^p(B_r(z_0))}. \tag{3.9}$$

Step 3. *L^p-estimate of I_1 and I_2.*
To estimate the L^p-norm of I_1 and I_2, we can use Theorem 3.3, getting:

$$\begin{aligned} &\|I_1\|_{L^p(B_r(z_0))} + \|I_2\|_{L^p(B_r(z_0))} \\ &\leq c \left\{ \|Lu\|_{L^p(B_r(z_0))} + \left\| \sum_{h,k=1}^{p_0} \left[a_{hk}(z_0) - a_{hk}(\cdot)\right] \partial^2_{x_h x_k} u \right\|_{L^p(B_r(z_0))} \right\} \\ &\leq c \left\{ \|Lu\|_{L^p(B_r(z_0))} + \omega(r) \sum_{h,k=1}^{p_0} \left\|\partial^2_{x_h x_k} u\right\|_{L^p(B_r(z_0))} \right\} \end{aligned} \tag{3.10}$$

with c independent of r and z_0.

Step 4. *Conclusion.*
By (3.5) and the estimates (3.6)-(3.10) in the previous steps, we get

$$\sum_{i,j=1}^{p_0} \left\|\partial^2_{x_i x_j} u\right\|_{L^p(B_r(z_0))} \leq c \left\{ \|Lu\|_{L^p(B_r(z_0))} + \omega(r) \sum_{h,k=1}^{p_0} \left\|\partial^2_{x_h x_k} u\right\|_{L^p(B_r(z_0))} \right\}$$

with c independent of r and z_0.
We now fix once and for all r_0 small enough so that $c\omega(r_0) < 1$, getting

$$\sum_{i,j=1}^{p_0} \left\|\partial^2_{x_i x_j} u\right\|_{L^p(B_r(z_0))} \leq c\, \|Lu\|_{L^p(B_r(z_0))}$$

for every $u \in C_0^\infty(B_r(z_0))$ with $r \leq r_0$, with c, r_0 independent of u and $z_0 \in S_T$.
■

Next, we have to prove the following crucial ingredient which has been used in the previous proof:

Theorem 3.3 *Let $k_0(z_0;\cdot)$ be the singular kernel defined in* (3.4)*. For every $p\in(1,\infty)$ there exists a positive constant c, independent of z_0, such that*

$$\|PV(f*k_0(z_0;\cdot))\|_{L^p(B_r(z_0))}\le c\|f\|_{L^p(B_r(z_0))} \tag{3.11}$$

for every $f\in C_0^\infty(B_r(z_0))$, $z_0\in S_T$ and $r>0$ such that $B_r(z_0)\subseteq S_{2T}$.

Proof. This theorem is analogous to Theorem 22 in [4], the novelty being the uniformity of the bound with respect to the point z_0 in the kernel $k_0(z_0;\cdot)$. As in [4], this theorem follows applying the abstract result contained in [1, Thm 3]. Without recalling the general setting of nondoubling spaces considered in [1], here we just list, for convenience of the reader, the assumptions that need to be checked on our kernel, in order to derive Theorem 3.3 from [1, Thm 3]. The constant c in (3.11) will depend only on the constants involved in the following bounds.

Let

$$k(z_0;w^{-1}\circ z)=a\left(z\right)k_0(z_0;w^{-1}\circ z)b\left(w\right)$$

where $a,b\in C_0^\infty\left(\mathbb{R}^{N+1}\right)$ with $\operatorname{sprt}a$, $\operatorname{sprt}b\subset B_r\left(z_0\right)$. Then the required properties are the following:

$$\left|k(z_0;w^{-1}\circ z)\right|+\left|k(z_0;z^{-1}\circ w)\right|\le\frac{c}{\|w^{-1}\circ z\|^{Q+2}} \tag{3.12}$$

for every $z_0\in S_T$, $z,w\in S_{2T}$ such that $\left\|w^{-1}\circ z\right\|\le1$;

$$\begin{aligned}&\left|k(z_0;w^{-1}\circ z)-k(z_0;w^{-1}\circ\overline{z})\right|+\\&\left|k(z_0;z^{-1}\circ w)-k(z_0;\overline{z}^{-1}\circ w)\right|\le c\frac{\|z^{-1}\circ\overline{z}\|}{\|w^{-1}\circ z\|^{Q+3}}\end{aligned} \tag{3.13}$$

for every $z_0\in S_T,z,\overline{z},w\in S_{2T}$ such that $\|z^{-1}\circ\overline{z}\|\le M\|w^{-1}\circ z\|$ and $\left\|w^{-1}\circ z\right\|\le1$;

$$\left|\int_{r_1\le\|\zeta^{-1}\circ z\|\le r_2}k(z_0;\zeta^{-1}\circ z)\,d\zeta\right|+\left|\int_{r_1\le\|\zeta^{-1}\circ z\|\le r_2}k(z_0;z^{-1}\circ\zeta)\,d\zeta\right|\le c \tag{3.14}$$

for every r_1,r_2 with $0<r_1<r_2$ and for all $z\in S_{2T}$ and $z_0\in S_T$;

$$h\left(z_0,\cdot\right)-h^*\left(z_0,\cdot\right)\in C^\gamma\left(B_r(z_0)\right) \tag{3.15}$$

for some positive γ, where

$$h\left(z_0,z\right)=\lim_{r\to0}\int_{r\le\|\zeta^{-1}\circ z\|}k(z_0;\zeta^{-1}\circ z)\,d\zeta; \tag{3.16}$$

$$h^*\left(z_0,z\right)=\lim_{r\to0}\int_{r\le\|\zeta^{-1}\circ z\|}k(z_0;z^{-1}\circ\zeta)\,d\zeta. \tag{3.17}$$

Now: estimates (3.12) and (3.13) follow from Theorem 4.1 and Remark 4.2 contained in the Appendix.

Let us prove (3.14). Actually, we will bound the first integral, the bound on the second being analogous. Moreover, we actually prove the following

$$\left|\int_{r_1\le\|\zeta^{-1}\circ z\|\le r_2} k_0(z_0;\zeta^{-1}\circ z)\,d\zeta\right| \le c, \tag{3.18}$$

which implies the analogous bound on k by the same argument contained in [4, Prop. 18]. To show (3.18), we proceed as in [4], page 803. Without loss of generality we assume $r_2 \le \rho_0$, where ρ_0 is the positive constant introduced at the beginning of the proof of Proposition 3.2; in fact, $k_0(z_0;w) = 0$ for $\|w\| > \rho_0$.

We have:

$$\int_{r_1\le\|\zeta^{-1}\circ z\|\le r_2} k_0(z_0;\zeta^{-1}\circ z)\,d\zeta = A(z_0;r_1,r_2) + B(z_0;r_1,r_2),$$

where

$$A(z_0;r_1,r_2) = \int_{r_1\le\|w\|\le r_2} \eta(w)\partial^2_{x_ix_j}\gamma(z_0;w)\,dw$$

and

$$B(z_0;r_1,r_2) = \int_{r_1\le\|w\|\le r_2} \eta(w)\partial^2_{x_ix_j}\gamma(z_0;w)\left(e^{\tau Tr(B)}-1\right)dw, \qquad w=(\xi,\tau).$$

Then, by (4.3)

$$B(z_0;r_1,r_2) \le c\int_{r_1\le\|w\|\le r_2}\frac{1}{\|w\|^Q}\,dw \le c\int_{\|w\|\le\rho_0}\frac{1}{\|w\|^Q}\,dw$$

with c independent of $z_0\in S_T$. Moreover, if $r_2 \le \frac{\rho_0}{2}$, then integrating by parts

$$\begin{aligned} A(z_0;r_1,r_2) &= \int_{\|w\|=r_2}\partial_{x_i}\gamma(z_0;w)\nu_j\,d\sigma(w) - \int_{\|w\|=r_1}\partial_{x_i}\gamma(z_0;w)\nu_j\,d\sigma(w) \\ &=: I(z_0;r_2) - I(z_0;r_1). \end{aligned}$$

Now we estimate $I(z_0;\rho)$ by proceeding as in [9], page 1280. We have

$$\begin{aligned} I(z_0;\rho) &= \int_{\|\zeta\|=1}\partial_{x_i}\gamma_\rho(z_0;\zeta)\nu_j\,d\sigma(\zeta) \\ &= \int_{\|\zeta\|=1}\left(\partial_{x_i}\gamma_\rho(z_0;\zeta) - \partial_{x_i}\gamma_0(z_0;\zeta)\right)\nu_j\,d\sigma(\zeta) \\ &\quad + \int_{\|\zeta\|=1}\partial_{x_i}\gamma_0(z_0;\zeta)\nu_j\,d\sigma(\zeta) \end{aligned}$$

where $\gamma_\rho(z_0;\cdot)$ is defined as in [9], (2.24).

The last integrand can be estimated by a constant independent of $z_0 \in S_T$, thanks to (4.2) and Remark 4.2. On the other hand, from (2.45) in [9] we get, for a suitable c independent of $z_0 \in S_T$:

$$\left| \int_{\|\zeta\|=1} \left(\partial_{x_i}\gamma_\rho(z_0;\zeta) - \partial_{x_i}\gamma_0(z_0;\zeta)\right)\nu_j\, d\sigma(\zeta) \right| \leq c\,\rho \int_{\|\zeta\|=1} \frac{1}{\sqrt{\tau}}\gamma(\zeta)\, d\sigma(\zeta),$$

$\zeta = (x,\tau)$, where γ is the fundamental solution with pole at the origin of

$$\mu \sum_{i=1}^{p_0} \partial_{x_i}^2 + \langle x, B_0 \nabla\rangle - \partial_t$$

for a suitable $\mu > 0$ independent of $z_0 \in S_T$. Note that the last integral is an absolute constant.

Suppose now $\frac{\rho_0}{2} \leq r_2 \leq \rho_0$. Then we can write

$$A\left(z_0; r_1, r_2\right) \leq \left| \int_{r_1 < \|w\| < \rho_0/2} k_0\left(z_0; w\right) dw \right| + \left| \int_{\rho_0/2 < \|w\| < r_2} k_0\left(z_0; w\right) dw \right|.$$

The first term can be bounded as above, while the second one is bounded by

$$\int_{\rho_0/2 \leq \|w\| \leq \rho_0} c\,\|w\|^{-(2+Q)}\; dw$$

with c independent of z_0, see (4.3). This completes the proof of (3.18).

Finally, let us prove the Hölder continuity of the function

$$h\left(z_0, \cdot\right) - h^*\left(z_0, \cdot\right)$$

defined in (3.16)-(3.17).[1]

$$\begin{aligned}
h\left(z_0, z\right) &= \lim_{r\to 0} a\left(z\right) \int_{r \leq \|\zeta^{-1}\circ z\|} k_0(z_0;\zeta^{-1}\circ z) b\left(\zeta\right)\; d\zeta = \\
&= \lim_{r\to 0} a\left(z\right) \int_{r\leq\|w\|} k_0(z_0;w) b\left(z\circ w^{-1}\right)\; e^{\tau Tr(B)} dw \\
&= a\left(z\right) \int_{\|w\|\leq\rho_0} k_0(z_0;w)\left[b\left(z\circ w^{-1}\right) - b\left(z\right)\right] e^{\tau Tr(B)} dw + \\
&+ a\left(z\right) b\left(z\right) \int_{\|w\|\leq\rho_0} k_0(z_0;w)\left[e^{\tau Tr(B)} - 1\right] dw \\
&+ \lim_{r\to 0} a\left(z\right) b\left(z\right) \int_{r\leq\|w\|} k_0(z_0;w) dw \\
&= h_1\left(z_0, z\right) + h_2\left(z_0, z\right) + h_3\left(z_0, z\right).
\end{aligned}$$

[1] We take this opportunity to notice that in [4] this check has not been explicitly done.

Now:

$$h_3(z_0, z) = a(z)\, b(z)\, c(z_0)$$

with $a(\cdot)$, $b(\cdot)$ smooth and $c(z_0)$ uniformly bounded in z_0 by the previous bound (3.18). Also,

$$h_2(z_0, z) = a(z)\, b(z)\, c_1(z_0)$$

with $c(z_0)$ uniformly bounded in z_0 by the same argument used above to bound $B(z_0; r_1, r_2)$. Let us come to $h_1(z_0, z)$. If Z is any right-invariant differential operator, then

$$\begin{aligned} Zh_1(z_0, z) &= Za(z) \int_{\|w\| \le \rho_0} k_0(z_0; w) \left[b\left(z \circ w^{-1}\right) - b(z) \right] e^{\tau Tr(B)} dw \\ &\quad + a(z) \int_{\|w\| \le \rho_0} k_0(z_0; w) \left[Zb\left(z \circ w^{-1}\right) - Zb(z) \right] e^{\tau Tr(B)} dw, \end{aligned}$$

hence

$$|Zh_1(z_0, z)| \le c \int_{\|w\| \le \rho_0} |k_0(z_0; w)|\, |w|\, dw \le c.$$

Since this procedure can be iterated, we get an upper bound on any derivative of the kind $|Z_1 Z_2 ... Z_k h_1(z_0, z)|$, hence (since the commutators of suitable right invariant vector fields span $\mathbb{R}^{N+1}$) also on $|\nabla h_1(z_0, z)|$. Therefore the function $h_1(z_0, \cdot)$ is Lipschitz continuous, uniformly with respect to z_0. The function $h^*(z_0, \cdot)$ can be handled similarly. This completes the proof of the conditions which are sufficient to apply [1, Thm 3] and deduce (3.11), with a constant c independent of z_0. ■

In order to deduce Theorem 3.1 from Proposition 3.2, we now need to recall a covering lemma, see Lemma 21 in [4] (note that this result is not standard since our space is not globally doubling):

Lemma 3.4 *For every $r_0 > 0$ and $K > 1$ there exist $r \in (0, r_0)$, a positive integer M and a sequence of points $\{z_i\}_{i=1}^{\infty} \subset S_T$ such that:*

$$S_T \subset \bigcup_{i=1}^{\infty} B_r(z_i); \tag{3.19}$$

$$\sum_{i=1}^{\infty} \chi_{B_{Kr}(z_i)}(z) \le M \quad \forall z \in S_T. \tag{3.20}$$

Proof of Theorem 3.1. Let us apply the previous lemma with r_0 as in Proposition 3.2; for a fixed $r \in (0, r_0)$, with $r/2$ satisfying (3.19), (3.20). Pick $A \in C_0^\infty(B_r(0))$, $A = 1$ in $B_{r/2}(0)$, $0 \le A \le 1$ and let $a_k(z) = A\left(z_k^{-1} \circ z\right)$.

Let now $u \in C_0^\infty(S_T)$. By (3.19) we can write

$$\begin{aligned} \left\| \partial^2_{x_i x_j} u \right\|^p_{L^p(S_T)} &\le \sum_{k=1}^{\infty} \left\| \partial^2_{x_i x_j} u \right\|^p_{L^p\left(B_{r/2}(z_k)\right)} = \sum_{k=1}^{\infty} \left\| \partial^2_{x_i x_j} (a_k u) \right\|^p_{L^p\left(B_{r/2}(z_k)\right)} \\ &\le \sum_{k=1}^{\infty} \left\| \partial^2_{x_i x_j} (a_k u) \right\|^p_{L^p(B_r(z_k))}. \end{aligned} \tag{3.21}$$

On the other hand, by (3.2) we have

$$\left\|\partial^2_{x_ix_j}(a_ku)\right\|_{L^p(B_r(z_k))} \le c\left\|L(a_ku)\right\|_{L^p(B_r(z_k))}$$
$$\le c\left\{\|a_kLu\|_{L^p(B_r(z_k))} + 2\sum_{l,m=1}^{p_0}\|\partial_{x_l}a_k\partial_{x_m}u\|_{L^p(B_r(z_k))} + \|uLa_k\|_{L^p(B_r(z_k))}\right\}. \tag{3.22}$$

By recalling that the operators ∂_{x_l}, $l = 1, ..., p_0$, and $Y_0 := \sum_{i,j=1}^N b_{ij}x_i\partial_{x_j}$ are left invariant with respect to the group law $\circ$, we have

$$\sup_{z\in B_r(z_k)}|\partial_{x_l}a_k(z)| = \sup_{z\in B_r(0)}|\partial_{x_l}A(z)| \le c, \qquad l = 1, ..., p_0,$$
$$\sup_{z\in B_r(z_k)}|Y_0a_k(z)| = \sup_{z\in B_r(0)}|Y_0A(z)| \le c$$

and

$$\sup_{z\in B_r(z_k)}\left|\partial^2_{x_ix_j}a_k(z)\right| = \sup_{z\in B_r(0)}\left|\partial^2_{x_ix_j}A(z)\right| \le c, \qquad i, j = 1, 2, ..., p_0.$$

As a consequence

$$\sup_{z\in B_r(z_k)}|La_k(z)| \le c$$

with c independent of k. Hence (3.22) gives

$$\left\|\partial^2_{x_ix_j}(a_ku)\right\|_{L^p(B_r(z_k))} \le c\left\{\|Lu\|_{L^p(B_r(z_k))} + 2\sum_{l,m=1}^{p_0}\|\partial_{x_m}u\|_{L^p(B_r(z_k))} + \|u\|_{L^p(B_r(z_k))}\right\},$$

c independent of k. Inserting the last inequality in (3.21) and recalling (3.20) we get

$$\left\|\partial^2_{x_ix_j}u\right\|^p_{L^p(S_T)} \le c\sum_{k=1}^{\infty}\left\{\|Lu\|^p_{L^p(B_r(z_k))} + \sum_{m=1}^{p_0}\|\partial_{x_m}u\|^p_{L^p(B_r(z_k))} + \|u\|^p_{L^p(B_r(z_k))}\right\}$$
$$\le cM\left\{\|Lu\|^p_{L^p(S_T)} + \sum_{m=1}^{p_0}\|\partial_{x_m}u\|^p_{L^p(S_T)} + \|u\|^p_{L^p(S_T)}\right\}.$$

This also gives

$$\sum_{i,j=1}^{p_0}\left\|\partial^2_{x_ix_j}u\right\|_{L^p(S_T)} \le cM\left\{\|Lu\|_{L^p(S_T)} + \sum_{m=1}^{p_0}\|\partial_{x_m}u\|_{L^p(S_T)} + \|u\|_{L^p(S_T)}\right\}$$

which, by the classical interpolation inequality

$$\|\partial_{x_m}u\|_{L^p(S_T)} \le \varepsilon\left\|\partial^2_{x_mx_m}u\right\|_{L^p(S_T)} + \frac{c}{\varepsilon}\|u\|_{L^p(S_T)},$$

yields (3.1). So we are done. ■

4 Appendix: uniform bounds on the fundamental solution of L_{z_0}

The aim of this section is to prove the following result, which has been exploited in the proof of Proposition 3.2 and Theorem 3.3:

Theorem 4.1 *There exists a positive constant c independent of $z_0 \in S_T$ such that*

$$|\gamma(z_0;\zeta)| \le \frac{c}{\|\zeta\|^Q}, \tag{4.1}$$

$$\left|\partial_{x_j}\gamma(z_0;\zeta)\right| \le \frac{c}{\|\zeta\|^{Q+1}} \qquad j=1,...,p_0, \tag{4.2}$$

$$\left|\partial^2_{x_i x_j}\gamma(z_0;\zeta)\right| \le \frac{c}{\|\zeta\|^{Q+2}} \qquad i,j=1,...,p_0, \tag{4.3}$$

for every $\zeta \in S_{2T}$.
Morever, if $H \subset \mathbb{R}^N$ is a compact set there exist constants c' and M, depending on H but not on z_0, such that

$$\begin{aligned}
\left|\gamma(z_0;w^{-1}\circ z)-\gamma(z_0;w^{-1}\circ\bar z)\right| &\le c'\frac{\|z^{-1}\circ\bar z\|}{\|w^{-1}\circ z\|^{Q+1}}, \\
\left|\partial_{x_j}\gamma(z_0;w^{-1}\circ z)-\partial_{x_j}\gamma(z_0;w^{-1}\circ\bar z)\right| &\le c'\frac{\|z^{-1}\circ\bar z\|}{\|w^{-1}\circ z\|^{Q+2}} \qquad j=1,...,p_0, \\
\left|\partial^2_{x_i x_j}\gamma(z_0;w^{-1}\circ z)-\partial^2_{x_i x_j}\gamma(z_0;w^{-1}\circ\bar z)\right| &\le c'\frac{\|z^{-1}\circ\bar z\|}{\|w^{-1}\circ z\|^{Q+3}} \qquad i,j=1,...,p_0,
\end{aligned} \tag{4.4}$$

for every z,$\bar z$,$w \in S_{2T}$ such that $\|z^{-1}\circ\bar z\| \le M\|w^{-1}\circ z\|$ and $w^{-1}\circ z \in H\times[-2T,2T]$.
The previous estimates still hold replacing $\gamma(z_0;z)$ with $\gamma(z_0;z^{-1})$.

Remark 4.2 *The estimates of Theorem 4.1 obviously hold if we replace $\gamma(z_0;\cdot)$ with $\gamma_0(z_0;\cdot)$. In this case we can exploit the homogeneity of γ_0 to obtain* (4.1)–(4.3) *for every ζ in the strip $\mathbb{R}^N\times[-1,1]$.*

The above theorem will follow by a careful inspection of several arguments contained in [9] and [12]. We first need to establish several lemmas.

In the following, $\mathcal{I}$ denotes the $N\times N$ matrix

$$\mathcal{I} := \begin{bmatrix} I_{p_0} & 0 \\ 0 & 0 \end{bmatrix}$$

where I_{p_0} is the $p_0\times p_0$ identity matrix. Moreover, for every $t>0$, $\widetilde{C}(t)$ is the $N\times N$ matrix defined as follows

$$\widetilde{C}(t) = \int_0^t E_0(s)\mathcal{I}E_0^T(s)\,ds \tag{4.5}$$

with $E_0(s)$ as in (2.4). Notice that $\widetilde{C}(t) > 0$ for every $t > 0$, or, equivalently, that the operator

$$\sum_{i=1}^{p_0} \partial^2_{x_i x_i} u(z) + \langle x, B_0 \nabla u(z) \rangle - \partial_t u(z)$$

is hypoelliptic (see [12]).

The following preliminary lemma holds.

Lemma 4.3 *The inequalities below hold true for all* $z_0 \in \mathbb{R}^{N+1}$*:*

$$\frac{1}{\Lambda} \langle \widetilde{C}(1) y, y \rangle \le \langle C_0(z_0; 1) y, y \rangle \le \Lambda \langle \widetilde{C}(1)\, y, y \rangle \qquad \forall y \in \mathbb{R}^N \tag{4.6}$$

and

$$\frac{1}{\Lambda^N} \det \widetilde{C}(1) \le \det C_0(z_0; 1) \le \Lambda^N \det \widetilde{C}(1). \tag{4.7}$$

Proof. We have that

$$\frac{1}{\Lambda} \mathcal{I} \le A(z_0) \le \Lambda \mathcal{I} \qquad \text{for all } z_0 \in \mathbb{R}^{N+1}.$$

Thus, (4.6) holds. Inequalities (4.7) are an easy consequence of (4.6). ∎

Lemma 4.4 *There exist* $M \ge 1, T > 0$ *such that for every* $x \in \mathbb{R}^N$, $z_0 \in \mathbb{R}^{N+1}$, $t \in [0, T]$,

$$\frac{1}{M} \langle \widetilde{C}(t) x, x \rangle \le \langle C(z_0; t) x, x \rangle \le M \langle \widetilde{C}(t) x, x \rangle \tag{4.8}$$

and

$$\frac{1}{M} \det \widetilde{C}(t) \le \det C(z_0; t) \le M \det \widetilde{C}(t). \tag{4.9}$$

Proof. It is a known fact (see [12, Proposition 2.3]) that

$$\begin{aligned} C_0\,(z_0; t) &= D(\sqrt{t}) C_0\,(z_0; 1)\, D(\sqrt{t}) \\ \tilde{C}\,(t) &= D(\sqrt{t}) \tilde{C}\,(1)\, D(\sqrt{t}), \qquad \forall\, t > 0. \end{aligned}$$

Then (4.6) implies

$$\frac{1}{\Lambda} \langle \widetilde{C}(t) x, x \rangle \le \langle C_0(z_0; t) x, x \rangle \le \Lambda \langle \widetilde{C}(t) x, x \rangle. \tag{4.10}$$

Therefore, to prove (4.8) it is enough to look for positive c_1, c_2 such that

$$\begin{aligned} c_1 \langle C_0\,(z_0; t)\, x, x \rangle \,(1 + O(t)) &\le \langle C\,(z_0; t)\, x, x \rangle \\ &\le c_2 \langle C_0\,(z_0; t)\, x, x \rangle \,(1 + O(t)) \qquad \text{as } t \to 0 \end{aligned} \tag{4.11}$$

with $O(t)$ uniform w.r.t. z_0.

This follows using the arguments in [12, p. 46]. Indeed, set $x = D\left(\frac{1}{\sqrt{t}}\right) y$ we get

$$\frac{\langle C(z_0;t)\,x,x\rangle}{\langle C_0(z_0;t)\,x,x\rangle} = 1 + \frac{\langle (C(z_0;t) - C_0(z_0;t))x,x\rangle}{\langle C_0(z_0;t)x,x\rangle}$$
$$= 1 + \frac{\langle D\left(\frac{1}{\sqrt{t}}\right)(C(z_0;t) - C_0(z_0;t))D\left(\frac{1}{\sqrt{t}}\right)y,y\rangle}{\langle C_0(z_0;1)y,y\rangle}.$$

Now, by the proof of Lemma 3.2 in [12] and a careful check of the block decomposition of the matrices $C(z_0;t)$ and $C_0(z_0;t)$, see Lemma 3.1 in [12], we get

$$\left\| D\left(\frac{1}{\sqrt{t}}\right)(C(z_0;t) - C_0(z_0;t))D\left(\frac{1}{\sqrt{t}}\right)\right\| \le ct \quad \text{as } t \to 0^+, \tag{4.12}$$

uniformly w.r.t. z_0. Thus, we get (4.11).

Let us now prove (4.9). By (4.10), we get

$$\frac{1}{\Lambda^N}\det \widetilde{C}(t) \le \det C_0(z_0;t) \le \Lambda^N \det \widetilde{C}(t).$$

Moreover, by (4.11) there exist positive constants c_3, c_4 such that

$$c_3(1 + O(t))\det C_0(z_0;t) \le \det C(z_0;t) \le c_4(1 + O(t))\det C_0(z_0;t)$$

as t goes to 0^+, uniformly w.r.t. $z_0 \in \mathbb{R}^{N+1}$. Thus, (4.9) follows. ■

Now, we turn to prove estimates for $C^{-1}(z_0;\cdot)$.

Lemma 4.5 *The following inequalities hold:*

(1) there exist $M \ge 1, T > 0$ such that for every $x \in \mathbb{R}^N$, $z_0 \in \mathbb{R}^{N+1}$, $t \in [0,T]$,

$$\frac{1}{M}\langle C_0^{-1}(z_0;t)x,x\rangle \le \langle C^{-1}(z_0;t)x,x\rangle \le M\langle C_0^{-1}(z_0;t)x,x\rangle \tag{4.13}$$

(2) let $\lambda_{\widetilde{C}}$ and $\Lambda_{\widetilde{C}}$ be the smallest and the largest eigenvalue of the symmetric positive definite matrix $\widetilde{C}(1)$, respectively. Then

$$\frac{1}{\Lambda\Lambda_{\widetilde{C}}}\left|D\left(\frac{1}{\sqrt{t}}\right)x\right|^2 \le \langle C_0^{-1}(z_0;t)x,x\rangle \le \frac{\Lambda}{\lambda_{\widetilde{C}}}\left|D\left(\frac{1}{\sqrt{t}}\right)x\right|^2, \tag{4.14}$$

for all $x \in \mathbb{R}^N$ and for all $z_0 \in \mathbb{R}^{N+1}$.

Proof. The proof of (4.13) follows the lines of the proof of (3.10) in [12], using (4.12) in place of (3.8) in [12].

As far as (4.14) it is concerned, we begin noticing that, see [12, p. 42],

$$C_0^{-1}(z_0;t) = D\left(\frac{1}{\sqrt{t}}\right) C_0^{-1}(z_0;1)\, D\left(\frac{1}{\sqrt{t}}\right), \qquad \forall\, t>0.$$

Thus we have

$$\begin{aligned}\langle C_0^{-1}(z_0;t)x,x\rangle &= \langle C_0^{-1}(z_0;1)D\left(\frac{1}{\sqrt{t}}\right)x, D\left(\frac{1}{\sqrt{t}}\right)x\rangle \\ &\le \max_{|y|=1}\langle C_0^{-1}(z_0;1)y,y\rangle \left|D\left(\frac{1}{\sqrt{t}}\right)x\right|^2 = \frac{\left|D\left(\frac{1}{\sqrt{t}}\right)x\right|^2}{\min\limits_{|y|=1}\langle C_0(z_0;1)y,y\rangle}.\end{aligned}$$

By (4.6)

$$\min_{|y|=1}\langle C_0(z_0;1)y,y\rangle \ge \frac{1}{\Lambda}\min_{|y|=1}\langle \tilde{C}(1)y,y\rangle = \frac{\lambda_{\tilde{C}}}{\Lambda}$$

and the last inequality in (4.14) follows. Analogously the first one can be proved. ■

Collecting the results in Lemma 4.4 and Lemma 4.5 we easily get the following:

Proposition 4.6 *Let* $\widetilde{C}$ *be defined as in* (4.5). *There exist positive constants* T *and* m*, depending only on the operator* L*, such that the following inequalities hold for every* $t\in[-2T,2T]$*, every* $z_0\in\mathbb{R}^{N+1}$ *and every* $x\in\mathbb{R}^N$*:*

(a) $\frac{1}{m}\langle \widetilde{C}(t)x,x\rangle \le \langle C(z_0;t)x,x\rangle \le m\langle \widetilde{C}(t)x,x\rangle$;

(b) $\frac{1}{m}\det \widetilde{C}(t) \le \det C(z_0;t) \le m \det \widetilde{C}(t)$;

(c) $\frac{1}{m}\left|D\left(\frac{1}{\sqrt{t}}\right)x\right|^2 \le \langle C^{-1}(z_0;t)x,x\rangle \le m\left|D\left(\frac{1}{\sqrt{t}}\right)x\right|^2$.

The above estimates, together with the procedure in [9, proof of Proposition 2.7], imply the uniform bounds in Theorem 4.1 for $\gamma(z_0;z)$. To prove analogous estimates for $\gamma(z_0;z^{-1})$ and its derivatives, one can proceed in a similar way.

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