BMO estimates for nonvariational operators with discontinuous coefficients structured on Hörmander's vector fields on Carnot groups*

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Abstract

We consider the class of operators

$$Lu = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j u$$

where $X_1, X_2, ..., X_q$ are homogeneous left invariant Hörmander's vector fields on $\mathbb{R}^N$ with respect to a structure of Carnot group, $q \leq N$, the matrix $\{a_{ij}\}$ is symmetric and uniformly positive on $\mathbb{R}^q$, the coefficients a_{ij} belong to $L^\infty \cap VLMO_{loc}(\Omega)$ ("vanishing logarithmic mean oscillation") with respect to the distance induced by the vector fields (in particular they can be discontinuous), Ω is a bounded domain of $\mathbb{R}^N$. We prove local estimates in $BMO_{loc} \cap L^p$ of the kind:

$$\|X_i X_j u\|_{BMO^p_{loc}(\Omega')} + \|X_i u\|_{BMO^p_{loc}(\Omega')} \leq$$
$$\leq c \left\{ \|Lu\|_{BMO^p_{loc}(\Omega)} + \|u\|_{BMO^p_{loc}(\Omega)} \right\}$$

for any $\Omega' \Subset \Omega$, $1 < p < \infty$.

Even in the uniformly elliptic case $X_i = \partial_{x_i}$, $q = N$ our estimates improve the known results.

Contents

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1 Introduction

Context and main result of the paper

Let $X_0, X_1, ..., X_q$ be a system of smooth vector fields,

$$X_i = \sum_{j=1}^{N} b_{ij}(u)\,\partial_{u_j}$$

defined in the whole $\mathbb{R}^N$, and assume they satisfy Hörmander's rank condition in $\mathbb{R}^N$: the vector fields X_i, and their commutators $[X_i, X_j], [X_k, [X_i, X_j],], ...$ up to some fixed length span $\mathbb{R}^N$ at any point. Then, a famous theorem proved by Hörmander in 1967 (see [26]), states that the second order differential operator

$$L = \sum_{i=1}^{q} X_i^2 + X_0 \tag{1.1}$$

is hypoelliptic, that is $u \in C^\infty(\Omega)$ whenever u is a distributional solution to $Lu = f$ in an open set $\Omega \subset \mathbb{R}^N$ with $f \in C^\infty(\Omega)$. In 1975 Folland [23] studied the class of Hörmander's operators (1.1) which admits an underlying structure of *homogeneous group*. This means that the vector fields X_i are left invariant with respect to a Lie group operation in $\mathbb{R}^N$ (which we think as "translations") and the operator L is homogeneous of degree 2 with respect to a one-parameter family of Lie group automorphisms (which we think as "dilations"). Then, Folland proved that there exists a global fundamental solution Γ for L, which is translation invariant and homogeneous of degree $2 - Q$ with respect to the dilations, where Q is the so called *homogeneous dimension* of the group. This fact allows to apply the theory of singular integrals in homogeneous groups, and derive from representation formulas suitable a priori estimates for the second order derivatives X_iX_ju $(i, j = 1, 2, ..., q)$ or X_0u (note that the "drift" vector field X_0 has weight two in the operator L). Later, in [30], Rothschild and Stein showed how the analysis of a general operator (1.1), also in absence of an underlying structure of homogeneous group, can be performed by a suitable technique of "lifting and approximation" which reduces the study of L to that of an operator of the kind studied by Folland.

In the last decade, more general families of second order differential operators modeled on Hörmander's vector fields have been studied, namely operators of the kinds

$$\mathcal{L}=\sum_{i,j=1}^{q} a_{ij}\left(x\right) X_i X_j \tag{1.2}$$

$$\mathcal{L}=\sum_{i,j=1}^{q} a_{ij}\left(x\right) X_i X_j - \partial_t \tag{1.3}$$

$$\mathcal{L}=\sum_{i,j=1}^{q} a_{ij}\left(x\right) X_i X_j + X_0 \tag{1.4}$$

where the matrix $\{a_{ij}\left(x\right)\}_{i,j=1}^{q}$ is symmetric positive definite, the coefficients are bounded and satisfy suitable mild regularity assumptions, for instance they belong to Hölder or VMO spaces defined with respect to the distance induced by the vector fields. Since the a_{ij}'s are not C^∞, these operators are no longer hypoelliptic. Nevertheless, a priori estimates on second order derivatives with respect to the vector fields are a natural result which does not in principle require smoothness of the coefficients. Namely, a priori estimates in L^p (with coefficients a_{ij} in $VMO \cap L^\infty$) have been proved in [7] for operators (1.2), in [6] for operators (1.4) but in homogeneous groups, and in [13] for operators (1.4) in the general case; a priori estimates in C^α spaces (with coefficients a_{ij} in C^α) have been proved in [9] for operators (1.3), in [25] for operators (1.4) but in homogeneous groups, and in [13] for operators (1.4) in the general case. See also the recent monograph [10] for more results on these classes of operators and for a larger bibliographic account.

A somewhat endpoint case of L^p estimates consists in BMO type estimates, which is the issue that we address in this paper. We will prove, for operators (1.2) in homogeneous groups, with coefficients $a_{ij}=a_{ji}$ satisfying the condition

$$\Lambda|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}\left(x\right)\xi_i\xi_j \leq \Lambda^{-1}|\xi|^2 \;\forall \xi \in \mathbb{R}^q, \text{a.e. } x \in \Omega$$

and having "vanishing logarithmic mean oscillation" ($VLMO$) in a bounded domain Ω, a priori estimates of the kind

$$\begin{aligned}&\sum_{i,j=1}^{q} \|X_i X_j u\|_{BMO_{loc}^{p}(\Omega')} + \sum_{i=1}^{q} \|X_j u\|_{BMO_{loc}^{p}(\Omega')}\\ &\leq c\left\{\|Lu\|_{BMO_{loc}^{p}(\Omega)} + \|u\|_{BMO_{loc}^{p}(\Omega)}\right\}\end{aligned}$$

for any $p \in (1,\infty)$, $\Omega' \Subset \Omega$ (see Theorem 2.10 for the precise statement; also, the precise meaning of these norms will be defined later). Let us stress that the $VLMO$ assumption allows some kind of discontinuity of the coefficients a_{ij}.

Comparison with the existing literature

Remarkably, this estimate appears to be new even in the nonvariational uniformly elliptic case ($q = n$, $X_i = \partial_{x_i}$ for $i = 1, 2, ..., n$). Actually, a few papers are devoted to BMO estimates for the second derivatives of the solutions to nonvariational uniformly elliptic equations: we can quote the old paper by Peetre [29], establishing local BMO estimates for elliptic equations with uniformly continuous coefficients (with a continuity modulus $o(1/|\log t|)$), and the more recent ones by Chang-Dafni-Stein [16], containing global BMO estimates for the laplacian, and by Chang-Li [17], dealing with elliptic operators with Dini-continuous coefficients. Elliptic equations in divergence form with $VLMO$ coefficients have been studied by Acquistapace [1], Huang [27] (in which also some nondivergence form equation has studied), while BMO estimates for some nonlinear equations have been established by Caffarelli-Huang [14].

For operators modeled on Hörmander vector fields and written in divergence form, BMO estimates have been proved by Di Fazio-Fanciullo in [21], while Bramanti-Brandolini in [8] have proved BMO type estimates in a scale of spaces BMO_ϕ for operators (1.2) built on general Hörmander's vector fields, assuming a certain modulus of continuity of the coefficients a_{ij}.

Problems and strategy

Although the continuity requirement on the coefficients asked in [8] is not a strong one, it represents a significant difference with the L^p theory developed in [6], [7] under the VMO assumption and with the BMO theory developed in the present paper under the $VLMO$ assumption.

The reason of this difference has its roots in the real variable machinery which is applied to prove these estimates, namely suitable extensions of the famous L^p estimate for the commutator of a Calderón-Zygmund operator with the multiplication by a BMO function, proved by Coifman-Rochberg-Weiss in [20], which was first applied to the proof of L^p a priori estimates for uniformly elliptic operators with VMO coefficients by Chiarenza-Frasca-Longo in [18], [19]. To put the real analysis problem into its context, let us recall some facts. It is known that, under fairly broad assumptions, a singular integral operator maps L^∞ into BMO. Under much more stringent assumptions it can be proved that it maps BMO into BMO. This was shown in [29] for convolution type operators in $\mathbb{R}^N$, and in [8], in spaces of homogeneous type of finite measure, for singular integrals satisfying a strong cancellation property. The multiplication operator for a function a maps BMO into BMO provided $a \in L^\infty \cap LMO$ (where LMO stands for "logarithmic bounded mean oscillation"), as proved in [31]. We also need a result stating that the commutator of a singular integral operators with the multiplication for a maps BMO into BMO, with operator norm bounded by the LMO seminorm of a. Actually, we want the operator norm of the commutator to be small whenever a has small *oscillation*, but not small absolute size. A result of this kind has been proved by Sun-Su in [33] for singular integral operators of convolution type in $\mathbb{R}^N$, satisfying a strong cancellation property. In this case, the commutator is proved to map $BMO \cap L^p$ into itself ($1 < p < \infty$) continuously, with operator norm bounded by the LMO seminorm of a:

$$[[T, a] f]_{BMO} + \|[T, a] f\|_{L^p} \leq c\, [a]_{LMO} \left([f]_{BMO} + \|f\|_{L^p}\right).$$

This result is clever under several regards. First, it exploits the idea of bounding the $BMO \cap L^p$ norm of the commutator with the analogous norm of f, and not separately the BMO seminorm of the commutator with the BMO seminorm of f; secondly, it relies on the very strong cancellation properties enjoyed by classical Calderón-Zygmund convolution-type kernels. The present paper starts from the idea of extending this commutator theorem to the context of convolution-type singular integrals on homogeneous groups, which should be useful to handle operators (1.2) in this context, in view of the results and techniques of [23], [6], [8]. However, in contrast with the global, convolution nature of our singular integrals, we are interested in the study of an operator (1.2) on a bounded domain Ω; this means that we don't want to assume the coefficients a_{ij} defined on the whole $\mathbb{R}^N$, nor rely on an extension result for LMO in this abstract context. Therefore, the commutator estimate that we prove has to be established directly in a *local* form. This forces us to go through the whole argument in [33] and reshape it on a new kind of local $BMO_{loc}(\Omega_1, \Omega_2)$ spaces, defined averaging the function over the balls centered at points of some open set Ω_1 and contained in a larger open set $\Omega_2 \subseteq \Omega$. This fact also serves another scope, namely avoiding to handle the sets $B \cap \Omega$, which would require some extra assumption on Ω in order to use the doubling condition. Moreover, it would seem to us rather unnatural to express the assumption on the coefficients a_{ij} in a form which involves their boundary behavior, since after all we are just proving interior estimates. Under this respect, the present paper moves in the spirit of the recent research about "local real analysis in locally homogeneous spaces" carried out in [12].

Once the real analysis part of this research is set into its proper frame, one can try to follow as close as possible the general line first drawn in [6]. In doing so, another major problem arises, namely the necessity of getting some new uniform upper bound related to the fundamental solution $\Gamma(x_0, u)$ of the "frozen operator"

$$L_0 = \sum_{i,j=1}^{q} a_{ij}(x_0) X_i X_j.$$

Actually, to apply the real variable machinery to the concrete singular integral operators which appear in our representation formulas, we have to resort to the technique of expansion of $\Gamma(x_0, \cdot)$ in spherical harmonics, first employed by Calderón-Zygmund in [15] and already used in all the aforementioned papers dealing with L^p estimates for nonvariational operators structured on Hörmander's vector fields. To get a control on the coefficients of this expansion, we need some upper bound on the u-derivatives of any order of $\Gamma(x_0, u)$, say for $|u| = 1$, uniform with respect to x_0. In [6] the following estimate was proved:

$$\sup_{x\in\Omega,\|u\|=1}\left|\left(\frac{\partial}{\partial u}\right)^{\beta}\Gamma(x;u)\right| \leq c(\beta), \tag{1.5}$$

for any multiindex β. In the present situation we also need a control on the LMO norm, and not just the L^∞ norm, of the coefficients of the expansion. To get this, the bound (1.5) is not enough, and we need to establish the following:

$$\sup_{x_1,x_2\in\Omega,|u|=1}\left|\left(\frac{\partial}{\partial u}\right)^{\beta}\Gamma(x_1,u)-\left(\frac{\partial}{\partial u}\right)^{\beta}\Gamma(x_2,u)\right| \leq c_\beta \|A(x_1)-A(x_2)\| \tag{1.6}$$

where $\|A(\cdot)\|$ is the matrix norm of the coefficients $\{a_{ij}\}$. To establish a bound on the derivatives of any order of a fundamental solution, uniform with respect to some parameter, is always a difficult task when, as happens for operators structured on Hörmander's vector fields, we cannot rely on any kind of explicit formula for the fundamental solution. We will get the bound (1.6) in § 6, adapting results and techniques contained in series of papers by Bonfiglioli-Lanconelli-Uguzzoni (see [2], [3], [4]) in the context of Gaussian bounds for operators (1.3). We point out that the reason why we did not consider in this paper operators with drift X_0 is only related to this part of the proof. Namely, the papers [2], [3], [4] deal with operators (1.2) or (1.3), but not (1.4), therefore proving (1.6) in presence of a drift would require a much deeper revision of the techniques used in those papers, and perhaps a completely different approach.

Plan of the paper

Section 2 contains some known facts, the definition and basic properties of *local* BMO-type spaces and the statement of our assumptions and main result.

In section 3 we write the representation formulas that we need for X_iX_ju in terms of Lu. These formulas involve singular integrals with "variable kernels" and their commutators. By the classical technique of expansion in spherical harmonics we rewrite these operators in series of singular integral operators of convolution type. We state some uniform bound on the fundamental solution of the frozen operator and show their use in proving suitable bounds on the coefficients of the expansion in spherical harmonics. Section 4 contains the core of the real analysis machinery: BMO_{loc} estimates for singular integrals and their commutators are established, first for convolution kernels and then in the general case, together with a number of other useful results, in particular a local version of the one stating that $LMO \cap L^\infty$ multiplies BMO. Section 5 contains the proof of our main result, in three steps: first, exploiting all the results of the previous sections, we prove local estimates for functions with small compact support; second, exploiting several techiques and results from [8], we prove local estimates for functions with small noncompact support; third, we conclude the proof of the result on any bounded domain. Finally, the Appendix contains the proof of the uniform bound on the fundamental solution of the frozen operator.

Although this bound is crucial in the paper, we have preferred postponing its proof to the Appendix because the techniques employed there are completely different from those of the previous sections.

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2 Preliminaries

2.1 Carnot groups, vector fields and their metric

Here we recall a number of known definitions and facts about homogeneous groups and left invariant vector fields. For the justification of our assertions, further details and examples, we refer to [32, p. 618-622], [5, § 1.3], [23], [6].

We call *Carnot group* or *stratified homogeneous group* the space $\mathbb{R}^N$ equipped with a Lie group structure, ("translations") together with a family of "dilations" that are group automorphisms and are given by

$$D(\lambda) : (x_1, \dots, x_N) \mapsto (\lambda^{\omega_1} x_1, \dots, \lambda^{\omega_N} x_N) \tag{2.1}$$

for $\lambda > 0$, where

$$(\omega_1, ..., \omega_N) = (1, 1, ..., 1, 2, 2, ..., 2, ..., s, s, ..., s)$$

for some positive integer s. We denote by $\circ$ the translation, and assume that the origin is the group identity and the Euclidean opposite is the group inverse. We will denote by G the space $\mathbb{R}^N$ with this structure of homogeneous group, and we will write $c(G)$ for a constant depending on the numbers N, $\omega_1, \dots, \omega_N$ and the group law $\circ$.

We say that a differential operator Y on $\mathbb{R}^N$ is *homogeneous of degree* $\beta > 0$ if

$$Y\left(f\left(D(\lambda)x\right)\right) = \lambda^{\beta}\left(Yf\right)\left(D(\lambda)x\right)$$

for every test function f, $\lambda > 0$, $x \in \mathbb{R}^N$. Also, we say that a function f is *homogeneous of degree* $\alpha \in \mathbb{R}$ if

$$f\left(D(\lambda)x\right) = \lambda^{\alpha} f(x) \quad \forall \lambda > 0, x \in \mathbb{R}^N.$$

Clearly, if Y is a differential operator homogeneous of degree β and f is a homogeneous function of degree α, then Yf is homogeneous of degree $\alpha - \beta$.

Let us consider now the Lie algebra ℓ associated to the group G, that is, the Lie algebra of left-invariant vector fields, with the Lie bracket given by the commutator of vector fields

$$[X, Y] = XY - YX.$$

We can fix a basis $X_1, \dots, X_N$ in ℓ choosing X_i as the left invariant vector field which agrees with ∂_{x_i} at the origin. It turns out that X_i is homogeneous of degree ω_i. In particular, let $X_1, X_2, ..., X_q$ be the elements of the basis which are homogeneous of degree 1. The Lie algebra ℓ turns out to be nilpotent and stratified:

$$\ell = \bigoplus_{i=1}^{s} V_i \quad \text{with } [V_1, V_j] = V_{j+1} \text{ for } j \leq s-1, [V_1, V_j] = \{0\} \text{ otherwise;}$$
$$V_1 = \text{span}\,(X_1, X_2, ..., X_q)\,.$$

The number s is called the *step* of the Lie algebra. We also say that the vector fields $X_1, X_2, ..., X_q$ satisfy Hörmander's condition at step s.

Let us introduce the *control distance* d induced by the vector fields. This is a concept which can be defined for any system of Hörmander's vector fields (even in absence of a group structure) but which possesses more properties in Carnot groups.

Definition 2.1 *For any $\delta > 0$, $x, y \in \mathbb{R}^N$, let $C_\delta\,(x, y)$ be the class of absolutely continuous mappings $\varphi : [0, 1] \to \Omega$ which satisfy*

$$\varphi'(t) = \sum_{i=1}^{q} a_i(t)\,(X_i)_{\varphi(t)} \;\; a.e. \;\; t \in (0, 1)$$

with $|a_i(t)| \leq \delta$ for $i = 1, ..., q$, $\varphi\,(0) = x, \varphi\,(1) = y$. We define

$$d(x, y) = \inf\,\{\delta > 0 : C_\delta\,(x, y) \neq \varnothing\}\,.$$

Note that the finiteness of $d\,(x, y)$ for any two points $x, y \in \Omega$ is not a trivial fact, but depends on a connectivity result known as "Chow's theorem"; it can be proved that d is actually a distance in $\mathbb{R}^N$, finite for any couple of points. Moreover:

Proposition 2.2 *The distance d is translation invariant and 1-homogeneous for dilations on the group.*

This fact is probably known, but we are unable to give a precise reference, so we will prove it.

Proof. Let $B\,(x, r)$ be the d-ball of center x and radius r. Let us prove that

$$y \in B\,(x, r) \Longrightarrow x^{-1} \circ y \in B\,(0, r)$$
$$y \in B\,(0, r) \Longrightarrow D\,(\lambda)\,y \in B\,(0, \lambda r)\,.$$

Let $y \in B\,(x, r)$, then for every $\delta < r$ there exists $\varphi \in C_\delta\,(x, y)$. Let

$$\varphi_x\,(t) = x^{-1} \circ \varphi\,(t)\,,$$

and let us show that $\varphi_x \in C_\delta\left(0, x^{-1} \circ y\right)$. Clearly $\varphi_x\,(0) = 0$ and $\varphi_x\,(1) = x^{-1} \circ y$. Moreover, let J_x denote the Jacobian matrix of the function $y \mapsto x^{-1} \circ y$, then

$$\varphi_x'(t) = J_x(\varphi(t)) \cdot \varphi'(t) = J_x(\varphi(t)) \cdot \sum_{i=0}^{q} a_i(t) (X_i)_{\varphi(t)}$$

$$= \sum_{i=0}^{q} a_i(t) J_x(\varphi(t)) \cdot (X_i)_{\varphi(t)} = \sum_{i=0}^{q} a_i(t) (X_i)_{x^{-1}\circ\varphi(t)} = \sum_{i=0}^{q} a_i(t) (X_i)_{\varphi_x(t)}$$

where the up to last identity follows by the translation invariance of the X_i's (see [5, Prop. 1.2.3 p.14]). So the first assertion is proved.

Let now $y \in B(0, r)$, then for every $\delta < r$ there exists $\varphi \in C_\delta(0, y)$. Let

$$\varphi_\lambda(t) = D(\lambda)\varphi(t).$$

Then $\varphi_\lambda(0) = 0; \varphi_\lambda(1) = D(\lambda) y$. Recall that

$$X_i = \sum_{j=1}^{N} b_{ij}(u) \partial_{u_j}$$

with b_{ij} homogeneous of degree $\omega_j - 1$, for $i = 1, 2, ..., q$. Hence

$$b_{ij}(\varphi_\lambda(t)) = \lambda^{\alpha_j - 1} b_{ij}(\varphi(t))$$

and

$$\varphi_\lambda'(t)_j = \lambda^{\omega_j} \varphi_j'(t) = \lambda^{\omega_j} \sum_{i=1}^{q} a_i(t) b_{ij}(\varphi(t)) =$$

$$= \sum_{i=0}^{q} \lambda a_i(t) b_{ij}(\varphi_\lambda(t))$$

with

$$|\lambda a_i(t)| \le \lambda\delta,$$

which shows that $\varphi_\lambda \in C_{\lambda\delta}(x, y)$. Since this holds for any $\delta < r$, we conclude $y \in B(0, \lambda r)$. ■

We can also define in $\mathbb{R}^N$ a *homogeneous norm* $\|\cdot\|$ as follows. For any $x \in \mathbb{R}^N$, $x \neq 0$, set

$$\|x\| = \rho \Leftrightarrow \left| D\left(\frac{1}{\rho}\right) x \right| = 1,$$

where $|\cdot|$ denotes the Euclidean norm; also, let $\|0\| = 0$. Then:

$\|D(\lambda)x\| = \lambda \|x\|$ for every $x \in \mathbb{R}^N$, $\lambda > 0$;

the set $\{x \in \mathbb{R}^N : \|x\| = 1\}$ coincides with the euclidean unit sphere $\sum_N$;

the function $x \mapsto \|x\|$ is smooth outside the origin;

there exists $c(G) \geq 1$ such that for every $x,\ y \in \mathbb{R}^N$

$$\|x \circ y\| \leq c(\|x\| + \|y\|) \text{ and } \left\|x^{-1}\right\| = \|x\|\,;$$
$$\frac{1}{c}\,|y| \leq \|y\| \leq c\,|y|^{1/s} \text{ if } \|y\| \leq 1,$$

hence the function

$$\rho(x,y) = \left\|y^{-1} \circ x\right\| \tag{2.2}$$

is a *quasidistance*, that is:

$$\rho\,(x,y) \geq 0 \text{ and } \rho\,(x,y) = 0 \text{ if and only if } x = y;$$
$$\rho\,(y,x) = \rho\,(x,y)$$
$$\rho\,(x,y) \leq c\,\left(\rho\,(x,z) + \rho\,(z,y)\right)$$

for every $x,\ y,\ z \in \mathbb{R}^N$ and some positive constant $c(G) \geq 1$. If we define the ρ-balls as

$$B_\rho\,(x,r) = \left\{y \in \mathbb{R}^N : \rho\,(x,y) < r\right\},$$

then $B_\rho\,(0,r) = D\,(r)\,B_\rho\,(0,1)$. The Lebesgue measure in $\mathbb{R}^N$ is the Haar measure of G, hence

$$|B_\rho(x,r)| = |B_\rho(0,1)|\,r^Q \;\forall x \in \mathbb{R}^N, r > 0,$$

where

$$Q = \omega_1 + \ldots + \omega_N$$

(with ω_i as in (2.1)) is the *homogeneous dimension* of $\mathbb{R}^N$.

A consequence of the Prop. 2.2 is that the function $d\,(x,0)$ is another *homogeneous norm* in the sense of Folland (see [23]), equivalent to the function $\|x\|$ defined above. In particular, d and ρ are globally equivalent.

2.2 Function spaces defined by local mean oscillations

Throughout the following, we will assume that $X_1, X_2, ..., X_q$ is a system of left invariant homogeneous Hörmander's vector fields on a Carnot group in $\mathbb{R}^N$, as described in § 2.1, and d is the control distance induced by the vector fields in $\mathbb{R}^N$.

Let Ω be a bounded open connected subset of $\mathbb{R}^N$. In the following definitions, the balls are always taken with respect to the distance d.

Definition 2.3 (BMO^p **spaces)** *For* $p \in [1,\infty)$ *we say that* $f \in BMO^p\,(\Omega)$ *if*

$$\|f\|_{BMO^p(\Omega)} = [f]_{BMO(\Omega)} + \|f\|_{L^p(\Omega)} < \infty$$

with

$$[f]_{BMO(\Omega)} = \sup_{x\in\Omega, r>0} \frac{1}{|B\,(x,r) \cap \Omega|} \int_{B(x,r)\cap\Omega} \left|f\,(y) - f_{B(x,r)\cap\Omega}\right| dy$$

where $f_A = \frac{1}{|A|}\int f$.

Definition 2.4 (Local BMO spaces) *Let $f \in L^1_{loc}(\Omega)$. We say that $f \in BMO_{loc}(\Omega)$ if*

$$[f]_{BMO_{loc}(\Omega)} = \sup_{B(x,r)\subset\Omega} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy < +\infty.$$

Note that the requirement $B(x,r) \subset \Omega$ is meaningful because the distance d is defined in the whole $\mathbb{R}^N$.

Let $f \in L^1_{loc}(\Omega)$ and $\Omega_1 \Subset \Omega_2 \subseteq \Omega$. We say that $f \in BMO_{loc}(\Omega_1, \Omega_2)$ if

$$[f]_{BMO_{loc}(\Omega_1,\Omega_2)} = \sup_{x\in\Omega_1, B(x,r)\subset\Omega_2} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy < +\infty\,.$$

Note that $f \in BMO_{loc}(\Omega)$ if and only if $f \in BMO_{loc}(\Omega_1, \Omega)$ for any $\Omega_1 \Subset \Omega$.

Moreover, for any $\Omega_1 \Subset \Omega_2 \subseteq \Omega$ we have the following inclusions

$$BMO_{loc}(\Omega) \subset BMO_{loc}(\Omega_1, \Omega_2)$$

with

$$[f]_{BMO_{loc}(\Omega_1,\Omega_2)} \leq [f]_{BMO_{loc}(\Omega)}\,,$$

and

$$BMO_{loc}(\Omega_2) \subset BMO_{loc}(\Omega_1, \Omega_2) \subset BMO_{loc}(\Omega_1)$$

with

$$[f]_{BMO_{loc}(\Omega_1)} \leq [f]_{BMO_{loc}(\Omega_1,\Omega_2)} \leq [f]_{BMO_{loc}(\Omega_2)}\,.$$

Definition 2.5 (Local LMO spaces) *Let $\Omega_1 \Subset \Omega_2 \subseteq \Omega$. We say that $f \in LMO_{loc}(\Omega_1, \Omega_2)$ if*

$$[f]_{LMO_{loc}(\Omega_1,\Omega_2)} = \\ = \sup_{x\in\Omega_1, B(x,r)\subset\Omega_2} \frac{1 + \log \frac{diam\Omega_2}{r}}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy < +\infty.$$

Analogously we define the space $LMO_{loc}(\Omega)$ taking the sup over all the balls $B(x,r) \subset \Omega$.

We say that $f \in VLMO_{loc}(\Omega)$ if $f \in LMO_{loc}(\Omega)$ and

$$\eta_f(r) = \sup_{B(x,\rho)\subset\Omega, \rho\leq r} \frac{1 + \log \frac{diam\Omega}{\rho}}{|B(x,\rho)|} \int_{B(x,\rho)} |f(y) - f_{B(x,\rho)}| dy \to 0 \text{ for } r \to 0.$$

Let $\overline{x} \in \Omega, 0 < r_1 < r_2$ such that $B(\overline{x}, r_2) \subset \Omega$. Then

$$VLMO_{loc}(\Omega) \subset LMO_{loc}(B(\overline{x}, r_1), B(\overline{x}, r_2))$$

with

$$[f]_{LMO_{loc}(B(\overline{x},r_1),B(\overline{x},r_2))} \leq \eta_f(2r_2)\,. \tag{2.3}$$

Definition 2.6 (Local BMO^p spaces) *Let $p \in (1, \infty)$ and $\Omega_1 \Subset \Omega_2 \subseteq \Omega$. We say that $f \in BMO^p_{loc}(\Omega_1, \Omega_2)$ if*

$$\|f\|_{BMO^p_{loc}(\Omega_1,\Omega_2)} \equiv \|f\|_{L^p(\Omega_2)} + [f]_{BMO_{loc}(\Omega_1,\Omega_2)} < \infty.$$

Also, we say that $f \in BMO^p_{loc}(\Omega_1)$ if

$$\|f\|_{BMO^p_{loc}(\Omega_1)} \equiv \|f\|_{L^p(\Omega_1)} + [f]_{BMO_{loc}(\Omega_1)} < \infty.$$

We have

$$BMO^p_{loc}(\Omega_2) \subset BMO^p_{loc}(\Omega_1, \Omega_2) \subset BMO^p_{loc}(\Omega_1).$$

Let us note that the spaces $BMO_{loc}(\Omega_1, \Omega_2)$, $BMO^p_{loc}(\Omega_1, \Omega_2)$ and so on, are increasing with respect to both Ω_1 and Ω_2. The following fact, which will be useful several times, says that for compactly supported functions also an inclusion in the reverse order holds, in some sense.

We will often use $BMO^p_{loc}(\Omega_1, \Omega_2)$ spaces with Ω_1, Ω_2 two concentric balls; just to shorten notations, we will set

$$BMO_{loc}(B(\overline{x}, R_1; R_2)) \equiv BMO_{loc}(B(\overline{x}, R_1), B(\overline{x}, R_2)).$$

Lemma 2.7 *Let $f \in BMO^p_{loc}(B(\overline{x}, R; KR))$ for some $K > 3$, with $\text{sprt} f \subset B(\overline{x}, R)$. Then*

$$[f]_{BMO_{loc}(B(\overline{x},R;KR))} \leq 2\left([f]_{BMO_{loc}(B(\overline{x},R;3R))} + \frac{1}{|B(\overline{x}, R)|^{1/p}} \|f\|_{L^p(B(\overline{x},R))}\right).$$

Proof. Let us consider a ball $B(x_0, r)$ with $x_0 \in B(\overline{x}, R)$ and $B(x_0, r) \subset B(\overline{x}, KR)$.

If $B(x_0, r) \subset B(\overline{x}, 3R)$, then obviously

$$\frac{1}{|B(x_0, r)|}\int_{B(x_0,r)} |f(y) - f_{B(x_0,r)}|dy \leq [f]_{BMO_{loc}(B(\overline{x},R;3R))},$$

so let us assume $B(x_0, r) \subsetneq B(\overline{x}, 3R)$. This means that $B(x_0, r) \supset B(\overline{x}, R)$, hence for any $c \in \mathbb{R}$

$$\frac{1}{|B(x_0, r)|}\int_{B(x_0,r)} |f(y) - f_{B(x_0,r)}|dy \leq \frac{2}{|B(x_0, r)|}\int_{B(x_0,r)} |f(y) - c|dy$$
$$= \frac{2}{|B(x_0, r)|}\left(\int_{B(\overline{x},R)} |f(y) - c|dy + |c|\,|B(x_0, r) \setminus B(\overline{x}, R)|\right)$$
$$\leq \frac{2}{|B(\overline{x}, R)|}\int_{B(\overline{x},R)} |f(y) - c|dy + 2|c|\frac{|B(x_0, r) \setminus B(\overline{x}, R)|}{|B(x_0, r)|}$$

choosing $c = f_{B(\overline{x},R)}$

$$\leq \frac{2}{|B(\overline{x},R)|}\int_{B(\overline{x},R)} |f(y) - f_{B(\overline{x},R)}|dy + 2\left|f_{B(\overline{x},R)}\right|$$

$$\leq 2\left([f]_{BMO_{loc}(B(\overline{x},R;3R))} + \frac{1}{|B(\overline{x},R)|^{1/p}}\|f\|_{L^p(B(\overline{x},R))}\right).$$

■

In the sequel we will need the following proposition that gives us a comparison between local and global BMO^p spaces.

Proposition 2.8 *Let $s > 0, p \in [1,\infty)$ and $\overline{x} \in \mathbb{R}^N$ be fixed.*
(a) If $f \in BMO^p_{loc}(B(\overline{x},s;3s))$, then $f \in BMO^p(B(\overline{x},s))$ with

$$\|f\|_{BMO^p(B(\overline{x},s))} \leq c\,\|f\|_{BMO^p_{loc}(B(\overline{x},s;3s))}$$

for some constant $c = c(G)$.
(b) If $f \in BMO^p(B(\overline{x},R))$ for some $R > s$ and $sprt f \subset B(\overline{x},s)$, then the function $\widetilde{f}$ obtained extending f to zero outside $B(\overline{x},R)$ belongs to $BMO^p_{loc}(B(\overline{x},s;3s))$, with

$$\left\|\widetilde{f}\right\|_{BMO^p_{loc}(B(\overline{x},s;3s))} \leq \|f\|_{BMO^p(B(\overline{x},R))} + \frac{c}{(R-s)^{Q/p}}\|f\|_{L^p(B(\overline{x},s))}$$

with $c = c(G,p)$.

Proof. (a) Pick $x \in B(\overline{x},s)$ and $r > 0$ such that $B(x,r) \subset B(\overline{x},3s)$. Then, for a constant c to be chosen later,

$$\frac{1}{|B(x,r)\cap B(\overline{x},s)|}\int_{B(x,r)\cap B(\overline{x},s)} |f(y) - c|\,dy$$
$$\leq \frac{1}{c_1|B(x,r)|}\int_{B(x,r)} |f(y) - c|\,dy$$

because

$$|B(x,r)\cap B(\overline{x},s)| \geq c_1|B(x,r)|$$

by the regularity of metric balls (see Lemma 4.2 in [8]), since r is bounded (being $B(x,r) \subset B(\overline{x},3s)$). From the proof in [8, Lemma 4.2] one reads that c_1 is a constant only depending on the doubling constant, that is on G. Choosing now $c = f_{B(x,r)}$ we get

$$\frac{1}{|B(x,r)\cap B(\overline{x},s)|}\int_{B(x,r)\cap B(\overline{x},s)} |f(y) - c| \leq \frac{1}{c_1}[f]_{BMO_{loc}(B(\overline{x},s;3s))}.$$

If, on the other hand, $x \in B(\overline{x},s)$ but $B(x,r) \subsetneq B(\overline{x},3s)$, this means that $B(x,r) \supset B(\overline{x},s)$, hence

$$\frac{1}{|B(x,r)\cap B(\overline{x},s)|}\int_{B(x,r)\cap B(\overline{x},s)} |f(y) - c|\,dy$$
$$= \frac{1}{|B(\overline{x},s)|}\int_{B(\overline{x},s)} |f(y) - c|\,dy \leq [f]_{BMO_{loc}(B(\overline{x},s;3s))}$$

(choosing $c = f_{B(\overline{x},s)}$).

(b) Pick $x \in B(\overline{x}, s)$ and $r > 0$ such that $B(x,r) \subset B(\overline{x}, 3s)$. Then

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} \left|\widetilde{f}(y) - c\right| dy$$
$$= \frac{1}{|B(x,r)|}\left\{\int_{B(x,r)\cap B(\overline{x},R)} |f(y) - c|\, dy + \int_{B(x,r)\setminus B(\overline{x},R)} |c|\, dy\right\}$$

choosing $c = f_{B(x,r)\cap B(\overline{x},R)}$

$$= \frac{1}{|B(x,r)|}\left\{\int_{B(x,r)\cap B(\overline{x},R)} \left|f(y) - f_{B(x,r)\cap B(\overline{x},R)}\right| dy + \left|f_{B(x,r)\cap B(\overline{x},R)}\right| |B(x,r)\setminus B(\overline{x},R)|\right\}$$
$$\equiv I + II.$$

Now,

$$I \leq \frac{1}{|B(x,r)\cap B(\overline{x},R)|}\int_{B(x,r)\cap B(\overline{x},R)} \left|f(y) - f_{B(x,r)\cap B(\overline{x},R)}\right| dy$$
$$\leq [f]_{BMO(B(\overline{x},R))}.$$

On the other hand,

$$II = \left|f_{B(x,r)\cap B(\overline{x},R)}\right| \frac{|B(x,r)\setminus B(\overline{x},R)|}{|B(x,r)|}.$$

The term II does not vanish only if $B(x,r)$ contains a point outside $B(\overline{x},R)$ (otherwise $|B(x,r)\setminus B(\overline{x},R)| = 0$). Since $x \in B(\overline{x},s)$, this implies $r > R - s$ and

$$|B(x,r)\cap B(\overline{x},R)| \geq c\,(R-s)^{Q}.$$

Then, by Hölder inequality

$$II \leq \left|f_{B(x,r)\cap B(\overline{x},R)}\right| \leq \left(\frac{1}{|B(x,r)\cap B(\overline{x},R)|}\int_{B(x,r)\cap B(\overline{x},R)} |f(y)|^{p}\, dy\right)^{1/p}$$
$$\leq \frac{c}{(R-s)^{Q/p}}\, \|f\|_{L^{p}(B(\overline{x},R))}$$

Then

$$[\widetilde{f}]_{BMO_{loc}(B(\overline{x},s;3s))} \leq [f]_{BMO(B(\overline{x},R))} + \frac{c}{(R-s)^{Q/p}}\, \|f\|_{L^{p}(B(\overline{x},s))} \tag{2.4}$$

from which (b) follows. ■

Definition 2.9 (Sobolev and BMO Sobolev spaces) *We say that* $u \in S^{2,p}(\Omega)$ *if*

$$\|u\|_{S^{2,p}(\Omega)} \equiv \sum_{i,j=1}^{q} \|X_i X_j u\|_{L^{p}(\Omega)} + \sum_{i=1}^{q} \|X_i u\|_{L^{p}(\Omega)} + \|u\|_{L^{p}(\Omega)} < \infty,$$

where the derivatives $X_i u$ are defined in the usual weak (distributional) sense.

We say that $u \in S^{2,p,*}_{loc}(\Omega)$ if

$$\|u\|_{S^{2,p,*}_{loc}(\Omega)} \equiv$$
$$\equiv \sum_{i,j=1}^{q} \|X_i X_j u\|_{BMO^p_{loc}(\Omega)} + \sum_{i=1}^{q} \|X_i u\|_{BMO^p_{loc}(\Omega)} + \|u\|_{BMO^p_{loc}(\Omega)} < \infty.$$

We say that $u \in S^{2,p,*}(\Omega)$ if

$$\|u\|_{S^{2,p,*}(\Omega)} \equiv$$
$$\equiv \sum_{i,j=1}^{q} \|X_i X_j u\|_{BMO^p(\Omega)} + \sum_{i=1}^{q} \|X_i u\|_{BMO^p(\Omega)} + \|u\|_{BMO^p(\Omega)} < \infty.$$

Analogously, for $\Omega_1 \Subset \Omega_2 \subseteq \Omega$, we can define the spaces $S^{2,p,*}_{loc}(\Omega_1, \Omega_2)$, replacing $BMO^p(\Omega)$ norms with $BMO^p_{loc}(\Omega_1, \Omega_2)$.

2.3 Assumptions and main results

We now keep the assumptions stated at beginning of § 2.2 about the vector fields $X_1, X_2, ..., X_q$, the domain $\Omega \subset \mathbb{R}^N$ and the distance d. Let us consider operators of the form

$$Lu \equiv \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j u \tag{2.5}$$

where $a_{ij} = a_{ji}$ satisfy the "ellipticity condition":

$$\Lambda |\xi|^2 \le \sum_{i,j=1}^{q} a_{ij}(x) \xi_i \xi_j \le \Lambda^{-1} |\xi|^2 \ \forall \xi \in \mathbb{R}^q, x \in \Omega.$$

Moreover

$$a_{ij} \in VLMO_{loc}(\Omega) \cap L^\infty(\Omega).$$

We also assume that the homogeneous dimension (see § 2.1) is $Q \ge 3$. This fact (necessary to apply Folland's results in [23]) simply rules out the case of uniformly elliptic equations in two variables. We stress the fact that, instead, uniformly elliptic operators in $n \ge 3$ variables *are* covered by the present theory. Our main result is the following:

Theorem 2.10 *Under the above assumptions, for any $\Omega_1 \Subset \Omega_2 \Subset \Omega$, $1 < p < \infty$ we have*

$$\|u\|_{S^{2,p,*}_{loc}(\Omega_1,\Omega_2)} \le c \left\{ \|Lu\|_{BMO^p_{loc}(\Omega_2,\Omega)} + \|u\|_{BMO^p_{loc}(\Omega_2,\Omega)} \right\}$$

for any $u \in S^{2,p,*}_{loc}(\Omega)$. The constant c only depends on the group G, the numbers p and Λ, the $VLMO_{loc}$ moduli of the coefficients, Ω_1, Ω_2. The previous inequality also implies the following

$$\|u\|_{S^{2,p,*}(\Omega_1)} \le c \left\{ \|Lu\|_{BMO^p_{loc}(\Omega)} + \|u\|_{BMO^p_{loc}(\Omega)} \right\}.$$

We will also prove a similar local estimate stated for "standard" BMO^p spaces, see Theorem 5.3.

3 Representation formulas and reduction to singular integrals of convolution type

In order to state suitable representation formulas for the operator L in (2.5), we proceed as follows. For any $x_0 \in \Omega$, let us "freeze" at x_0 the coefficients $a_{ij}(x)$, and consider

$$L_0 = \sum_{i,j=1}^{q} a_{ij}(x_0) X_i X_j.$$

As shown in [6], this operator can be rewritten as a sum of squares of Hörmander's vector fields; in particular, it is hypoelliptic. Moreover, L_0 is left invariant, homogeneous of degree 2, and coincides with its formal transpose; hence Folland's theory in [23] applies, and assures the existence of a $(2-Q)$-homogeneous fundamental solution, smooth outside the pole. Let us denote it by $\Gamma(x_0;\cdot)$, to indicate its dependence on the frozen coefficients $a_{ij}(x_0)$. Also, set for $i,j = 1,\ldots,q$,

$$\Gamma_{ij}(x_0;u) = X_i X_j \,[\Gamma(x_0;\cdot)](u).$$

The next theorem summarizes the basic properties of $\Gamma(x_0;\cdot)$ that we will need in the following. All of them are due to Folland [23, Thm. 2.1 and Corollary 2.8] or Folland-Stein [24, Proposition 8.5] (see also [6]).

Theorem 3.1 *Assume that the homogeneous dimension of G is $Q \geq 3$. For every $x_0 \in \Omega$ the operator L_0 has a unique fundamental solution $\Gamma(x_0;\cdot)$ such that:*

(*a*) $\Gamma(x_0;\cdot) \in C^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$;
(*b*) $\Gamma(x_0;\cdot)$ *is homogeneous of degree* $(2-Q)$;
(*c*) *for every test function f and every $v \in \mathbb{R}^N$,*

$$f(v) = \int_{\mathbb{R}^N} \Gamma\left(x_0\,;\,u^{-1}\circ v\right)\, L_0 f(u)\,du;$$

moreover, for every $i,j = 1,\ldots,q$, there exist constants $\alpha_{ij}(x_0)$ such that

$$X_i X_j f(v) = P.V.\int_{\mathbb{R}^N} \Gamma_{ij}\left(x_0;\,u^{-1}\circ v\right)\, L_0 f(u)du + \alpha_{ij}(x_0)\cdot L_0 f(v); \quad (3.1)$$

(*d*) $\Gamma_{ij}(x_0\,;\cdot) \in C^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$;
(*e*) $\Gamma_{ij}(x_0\,;\cdot)$ *is homogeneous of degree* $-Q$;
(*f*) *for every* $R > r > 0$,

$$\int_{r<d(0,x)<R} \Gamma_{ij}(x_0;\,x)\,dx = \int_{d(0,x)=1} \Gamma_{ij}(x_0;\,x)\,d\sigma(x) = 0.$$

Here and in the following,

$$P.V.\int (...)\, dy = \lim_{\varepsilon\to 0}\int_{d(x,y)>\varepsilon} (...)\, dy = \lim_{\varepsilon\to 0}\int_{\rho(x,y)>\varepsilon} (...)\, dy.$$

The cancellation properties stated at point (f) still hold with $d(0,x)$ replaced with $\|x\|$.

A second fundamental result we need contains a bound on the derivatives of Γ, uniform with respect to x_0, and is proved in [6, Thm. 12]:

Theorem 3.2 *For every multi-index β, there exists a constant $c = c(\beta, G, \Lambda)$ such that*

$$\sup_{\substack{\|u\|=1\\ x\in\Omega}} \left| \left(\frac{\partial}{\partial u}\right)^{\beta} \Gamma_{ij}\,(x\,;\,u)\right| \leq\; c,$$

for any $i, j = 1, \ldots, q$; moreover, for the α_{ij}'s appearing in (3.1), the uniform bound

$$\sup_{x\in\Omega} |\alpha_{ij}(x)| \leq c_2 \tag{3.2}$$

holds for some constant $c_2 = c_2\,(G, \Lambda)$.

The above theorem will be useful but not sufficient for our aims. In § 6 we will also prove the following uniform bound:

Theorem 3.3 *For any nonnegative integer p, there exists a constant $c_{\Lambda,p}$ such that for any $x_1, x_2, y \in \mathbb{R}^N$ we have:*

$$\begin{aligned} &\left|X_{i_1}X_{i_2}...X_{i_p}\Gamma\,(x_1, y) - X_{i_1}X_{i_2}...X_{i_p}\Gamma\,(x_2, y)\right| \\ &\leq c_{\Lambda,p}\,\|A\,(x_1) - A\,(x_2)\|\,\|y\|^{2-Q-p} \end{aligned} \tag{3.3}$$

where the differential operators X_{i_j} $(i_j \in \{1, 2, ..., q\})$ act on the y-variable and $A = \{a_{ij}\}_{i,j=1}^{q}$.

Here $\|y\|$ is the homogeneous norm in G, while $\|A\,(x_1) - A\,(x_2)\|$ denotes the usual matrix norm in $\mathbb{R}^{2q}$.

By the representation formula (3.1), writing $L_0 = L + (L_0 - L)$ and then letting x be equal to x_0, we get the following:

Theorem 3.4 *Let $u \in C_0^{\infty}\,(\Omega)$. Then, for $i, j = 1, \ldots, q$ and every $x \in \Omega$*

$$\begin{aligned} X_iX_ju\,(x) = P.V.\int_{\Omega} \Gamma_{ij}\left(x; y^{-1}\circ x\right) \Bigg\{ &\sum_{h,k=1}^{q} [a_{hk}(x) - a_{hk}(y)]\; X_hX_k\,u(y) + \\ &+Lu(y)\Bigg\}\, dy + \alpha_{ij}(x)\cdot Lu(x). \end{aligned} \tag{3.4}$$

The previous formula still holds, for a.e. x, if $u = v\phi$ with $v \in S^{2,p}\,(\Omega)$ and $\phi \in C_0^{\infty}\,(\Omega)$.

In order to rewrite the above formula in a more compact form, let us introduce the following singular integral operators:

$$K_{ij} f(x) = P.V. \int_{\Omega} \Gamma_{ij} \left(x; y^{-1} \circ x \right) f(y) \, dy. \tag{3.5}$$

Moreover, for an operator K and a function $a \in L^{\infty}(\Omega)$, define the commutator

$$C[K, a](f) = K(af) - a \cdot K(f). \tag{3.6}$$

Then (3.4) becomes

$$X_i X_j \, u = K_{ij} (Lu) - \sum_{h,k=1}^{q} C[K_{ij}, a_{hk}] (X_h X_k \, u) + \alpha_{ij} \cdot Lu \tag{3.7}$$

for any $u \in C_0^{\infty}(\Omega), i, j = 1, \dots, q$.

Next, we are going to expand the "variable kernel" $\Gamma_{ij}(x; u)$ in series of spherical harmonics. At this point it is more convenient to use the ρ-balls (defined by the quasidistance (2.2)), which have the property that $B_{\rho}(0, r) = D(r) B_E(0, 1)$, where B_E stands for the Euclidean ball.

Let us denote by

$$\{Y_{km}(y)\}_{\substack{k=1,\dots,g_m \\ m=0,1,\dots,\infty}}$$

a complete orthonormal system of $L^2(\Sigma_N)$ consisting in spherical harmonics; here m is the degree of the harmonic homogeneous polynomial Y_{km}, and g_m the dimension of the space of harmonic homogeneous polynomial of degree m in N variables. Then, as in [6], for any fixed $x \in \Omega$, $y \in \Sigma_N$, we can expand:

$$\Gamma_{ij}(x; y) = \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} c_{ij}^{km}(x) \frac{Y_{km}(y')}{\|y\|^Q} \quad \text{for } i, j = 1, \dots, q$$

where $y' = D\left(\|y\|^{-1} \right) y$, so that $y' \in \Sigma_N$.

We explicitly note that for $m = 0$ the coefficients in the above expansion are zero, because of the vanishing property of $\Gamma_{ij}(x; \cdot)$. Also, note that the integral of $Y_{km}(y)$ over Σ_N, for $m \geq 1$, is zero. Then

$$K_{ij}(f)(x) = \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} c_{ij}^{km}(x) \, T_{km} f(x) \tag{3.8}$$

with

$$\begin{aligned} T_{km} f(x) &= P.V. \int H_{km}(y^{-1} \circ x) \, f(y) \, dy \\ H_{km}(x) &= \frac{Y_{km}(x')}{\|x\|^Q}. \end{aligned} \tag{3.9}$$

We will use the following bounds about spherical harmonics:

$$g_m \leq c(N) \cdot m^{N-2} \quad \text{for every } m = 1, 2, \ldots \tag{3.10}$$

$$\left| \left(\frac{\partial}{\partial x} \right)^{\beta} Y_{km}(x) \right| \leq c(N) \cdot m^{\left(\frac{N-2}{2} + |\beta|\right)} \text{ for } x \in \Sigma_N, k = 1, \ldots, g_m, m = 1, 2, \ldots. \tag{3.11}$$

Moreover, if $f \in C^{\infty}(\Sigma_N)$ and if $f(x) \sim \sum_{k,m} b_{km} Y_{km}(x)$ is the Fourier expansion of $f(x)$ with respect to $\{Y_{km}\}$, that is

$$b_{km} = \int_{\Sigma_N} f(x)\, Y_{km}(x)\, d\sigma(x)$$

then, for every positive integer n there exists c_n such that

$$|b_{km}| \leq c_n \cdot m^{-2n} \sup_{\substack{|\beta|=2n \\ x \in \Sigma_N}} \left| \left(\frac{\partial}{\partial x} \right)^{\beta} f(x) \right|. \tag{3.12}$$

In view of Theorem 3.2, we get by (3.12) the following bound on the coefficients $c_{ij}^{km}(x)$ appearing in the expansion (3.8): for every positive integer n there exists a constant $c = c(n, G, \Lambda)$ such that

$$\sup_{x \in \Omega} \left| c_{ij}^{km}(x) \right| \leq c(n, G, \Lambda) \cdot m^{-2n} \tag{3.13}$$

for every $m = 1, 2, \ldots$; $k = 1, \ldots, g_m$; $i, j = 1, \ldots, q$.

We will also need a bound on the LMO_{loc} seminorm of these coefficients and the functions α_{ij}:

Theorem 3.5 *For every $n > 0$ there exists $c_n > 0$ such that*

$$\left[c_{ij}^{km}\right]_{LMO_{loc}(B(\overline{x}, R_1; R_2))} \leq c_n \cdot m^{-2n} \cdot [A]_{LMO_{loc}(B(\overline{x}, R_1; R_2))}$$

for any $k, m, i, j, R_1 < R_2$ (with c_n independent of R_1, R_2). We have set

$$[A]_{LMO_{loc}(B(\overline{x}, R_1; R_2))} = \sup_{h,l} [a_{hl}]_{LMO_{loc}(B(\overline{x}, R_1; R_2))}.$$

Also,

$$[\alpha_{ij}]_{LMO_{loc}(B(\overline{x}, R_1; R_2))} \leq c\, [A]_{LMO_{loc}(B(\overline{x}, R_1; R_2))}. \tag{3.14}$$

For the proof of the above Theorem we need the following:

Lemma 3.6 *With the above notation, we have:*

$$\alpha_{ij}(x) = -\int_{\Sigma_N} X_j \Gamma(x, y) \sum_{k=1}^{n} b_{ik}(y) \nu_k \, d\sigma(y)$$

where $\nu = (\nu_1, \nu_2, , ...\nu_n)$ is the outer normal to Σ_N (hence $\nu_k = y_k$, but this is irrelevant) and $X_i = \sum_{k=1}^{n} b_k(y) \partial_{y_k}$.

Proof. We follow an argument in [22, proof of Proposition 2.11]. Let η be a cutoff function such that $0 \leq \eta \leq 1$, $\eta(y) = 1$ for $\|y\| \geq 1, \eta(y) = 0$ for $\|y\| \leq 1/2$, and let $\eta_\varepsilon(y) = \eta(D(1/(\varepsilon))y)$. Reasoning like in the quoted proof, it is enough to show that

$$A_{ij}^{\varepsilon}(x_0, x) \equiv \int_{\|y^{-1}\circ x\|<\varepsilon} X_i\left[\eta_\varepsilon\left(y^{-1}\circ x\right) X_j\Gamma\left(x_0, y^{-1}\circ x\right)\right] dy \rightarrow$$

$$\rightarrow -\int_{\Sigma_N} X_j\Gamma(x_0, y) \sum_{k=1}^{n} b_k(y)\nu_k \, d\sigma(y)$$

for $\varepsilon \to 0$. Namely, $\alpha_{ij}(x) = \lim_{\varepsilon\to 0} A_{ij}^{\varepsilon}(x, x)$. Actually the frozen point x_0 is irrelevant in this calculation, so we will drop it, writing $A_{ij}^{\varepsilon}(x), \Gamma(y)$, etc. We can write:

$$A_{ij}^{\varepsilon}(x) = \int_{\|y^{-1}\circ x\|<\varepsilon} \left[X_i\eta_\varepsilon \cdot X_j\Gamma + \eta_\varepsilon \cdot X_iX_j\Gamma\right]\left(y^{-1}\circ x\right) dy =$$

$$= \int_{\|y\|<\varepsilon} \left[X_i\eta_\varepsilon \cdot X_j\Gamma + \eta_\varepsilon \cdot X_iX_j\Gamma\right](y)\, dy.$$

Then, since $X_i[\eta_\varepsilon(y)] = X_i[\eta(D(1/\varepsilon)y)] = \frac{1}{\varepsilon}(X_i\eta)(D(1/\varepsilon)y)$ and Γ is $2-Q$-homogeneous, the change of variables $y = D(\varepsilon)w$ gives

$$A_{ij}^{\varepsilon}(x) = \int_{\|w\|<1} \left[X_i\eta \cdot X_j\Gamma + \eta \cdot X_iX_j\Gamma\right](w)\, dw = \int_{\|w\|<1} X_i\left[\eta \cdot X_j\Gamma\right](w)\, dw.$$

Next, we apply the divergence theorem, recalling that $X_if(w) = \sum_{k=1}^{n} b_{ik}(w)\partial_{w_k} f(w) = \sum_{k=1}^{n} \partial_{w_k}\left[b_{ik}f\right](w)$. Then, letting ν_k be the k-th component of the outer normal at the surface $\|w\| = 1$,

$$A_{ij}^{\varepsilon}(x) = -\int_{\|w\|=1}\left[\eta X_j\Gamma \sum_{k=1}^{n} b_{ik}\nu_k\right](w)\, d\sigma(w) = -\int_{\|w\|=1}\left[X_j\Gamma\sum_{k=1}^{n} b_{ik}\nu_k\right](w)\, d\sigma(w),$$

which gives the desired result. ■

Proof of Theorem 3.5 from Theorem 3.3. First of all, since the vector fields $X_1, ..., X_q$ satisfy Hörmander's condition, any derivative $D_y^\beta \Gamma_{ij}(x, y)$ can be expressed as a linear combination of derivatives $X_{i_1}X_{i_2}...X_{i_p}\Gamma(x, y)$, for p large enough. Therefore, (3.3) implies also the following

$$\left|D_y^\beta\Gamma_{ij}(x_1, y) - D_y^\beta\Gamma_{ij}(x_2, y)\right| \leq c_{\Lambda,\beta}\left\|A(x_1) - A(x_2)\right\| \|y\|^{2-Q-c(\beta)}. \quad (3.15)$$

Then, let B_r be any ball centered at some point of $B(\overline{x}, R_1)$ and contained in $B(\overline{x}, R_2)$. Since

$$c_{ij}^{km}(x) = \int_{\Sigma_N} \Gamma_{ij}(x, y)\; Y_{km}(y)\, d\sigma(y),$$

we can write:

$$c_{ij}^{km}(x)-\left(c_{ij}^{km}\right)_{B_r}=\int_{\Sigma_N}\left[\Gamma_{ij}(x,y)-\left(\frac{1}{|B_r|}\int_{B_r}\Gamma_{ij}(x,y)\,dx\right)\right]Y_{km}(y)d\sigma(y)$$
$$\equiv\int_{\Sigma_N}g_{ij}(x,y)\,Y_{km}(y)d\sigma(y).$$

Then, by (3.12), we know that for every positive integer n there exists c_n such that

$$\left|c_{ij}^{km}(x)-\left(c_{ij}^{km}\right)_{B_r}\right|\leq c_n\cdot m^{-2n}\sup_{|\beta|=2n,\ y\in\Sigma_N}\left|\left(\frac{\partial}{\partial y}\right)^{\beta}g_{ij}(x,y)\right|$$
$$=c_n\cdot m^{-2n}\sup_{|\beta|=2n,\ y\in\Sigma_N}\left|D_y^{\beta}\Gamma_{ij}(x,y)-\frac{1}{|B_r|}\int_{B_r}D_y^{\beta}\Gamma_{ij}(u,y)\,du\right|. \tag{3.16}$$

By (3.15) we have:

$$\frac{1}{|B_r|}\int_{B_r}\left|D_y^{\beta}\Gamma_{ij}(x,y)-\frac{1}{|B_r|}\int_{B_r}D_y^{\beta}\Gamma_{ij}(u,y)\,du\right|dx$$
$$=\frac{1}{|B_r|}\int_{B_r}\left|\frac{1}{|B_r|}\int_{B_r}\left[D_y^{\beta}\Gamma_{ij}(x,y)-D_y^{\beta}\Gamma_{ij}(u,y)\right]du\right|dx$$
$$\leq\frac{1}{|B_r|}\int_{B_r}\frac{1}{|B_r|}\int_{B_r}\left|D_y^{\beta}\Gamma_{ij}(x,y)-D_y^{\beta}\Gamma_{ij}(u,y)\right|dudx$$
$$\leq c_{\Lambda,\beta}\|y\|^{2-Q-c(\beta)}\frac{1}{|B_r|}\int_{B_r}\frac{1}{|B_r|}\int_{B_r}\|A(x)-A(u)\|\,dudx$$
$$\leq 2c_{\Lambda,\beta}\|y\|^{2-Q-c(\beta)}\frac{1}{|B_r|}\int_{B_r}\left\|A(x)-\frac{1}{|B_r|}\int_{B_r}A(u)\,du\right\|dx$$

(where $\|\cdot\|$ inside the last integral just denotes the matrix norm). Therefore by (3.16) we have:

$$\frac{1}{|B_r|}\int_{B_r}\left|c_{ij}^{km}(x)-\left(c_{ij}^{km}\right)_{B_r}\right|dx\leq c_n\cdot m^{-2n}\cdot\frac{1}{|B_r|}\int_{B_r}\left\|A(x)-\frac{1}{|B_r|}\int_{B_r}A(u)\,du\right\|dx,$$

and

$$\left[c_{ij}^{km}\right]_{LMO}\leq c_r\cdot m^{-2r}\cdot[A]_{LMO}\,.$$

Next, we prove (3.14). By Lemma 3.6 we can write

$$\alpha_{ij}(x)-(\alpha_{ij})_{B_r}=-\int_{\Sigma_N}\left[\Gamma_j(x,y)-\frac{1}{|B_r|}\int_{B_r}\Gamma_j(w,y)\,dw\right]\sum_{k=1}^{n}b_{ik}(y)\nu_k\,d\sigma(y)$$

so that, by (3.3),

$$\begin{aligned}\left|\alpha_{ij}(x)-(\alpha_{ij})_{B_r}\right| &\leq c\int_{\Sigma_N}\left|\Gamma_j(x,y)-\frac{1}{|B_r|}\int_{B_r}\Gamma_j(w,y)\,dw\right|d\sigma(y)\leq\\ &\leq c\int_{\Sigma_N}\left|\frac{1}{|B_r|}\int_{B_r}[\Gamma_j(x,y)-\Gamma_j(w,y)]\,dw\right|d\sigma(y)\leq\\ &\leq c\int_{\Sigma_N}\frac{1}{|B_r|}\int_{B_r}|\Gamma_j(x,y)-\Gamma_j(w,y)|\,dw\,d\sigma(y)\leq\\ &\leq c\int_{\Sigma_N}\frac{1}{|B_r|}\int_{B_r}\|A(x)-A(w)\|\,\|y\|^{1-Q}\,dw\,d\sigma(y)=\\ &=c\frac{1}{|B_r|}\int_{B_r}\|A(x)-A(w)\|\,dw\end{aligned}$$

and

$$\frac{1}{|B_r|}\int_{B_r}\left|\alpha_{ij}(x)-(\alpha_{ij})_{B_r}\right|dx\leq c\frac{1}{|B_r|}\int_{B_r}\frac{1}{|B_r|}\int_{B_r}\|A(x)-A(w)\|\,dwdx\leq$$

as before

$$\leq c\frac{1}{|B_r|}\int_{B_r}\left\|A(x)-\frac{1}{|B_r|}\int_{B_r}A(\cdot)\right\|dx,$$

which gives the desired bound on LMO norm of α_{ij} in terms of that of the matrix A. ■

4 Singular integral estimates

The main object of this section is to prove the following two theorems, which will be the key tool in order to derive our local BMO^p estimates from the representation formula (3.7).

Theorem 4.1 (Singular integral estimate) *If K_{ij} are the singular integral operators defined in* (3.5)*, then for any $p\in(1,\infty)$ there exists $C>0$ such that:*

$$\begin{aligned}[K_{ij}f]_{BMO_{loc}(B(\overline{x},R;3R))} &\leq C\left(1+[A]_{LMO_{loc}(\Omega)}\right)\cdot\\ &\cdot\left([f]_{BMO_{loc}(B(\overline{x},R;3R))}+\frac{\|f\|_{L^p(B(\overline{x},R))}}{|B(\overline{x},R)|^{1/p}}\right)\end{aligned}$$

for any ball $B(\overline{x},3R)\subset\Omega, f\in BMO^p_{loc}(B(\overline{x},R;3R))$ with sprtf $\subset B(\overline{x},R)$. The number C depends on p,G,Λ.

Theorem 4.2 (Local commutator estimate) *Let $b\in LMO_{loc}(\Omega)$, K_{ij} as before, and $C[K_{ij},b]$ the commutator, defined as in* (3.6)*. Then for any $p\in(1,\infty)$ there exists a constant C and two absolute constants $K>H>3$, such* *that for any $R>0$ with $B(\overline{x},KR)\subset\Omega$, any $f\in BMO^p_{loc}(B(\overline{x},R;3R))$ with $sprt f\subset B(\overline{x},R)$,*

$$[C[K_{ij},b]f]_{BMO_{loc}(B(\overline{x},R;3R))}\leq C\left(1+[A]_{LMO_{loc}(\Omega)}\right)[b]_{LMO_{loc}(B(\overline{x},HR;KR))}\cdot$$
$$\cdot\left\{[f]_{BMO_{loc}(B(\overline{x},R;3R))}+\frac{1}{|B(\overline{x},R)|^{1/p}}\|f\|_{L^p(B(\overline{x},R))}\right\}.$$

The number C depends on p,G,Λ, but not on f,b,R. This means in particular that, if $b\in VLMO_{loc}(\Omega)$, then for any $\varepsilon>0$ there exists $R>0$ such that

$$C[b]_{LMO_{loc}(B(\overline{x},HR;KR))}<\varepsilon.$$

The above two theorems will be derived exploiting the expansion in spherical harmonics and the bounds on the corresponding coefficients discussed in the previous section, applying similar theorems regarding singular integrals of convolution type (modeled on spherical harmonics), and a multiplication theorem (see next Theorem 4.3). Therefore the plan of this section is the following: after establishing some basic estimates regarding BMO type norms (§ 4.1), and particularly the aforementioned multiplication theorem, we will state and prove singular integral estimates for convolution kernels (§ 4.2) and then we will prove Theorems 4.1 and 4.2 (§ 4.3).

4.1 Preliminary real analysis estimates

Throughout the following, we will need a localized version of two well-known facts, namely a multiplication theorem and John-Nirenberg theorem.

Theorem 4.3 *There exists an absolute constant $K>3$ such that if $f\in BMO^p_{loc}(B(\overline{x},R;KR))$ for some $1<p<\infty$, and $\psi\in L^\infty\cap LMO_{loc}(B(\overline{x},R;3R))$, then $\psi f\in BMO^p_{loc}(B(\overline{x},R;3R))$ and*

$$[f\psi]_{BMO_{loc}(B(\overline{x},R;3R))}\leq C\left(\|\psi\|_{L^\infty(B(\overline{x},3R))}+[\psi]_{LMO_{loc}(B(\overline{x},R;3R))}\right)\cdot$$
$$\cdot\left([f]_{BMO_{loc}(B(\overline{x},R;KR))}+\frac{\|f\|_{L^p(B(\overline{x},KR))}}{|B(\overline{x},R)|^{1/p}}\right),$$

with $C=C(G)$.

Proof. Let n be a positive integer such that $2^n r<3R\leq 2^{n+1}r$. We set $B_k=B(x_0,2^k r)$, with $k=0,1,...,n$. Since $\int_{B_\rho}|f-f_{B_\rho}|dx\leq 2\int_{B_t}|f-f_{B_t}|dx$ for $\rho<t$, we have that for $f\in BMO^p_{loc}(B(\overline{x},R;13R))$, $B(x_0,r)\subset B(\overline{x},3R)$,

$x_0 \in B(\overline{x}, R)$

$$
\begin{aligned}
|f_{B(x_0,r)}| &\leq \sum_{k=0}^{n} |f_{B_k} - f_{B_{k+1}}| + |f_{B_{n+1}}| \\
&\leq \sum_{k=0}^{n} \frac{1}{|B_k|} \int_{B_{k+1}} |f(x) - f_{B_{k+1}}| dx + \frac{c}{|B(\overline{x}, R)|^{1/p}} \|f\|_{L^p(B(\overline{x},7R))} \\
&\leq \frac{2^Q}{\log 2} \sum_{k=0}^{n} \int_{2^{k+1}r}^{2^{k+2}r} \frac{ds}{s} \frac{1}{|B_{k+1}|} \int_{B_{k+1}} |f(x) - f_{B_{k+1}}| dx + \frac{c}{|B(\overline{x}, R)|^{1/p}} \|f\|_{L^p(B(\overline{x},7R))} \\
&\leq c \sum_{k=0}^{n} \int_{2^{k+1}r}^{2^{k+2}r} \left(\frac{s}{2^{k+1}r} \right)^Q \frac{1}{|B(x_0,s)|} \int_{B(x_0,s)} |f(x) - f_{B(x_0,s)}| dx \frac{ds}{s} + \frac{c}{|B(\overline{x}, R)|^{1/p}} \|f\|_{L^p(B(\overline{x},7R))} \\
&\leq c \int_{2r}^{2^{n+2}r} \frac{ds}{s} [f]_{BMO_{loc}(B(\overline{x},R;13R))} + \frac{c}{|B(\overline{x}, R)|^{1/p}} \|f\|_{L^p(B(\overline{x},7R))} \\
&\leq c \log \frac{3R}{r} [f]_{BMO_{loc}(B(\overline{x},R;13R))} + \frac{c}{|B(\overline{x}, R)|^{1/p}} \|f\|_{L^p(B(\overline{x},7R))}. \qquad (4.1)
\end{aligned}
$$

Then (see also Lemma 2.4 in [8])

$$
\begin{aligned}
&\frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |\psi f - (\psi f)_{B(x_0,r)}| dx \\
&\leq \left| \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |\psi f - (\psi f)_{B(x_0,r)}| dx - \frac{|f_{B(x_0,r)}|}{|B(x_0,r)|} \int_{B(x_0,r)} |\psi - (\psi)_{B(x_0,r)}| dx \right| \\
&+ \frac{|f_{B(x_0,r)}|}{|B(x_0,r)|} \int_{B(x_0,r)} |\psi - (\psi)_{B(x_0,r)}| dx \\
&\leq \frac{2}{|B(x_0,r)|} \int_{B(x_0,r)} |\psi| |f - (f)_{B(x_0,r)}| dx \\
&+ c \left\{ \log \frac{3R}{r} [f]_{BMO_{loc}(B(\overline{x},R;13R))} + \frac{1}{|B(\overline{x}, R)|^{1/p}} \|f\|_{L^p(B(\overline{x},7R))} \right\} \cdot \\
&\cdot \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |\psi - (\psi)_{B(x_0,r)}| dx \\
&\leq c \|\psi\|_{L^\infty} [f]_{BMO_{loc}(B(\overline{x},R;3R))} + c [\psi]_{LMO_{loc}(B(\overline{x},R;3R))} [f]_{BMO_{loc}(B(\overline{x},R;13R))} \\
&+ c [\psi]_{LMO_{loc}(B(\overline{x},R;3R))} \frac{\|f\|_{L^p(B(\overline{x},7R))}}{|B(\overline{x}, R)|^{1/p}}.
\end{aligned}
$$

■

From Theorem 4.3 we can derive also the following (see also Lemma 4.12 in [8]):

Corollary 4.4 *Let* $\psi \in C^1(B(x,R))$ *such that, for some* $t < s < R$, $\psi = 1$ *in* $B(x,t)$, $\psi = 0$ *outside* $B(x,s)$, $\psi \leq 1$ *and* $|D\psi| \leq c/(s-t)$. *Then for any*

$f \in BMO_{loc}\left(B\left(\overline{x}, R; KR\right)\right)$ *one has*

$$[f\psi]_{BMO_{loc}(B(\overline{x},R;3R))} \leq \frac{C}{(s-t)}\left([f]_{BMO_{loc}(B(\overline{x},R;KR))} + \frac{\|f\|_{L^p(B(\overline{x},KR))}}{|B(\overline{x},R)|^{1/p}}\right).$$

The proof of the following John-Nirenberg Theorem is similar to the proof of Theorem A in [28].

Theorem 4.5 *There exist positive constants* c_1,c_2 *and* $\alpha > 1$ *such that for any* $f \in BMO_{loc}(B(\overline{x}, R; \alpha R))$, *any ball* $B(x_0, r) \subset B(\overline{x}, 3R)$ *with* $x_0 \in B(\overline{x}, R)$, *and* $\forall \lambda > 0$, *we have*

$$|\{x \in B : |f(x) - f_B| > \lambda\}| \leq c_1 \exp(-c_2\lambda/[f]_{BMO_{loc}(B(\overline{x},R;\alpha R))})|B|.$$

In a standard way it is also possible to prove

Corollary 4.6 *Let* $f \in BMO_{loc}(B(\overline{x}, R; \alpha R))$ *and* $1 < p < +\infty$. *Then there exists a constant* $c = c(p)$ *such that*

$$\sup_{x\in B(\overline{x},R), B(x,r)\subset B(\overline{x},3R)} \left(\frac{1}{|B\left(x,r\right)|}\int_{B(x,r)} |f(y) - f_{B(x,r)}|^p dy\right)^{1/p} \leq$$
$$\leq c[f]_{BMO_{loc}(B(\overline{x},R;\alpha R))}$$

with $\alpha > 1$ *as in the previous theorem.*

We now state and prove some further preliminary results which will be useful in the next subsection.

Lemma 4.7 *Let* $f \in BMO_{loc}\left(B\left(\overline{x}, R; 5R\right)\right)$ *and* $B = B(x_0, r) \subset B\left(\overline{x}, 3R\right)$, $x_0 \in B\left(\overline{x}, R\right)$. *Then* $\forall \beta > 0$

$$\int_{B(\overline{x},R)\setminus B(x_0,2r)} \frac{|f(y) - f_B|}{d(x_0, y)^{Q+\beta}} dy \leq \frac{C}{r^\beta}[f]_{BMO_{loc}(B(\overline{x},R;5R))},$$

where C *depends only on* Q *and* β.

Proof. Set $B_k = B(x_0, 2^k r)$, $k = 0, 1, 2, .., n$, with $2^n r \leq 2R < 2^{n+1}r$.

$$\left|f_{B_{k+1}} - f_{B_k}\right| = \left|\frac{1}{|B_k|}\int_{B_k} (f - f_{B_{k+1}})dy\right|$$
$$\leq \frac{2^Q}{|B_{k+1}|}\int_{B_{k+1}} |f(y) - f_{B_{k+1}}|dy \leq 2^Q[f]_{BMO_{loc}(B(\overline{x},R;5R))},$$

from which

$$\left|f_{B_{k+1}} - f_{B_0}\right| \leq (k+1)2^Q[f]_{BMO_{loc}(B(\overline{x},R;5R))}. \tag{4.2}$$

Then

$$\int_{B(\overline{x},R)\setminus B(x_0,2r)} \frac{|f(y)-f_B|}{d(x_0,y)^{Q+\beta}}dy \leq \int_{B(x_0,2^{n+1}r)\setminus B(x_0,2r)} \frac{|f(y)-f_B|}{d(x_0,y)^{Q+\beta}}dy$$
$$= \sum_{k=1}^{n} \int_{B_{k+1}\setminus B_k} \frac{|f(y)-f_B|}{d(x_0,y)^{Q+\beta}}dy \leq \sum_{k=1}^{n} \int_{B_{k+1}} \frac{|f(y)-f_B|}{(r2^k)^{Q+\beta}}dy$$

$$\leq \sum_{k=1}^{n} \frac{c}{(r2^k)^{\beta}} \frac{1}{|B_{k+1}|} \int_{B_{k+1}} |f(y)-f_B|dy \tag{4.3}$$
$$\leq \sum_{k=1}^{n} \frac{c}{(r2^k)^{\beta}} ([f]_{BMO_{loc}(B(\overline{x},R;5R))} + |f_{B_{k+1}} - f_B|)$$

by (4.2)

$$\leq \sum_{k=1}^{+\infty} \frac{c}{(r2^k)^{\beta}} (2+k)[f]_{BMO_{loc}(B(\overline{x},R;5R))} \leq \frac{c}{r^{\beta}} [f]_{BMO_{loc}(B(\overline{x},R;5R))}.$$

■

Lemma 4.8 *Let f be in $LMO_{loc}(B(\overline{x},R;5R))$ and $B = B(x_0,r) \subset B(\overline{x},3R)$, $x_0 \in B(\overline{x},R)$. Then $\forall \beta > 0$*

$$\int_{B(\overline{x},R)\setminus B(x_0,2r)} \frac{|f(y)-f_B|}{d(x_0,y)^{Q+\beta}}dy \leq \frac{C}{r^{\beta}\left(1+\log\frac{5R}{r}\right)}[f]_{LMO_{loc}(B(\overline{x},R;5R))},$$

where C depends only on Q and β.

Proof. The proof is similar to that of (4.1). With the same notation, we have

$$\left|f_{B_{k+1}} - f_{B_k}\right| \leq \frac{2^Q}{\log \frac{10R}{2^{k+1}r}}[f]_{LMO_{loc}(B(\overline{x},R;5R))} = \frac{c}{n-k+1}[f]_{LMO_{loc}(B(\overline{x},R;5R))}$$

from which

$$\left|f_{B_{k+1}} - f_{B_0}\right| \leq c\frac{k+1}{n-k+1}[f]_{LMO_{loc}(B(\overline{x},R;5R))}.$$

Then (4.3) gives, recalling that $2^n r \leq 2R < 2^{n+1}r$

$$
\begin{aligned}
&\int_{B(\overline{x},R)\setminus B(x_0,2r)} \frac{|f(y)-f_B|}{d(x_0,y)^{Q+\beta}}dy \leq \\
&\leq \sum_{k=1}^{n} \frac{c}{(r2^k)^\beta}\left(\frac{1}{|B_{k+1}|}\int_{B_{k+1}}|f(y)-f_{B_{k+1}}|dy+|f_{B_{k+1}}-f_{B_0}|\right) \\
&\leq \sum_{k=1}^{n} \frac{c}{(r2^k)^\beta}\left(\frac{1}{1+\log\left(\frac{5R}{2^k r}\right)}+\frac{k+1}{n-k+1}\right)[f]_{LMO_{loc}(B(\overline{x},R;5R))} \\
&\leq \frac{c}{r^\beta}[f]_{LMO_{loc}(B(\overline{x},R;5R))}\sum_{k=1}^{n}\frac{1}{2^{k\beta}}\left(\frac{k+2}{n-k+1}\right) \\
&\leq \frac{c}{r^\beta\left(1+\log\left(\frac{5R}{r}\right)\right)}[f]_{LMO_{loc}(B(\overline{x},R;5R))}\sum_{k=1}^{n}\frac{n+1}{2^{k\beta}}\left(\frac{k+2}{n-k+1}\right) \\
&\leq \frac{C}{r^\beta\left(1+\log\frac{5R}{r}\right)}[f]_{LMO_{loc}(B(\overline{x},R;5R))}
\end{aligned}
$$

since

$$
\begin{aligned}
\sum_{k=1}^{n}\frac{n+1}{2^{k\beta}}\left(\frac{k+2}{n-k+1}\right) &= \sum_{k=1}^{n}\frac{k+2}{2^{k\beta}}\left(1+\frac{k}{n-k+1}\right) \\
&\leq \sum_{k=1}^{\infty}\frac{(k+2)(k+1)}{2^{k\beta}} = c_\beta.
\end{aligned}
$$

■

4.2 Estimates for singular integrals of convolution type

In this section we prove the BMO-type estimates for singular integrals and their commutators. Here we deal with the convolution kernels H_{km} defined in (3.9). The explicit form of the kernel is not important; it will be enough to point out the relevant properties which we will use. So, let

$$
k(x,y) = k_0\left(y^{-1}\circ x\right)
$$

be one of our singular integral kernels, defined in the whole $\mathbb{R}^N$. The following properties hold true (see [6, Prop. 1]):

$$
|k(x,y)| \leq \frac{A}{d(x,y)^Q} \quad \forall x,y\in\mathbb{R}^N \tag{4.4}
$$

$$
|k(x,y)-k(x_0,y)|+|k(y,x)-k(y,x_0)| \leq B\frac{d(x_0,x)}{d(x_0,y)^{Q+1}} \tag{4.5}
$$

for any $x_0,x,y\in\mathbb{R}^N$ with $d(x_0,y)\geq 2d(x_0,x)$.

$$
\int_{r_1<d(x,y)<r_2} k(x,y)\,dy = 0 = \int_{r_1<d(x,y)<r_2} k(y,x)\,dy
$$

for any $0 < r_1 < r_2 < \infty$.

The last property requires some comments. It is known that the integral of $Y_{km}(y)$ over Σ_N, for $m \geq 1$, is zero. This also implies that

$$\int_{r_1<\rho(x,y)<r_2} k(x,y)\,dy = 0 = \int_{r_1<\rho(x,y)<r_2} k(y,x)\,dy$$

for any $0 < r_1 < r_2 < \infty$, since $B_\rho(0,r) = D(r)\,B_\rho(0,1) = D(r)\,B_E(0,1)$. It is less obvious that this vanishing property still holds with respect to d-balls. However, this is true in view of the homogeneity of d (see Proposition 2.2), that is a homogeneous norm.

Let now

$$Tf(x) = P.V.\int_{\mathbb{R}^N} k(x,y)\,f(y)\,dy.$$

All the quantitative estimates that we will prove in this section on the operator T will depend on k only through the numbers A, B in (4.4)-(4.5). We will show in the next section how to quantify this dependence in the case of our concrete kernels H_{km}.

Throughout this section, let $B(\overline{x},R)$ be a fixed d-ball such that $B(\overline{x},KR) \subset \Omega$ for some large $K > 0$ which will be chosen later. We are interested in studying $Tf(x)$ and its commutator for $\text{sprt}f \subset B(\overline{x},R)$ and $x \in B(\overline{x},3R)$, hence for $d(x,y) < 4R$. So, let $\psi(x,y) = \psi_0\left(y^{-1}\circ x\right)$ be a cutoff function such that

$$B(0,4R) \prec \psi_0 \prec B(0,5R).$$

Hence for $\text{sprt}f \subset B(\overline{x},R)$ and $x \in B(\overline{x},3R)$,

$$Tf(x) = \int_{\mathbb{R}^N} k(x,y)\,f(y)\,dy = \int_{\mathbb{R}^N} k(x,y)\,\psi(x,y)\,f(y)\,dy \equiv \widetilde{T}f(x).$$

We will also let

$$\widetilde{k}(x,y) = k(x,y)\,\psi(x,y).$$

Note that for any $x \in B(\overline{x},3R)\,, g \in L^1_{loc}\left(\mathbb{R}^N\right)$ we have

$$\widetilde{T}g(x) = \int_{B(\overline{x},8R)} k(x,y)\,\psi(x,y)\,g(y)\,dy.$$

Note that the cancellation property

$$\int_{r_1<d(x,y)<r_2} k(x,y)\,dy = 0 \quad \forall r_1 < r_2$$

implies that

$$\int_{r_1<d(x,y)<r_2} k(x,y)\,\psi(x,y)\,dy = 0 \quad \forall r_1 < r_2$$

and so

$$\widetilde{T}(c) = 0 \text{ in } B(\overline{x},3R)\,, \text{ for any constant } c. \tag{4.6}$$

Now, for $b \in LMO_{loc}(B(\overline{x}, R; KR))$, $f \in C_0^\infty(B(\overline{x}, R)), x \in B(\overline{x}, 3R)$

$$\begin{aligned} T_b f(x) &= [T, b](f)(x) = T(bf)(x) - b(x)Tf(x) \\ &= \int_{\mathbb{R}^N} k(x,y)[b(y) - b(x)] f(y)\, dy \\ &= \left[\widetilde{T}, b\right](f)(x) \equiv \widetilde{T}_b f(x). \end{aligned}$$

Note that for $x \in B(\overline{x}, 3R)$ and $g \in L^1_{loc}(\mathbb{R}^N)$ we have

$$\left[\widetilde{T}, b\right](g)(x) = \int_{B(\overline{x}, 8R)} k(x,y)\psi(x,y)[b(y) - b(x)] g(y)\, dy$$

which is meaningful provided b is defined in $B(\overline{x}, 8R)$ (hence, we will pick $K \geq 8$).

The aim of the previous definitions is the following: on the one hand, we want to define a "local" commutator, without the necessity of extending the function b to the whole space $\mathbb{R}^N$; but, on the other hand, we need to preserve the strong cancellation property (4.6), which will be essential in the sequel.

Theorem 4.9 *Let T be a singular integral operator as before. There exists an absolute constant $K > 3$ such that for any ball $B(\overline{x}, KR) \subset \Omega$ we have*

$$\begin{aligned} [Tf]_{BMO_{loc}(B(\overline{x},R;3R))} &\leq c[f]_{BMO_{loc}(B(\overline{x},R;KR))}; \\ [Tf]_{LMO_{loc}(B(\overline{x},R;3R))} &\leq c[f]_{LMO_{loc}(B(\overline{x},R;KR))} \end{aligned}$$

for any $f \in BMO_{loc}(B(\overline{x}, R; KR))$ (or LMO_{loc}, respectively) with sprt$f \subset B(\overline{x}, R)$, some constant c independent of R and f.

We recall that the singular integral operator T and the commutator T_b are continuous in L^p (see [7] and [11]). Moreover we note that in [8] a general continuity result for singular integral operators on the scale of spaces BMO_ϕ has been proved, which in particular applies to BMO and LMO. However, the bounds we need here are formulated in terms of local BMO and LMO spaces; moreover, the strong cancellation property we can rely on makes it easy to present a short self-contained proof.

Proof. We are going to prove the second inequality; the proof of the first follows by the same reasoning, just dropping all the "log" functions.

Let $B = B(x_0, r) \subset B(\overline{x}, 3R)$ with $x_0 \in B(\overline{x}, R)$. Since sprt$f \subset B(\overline{x}, R)$ for $x \in B(\overline{x}, 3R)$ $\widetilde{T}f(x) = Tf(x)$, so in the following we will always handle $\widetilde{T}f$ instead of Tf. Let us split

$$f(x) = f_{2B} + (f(x) - f_{2B})\chi_{2B}(x) + (f(x) - f_{2B})\chi_{(2B)^c}(x) = f_1 + f_2(x) + f_3(x).$$

By (4.6), $\widetilde{T}f_1 = 0$, hence for any $c \in \mathbb{R}$

$$\begin{aligned}
&\frac{1+\log\frac{6R}{r}|}{|B|}\int_B \left|\widetilde{T}f(x) - (\widetilde{T}f)_B\right| dx \\
&\leq 2\frac{1+\log\frac{6R}{r}}{|B|}\int_B \left|\widetilde{T}f(x) - c\right| dx \\
&\leq 2\frac{1+\log\frac{6R}{r}}{|B|}\left(\int_B \left|\widetilde{T}f_2(x)\right| dx + \int_B \left|\widetilde{T}f_3(x) - c\right| dx\right) \\
&\equiv I + II.
\end{aligned}$$

By Hölder and John-Nirenberg inequalities and the L^2 continuity of $\widetilde{T}$ we have

$$\begin{aligned}
I &\leq 2\left(1+\log\frac{6R}{r}\right)\left(\frac{1}{|B|}\int_B |\widetilde{T}f_2(x)|^2\right)^{1/2} \\
&\leq c\left(1+\log\frac{6R}{r}\right)\left(\frac{1}{|B|}\int_{\mathbb{R}^n} |f_2(x)|^2 dx\right)^{1/2} \\
&\leq c\left(1+\log\frac{12R}{2r}\right)\left(\frac{1}{|2B|}\int_{2B} |f(x) - f_{2B}|^2 dx\right)^{1/2} \\
&\leq c[f]_{LMO_{loc}(B(\overline{x},R;6R))}.
\end{aligned}$$

To bound II, pick $x^* \in B$ such that $\widetilde{T}f_3(x^*) < +\infty$ (this is true for a.e. $x^* \in B$ since $\widetilde{T}f_3 \in L^2(B)$) and choose $c = \widetilde{T}f_3(x^*)$. Then for any $x \in B$, by (4.6) we can write, by 4.5,

$$\begin{aligned}
\left|\widetilde{T}f_3(x) - \widetilde{T}f_3(x^*)\right| &= \left|\int \left[\widetilde{k}(x,y) - \widetilde{k}(x^*,y)\right] f_3(y)dy\right| \\
&= \left|\int_{B(\overline{x},8R)\setminus B(x_0,2r)} \left[\widetilde{k}(x,y) - \widetilde{k}(x^*,y)\right][f(y) - f_{2B}]\, dy\right| \\
&\leq c\int_{B(\overline{x},8R)\setminus B(x_0,2r)} \frac{d(x,x^*)}{d(x^*,y)^{Q+1}}|f(y) - f_{2B}|dy \\
&\leq cr\int_{B(\overline{x},8R)\setminus B(x_0,2r)} \frac{|f(y) - f_{2B}|}{d(x_0,y)^{Q+1}}dy.
\end{aligned}$$

Applying Lemma 4.8 with $\beta = 1$ we get

$$\int_{B(\overline{x},8R)\setminus B(x_0,2r)} \frac{|f(y) - f_B|}{d(x_0,y)^{Q+1}}dy \leq \frac{C}{r\left(1+\log\frac{3R}{r}\right)}[f]_{LMO_{loc}(B(\overline{x},R;KR))}$$

for some large constant K (independent of R), hence

$$II \leq c[f]_{LMO_{loc}(B(\overline{x},R;KR))}$$

and we are done. ■

Remark 4.10 *In the previous theorem the number* 3 *has nothing special: we will also need, in the following, a modified version of the previous estimate: given a constant* $K > 3$ *there exists a constant* $K' > K$ *such that for any ball* $B(\overline{x}, K'R) \subset \Omega$ *we have*

$$[Tf]_{BMO_{loc}(B(\overline{x},R;KR))} \leq c[f]_{BMO_{loc}(B(\overline{x},R;K'R))};$$
$$[Tf]_{LMO_{loc}(B(\overline{x},R;3R))} \leq c[f]_{LMO_{loc}(B(\overline{x},R;KR))}$$

for any $f \in BMO_{loc}(B(\overline{x}, R; K'R))$ *(or* LMO_{loc}*, respectively) with sprt*$f \subset B(\overline{x}, R)$.

Theorem 4.11 *Let* $b \in LMO_{loc}(\Omega)$*, then for any* $p \in (1, \infty)$ *there exists a constant* C *such that for any* $R > 0$ *such that* $B(\overline{x}, KR) \subset \Omega$*, any* $f \in BMO^p_{loc}(B(\overline{x}, R; 3R))$ *with sprt*$f \subset B(\overline{x}, R)$*,*

$$[T_b f]_{BMO_{loc}(B(\overline{x},R;3R))} \leq C[b]_{LMO_{loc}(B(\overline{x},4R;KR))} \cdot \cdot \left\{ [f]_{BMO_{loc}(B(\overline{x},R;3R))} + \frac{1}{|B(\overline{x}, R)|^{1/p}} \|f\|_{L^p(B(\overline{x},R))} \right\}. \tag{4.7}$$

The number C *depends on* p *and the constants of the singular kernel of* T*, but not on* f, b, R*. This means in particular that, if* $b \in VLMO_{loc}(\Omega)$*, then for any* $\varepsilon > 0$ *there exists* $R > 0$ *such that*

$$C[b]_{LMO_{loc}(B(\overline{x},4R;KR))} < \varepsilon.$$

Remark 4.12 *Again, the absolute constant* 3 *appearing in the previous estimates is not so important: replacing* 3 *with a larger number simply causes the constants* K *and* 4 *be replaced by larger constants.*

Also, note that by [12, Thm. 7.1],

$$\|T_b f\|_{L^p(B(\overline{x},3R))} \leq C[b]_{LMO_{loc}(B(\overline{x},4R;KR))} \|f\|_{L^p(B(\overline{x},3R))} \tag{4.8}$$

which, coupled with the above theorem, gives, for a function f *supported in* $B(\overline{x}, R)$*,*

$$\|T_b f\|_{BMO^p_{loc}(B(\overline{x},R;3R))} \leq C[b]_{LMO_{loc}(B(\overline{x},4R;KR))} \cdot \cdot \left\{ [f]_{BMO_{loc}(B(\overline{x},R;3R))} + \left(1 + \frac{1}{|B(\overline{x}, R)|^{1/p}}\right) \|f\|_{L^p(B(\overline{x},R))} \right\}.$$

Proof of Theorem 4.11. Since we want to bound $T_b f(x)$ for f supported in $B(\overline{x}, R)$ and $x \in B(\overline{x}, 3R)$, as remarked at the beginning of this section we have

$$T_b f(x) = \widetilde{T}_b f(x),$$

hence from now on we will work with $\widetilde{T}_b$.

Let $B = B(x_0, r) \subset B(\overline{x}, 3R)$ with $x_0 \in B(\overline{x}, R)$, and let us split, like in the proof of Theorem 4.9:

$$f(x) = f_{2B} + (f(x) - f_{2B})\chi_{2B}(x) + (f(x) - f_{2B})\chi_{(2B)^c}(x) = f_1 + f_2(x) + f_3(x)$$

from which

$$\widetilde{T}_b f - (\widetilde{T}_b f)_B = (\widetilde{T}_b f_1 - (\widetilde{T}_b f_1)_B) + (\widetilde{T}_b f_2 - (\widetilde{T}_b f_2)_B) + \\ + (\widetilde{T}_b f_3 - (\widetilde{T}_b f_3)_B) = I + II + III\,.$$

Then, since f_1 is constant, we have $\widetilde{T} f_1 = 0$ (see (4.6)), hence

$$I = \widetilde{T}(bf_1) - f_1(\widetilde{T}b)_B = f_1(\widetilde{T}b - (\widetilde{T}b)_B)$$

hence by (4.1) and Theorem 4.9 we have

$$\begin{aligned}
&\frac{1}{|B|}\int_B |I|dx = \frac{|f_{2B}|}{|B|}\int_B |\widetilde{T}b - (\widetilde{T}b)_B|dx \\
&\leq c\left\{\log\frac{3R}{r}[f]_{BMO_{loc}(B(\overline{x},R;13R))} + \frac{1}{|B(\overline{x},R)|^{1/p}}\|f\|_{L^p(B(\overline{x},R))}\right\}\cdot \\
&\cdot\frac{1}{|B|}\int_B |\widetilde{T}b - (\widetilde{T}b)_B|dx \\
&\leq c\left\{\frac{\log\frac{3R}{r}}{1+\log\frac{6R}{r}}[f]_{BMO_{loc}(B(\overline{x},R;13R))} + \frac{1}{|B(\overline{x},R)|^{1/p}}\|f\|_{L^p(B(\overline{x},R))}\right\}\cdot \\
&\cdot\frac{1+\log\frac{6R}{r}}{|B|}\int_B |\widetilde{T}b - (\widetilde{T}b)_B|dx \\
&\leq c[\widetilde{T}b]_{LMO_{loc}(B(\overline{x},R,3R))}\left\{[f]_{BMO_{loc}(B(\overline{x},R;13R))} + \frac{1}{|B(\overline{x},R)|^{1/p}}\|f\|_{L^p(B(\overline{x},R))}\right\} \\
&\leq c[b]_{LMO_{loc}(B(\overline{x},R,KR))}\left\{[f]_{BMO_{loc}(B(\overline{x},R;13R))} + \frac{1}{|B(\overline{x},R)|^{1/p}}\|f\|_{L^p(B(\overline{x},R))}\right\}.
\end{aligned} \tag{4.9}$$

Now we consider

$$\begin{aligned}
\frac{1}{|B|}\int_B |II|dx &= \frac{1}{|B|}\int_B |\widetilde{T}_b f_2 - (\widetilde{T}_b f_2)_B|dx \\
&\leq \frac{2}{|B|}\int_B |\widetilde{T}_b f_2|dx \leq 2|B|^{-1/2}\|\widetilde{T}_b f_2\|_{L^2(B(x_0,r))}
\end{aligned}$$

by Hölder inequality. Next, we apply the L^2 boundedness of $\widetilde{T}_b$. The local continuity result on the commutator of $\widetilde{T}_b$ in L^p proved in [12, Thm. 7.1] implies that

$$\|\widetilde{T}_b f_2\|_{L^2(B(\overline{x},3R))} \leq c[b]_{BMO_{loc}(B(\overline{x},4R;KR))}\|f_2\|_{L^2(B(\overline{x},3R))}$$

for some absolute constant $K > 4$, with c independent of R. Hence, since $B(x_0, r) \subset B(\overline{x}, 3R)$,

$$\begin{aligned}
&\frac{1}{|B|}\int_B |II|dx \leq \\
&\qquad \leq c|B|^{-1/2}[b]_{BMO_{loc}(B(\overline{x},4R;KR))}\|f_2\|_{L^2(B(x_0,2r))} \\
&\qquad = c[b]_{BMO_{loc}(B(\overline{x},4R;KR))}\left(\frac{1}{|2B|}\int_{2B}|f(x)-f_{2B}|^2dx\right)^{1/2} \\
&\qquad \leq c[b]_{BMO_{loc}(B(\overline{x},4R;KR))}[f]_{BMO_{loc}(B(\overline{x},R;KR))} \qquad (4.10)
\end{aligned}$$

where we have used also Corollary 4.6.

Last, we come to

$$\begin{aligned}
III &= \widetilde{T}(bf_3)(x) - b(x)\widetilde{T}f_3(x) - \frac{1}{|B|}\int_B (\widetilde{T}_b f_3)(z)dz = \\
&= \widetilde{T}((b-b_B)\,f_3) - (b-b_B)\,\widetilde{T}f_3 + \\
&- \left(\frac{1}{|B|}\int_B \widetilde{T}((b-b_B)f_3)(z)dz - \frac{1}{|B|}\int_B (b(z)-b_B)\widetilde{T}f_3(z)dz\right) \\
&= -(b(x)-b_B)(\widetilde{T}f_3(x) - \widetilde{T}f_3(x_0)) - (b(x)-b_B)\widetilde{T}f_3(x_0) + \\
&+ \frac{1}{|B|}\int_B (b(z)-b_B)(\widetilde{T}f_3(z)-\widetilde{T}f_3(x_0))dz + \\
&- \frac{1}{|B|}\int_B [\widetilde{T}((b-b_B)f_3)(z) - \widetilde{T}((b-b_B)f_3)(x)]dz \\
&\equiv III_1 + III_2 + III_3 + III_4\,.
\end{aligned}$$

First, we want to bound, for $x \in B$,

$$\begin{aligned}
\left|\widetilde{T}f_3(x) - \widetilde{T}f_3(x_0)\right| &= \left|\int_{\mathbb{R}^n\setminus 2B}(\widetilde{k}(x,y) - \widetilde{k}(x_0,y)(f(y)-f_{2B})dy\right| \leq \\
&\leq c\int_{B(\overline{x},7R)\setminus B(x_0,2r)} \frac{d(x_0.x)}{d(x_0,y)^{Q+1}}|f(y)-f_{2B}|dy \\
&= c\int_{B(\overline{x},R)\setminus B(x_0,2r)} \frac{d(x_0,x)}{d(x_0,y)^{Q+1}}|f(y)-f_{2B}|dy + \\
&+ c\int_{B(\overline{x},7R)\setminus(B(\overline{x},R)\cup B(x_0,2r))} \frac{d(x_0,x)}{d(x_0,y)^{Q+1}}|f(y)-f_{2B}|dy \\
&\equiv I + II
\end{aligned}$$

By Lemma 4.7, with $\beta = 1$

$$\begin{aligned}
I &\leq cr\left(\int_{B(\overline{x},R)\setminus B(x_0,2r)} \frac{|f(y)-f_B|}{d(x_0,y)^{Q+1}}dy + \int_{B(\overline{x},R)\setminus B(x_0,2r)} \frac{|f_B - f_{2B}|}{d(x_0,y)^{Q+1}}dy\right) \\
&\leq cr\left\{\frac{1}{r}[f]_{BMO_{loc}(B(\overline{x},R;5R))} + \frac{1}{r}|f_B - f_{2B}|\right\} \leq c[f]_{BMO_{loc}(B(\overline{x},R;5R))}.
\end{aligned}$$

Since $\mathrm{sprt} f \subset B(\overline{x}, R)$,

$$II = c|f_{2B}| \int_{B(\overline{x},7R)\setminus(B(\overline{x},R)\cup B(x_0,2r))} \frac{d(x_0,x)}{d(x_0,y)^{Q+1}} dy \leq c|f_{2B}|$$

by (4.1)

$$\leq c\log\frac{3R}{r}[f]_{BMO_{loc}(B(\overline{x},R;13R))} + \frac{c}{|B(\overline{x},R)|^{1/p}}\|f\|_{L^p(B(\overline{x},R))}.$$

Therefore

$$\left|\widetilde{T}f_3(x) - \widetilde{T}f_3(x_0)\right| \leq c\left(1+\log\frac{3R}{r}\right)[f]_{BMO_{loc}(B(\overline{x},R;13R))} + \\ + \frac{c}{|B(\overline{x},R)|^{1/p}}\|f\|_{L^p(B(\overline{x},R))}$$

which implies

$$\begin{aligned}
&\frac{1}{|B|}\int_B |III_1|\,dx \leq \frac{1}{|B|}\int_B |b(x)-b_B|\,dx\cdot \qquad (4.11)\\
&\cdot\left\{c\left(1+\log\frac{3R}{r}\right)[f]_{BMO_{loc}(B(\overline{x},R;13R))} + \frac{c}{|B(\overline{x},R)|^{1/p}}\|f\|_{L^p(B(\overline{x},R))}\right\}\\
&\leq c\,[f]_{BMO_{loc}(B(\overline{x},R;13R))}\,[b]_{LMO_{loc}(B(\overline{x},R;3R))} +\\
&+\frac{c}{|B(\overline{x},R)|^{1/p}}\|f\|_{L^p(B(\overline{x},R))}[b]_{BMO_{loc}(B(\overline{x},R;3R))}.
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{|B|}\int_B |III_3|dx &\leq c\,[f]_{BMO_{loc}(B(\overline{x},R;13R))}\,[b]_{LMO_{loc}(B(\overline{x},R;3R))} \qquad (4.12)\\
&+\frac{c}{|B(\overline{x},R)|^{1/p}}\|f\|_{L^p(B(\overline{x},R))}[b]_{BMO_{loc}(B(\overline{x},R;3R))}.
\end{aligned}$$

In order to bound III_2, we now start proving that $\widetilde{T}f_3(x_0)$ exists and satisfies the estimate

$$\left|\widetilde{T}f_3(x_0)\right| \leq c\left(1+\log\left(\frac{4R}{r}\right)\right)[f]_{BMO_{loc}(B(\overline{x},R;5R))}. \qquad (4.13)$$

Indeed, let $j = 1,2,...,n$ with $2^n r < 7R \leq 2^{n+1}r$, $B_j = B\left(x_0, 2^j r\right)$. Then (here we have to modify the technique previously used, to exploit the cancellation property of the kernel $\widetilde{k}$)

$$\begin{aligned}
\left|\widetilde{T}f_3(x_0)\right| &= \left|\int_{B(\overline{x},6R)\setminus B(x_0,2r)} \widetilde{k}(x_0,y)(f(y)-f_{2B})dy\right| =\\
&= \left|\int_{B(x_0,2^{n+1}r)\setminus B(x_0,2r)} (...)\,dy - \int_{B(x_0,2^{n+1}r)\setminus B(\overline{x},6R)} (...)\,dy\right| \equiv |A-B|.
\end{aligned}$$

$$|A| \le \sum_{j=1}^{n} \left| \int_{B_{j+1}\setminus B_j} \widetilde{k}(x_0,y)(f(y)-f_{2B})dy \right| = \sum_{j=1}^{n} \left| \int_{B_{j+1}\setminus B_j} \widetilde{k}(x_0,y)(f(y)-f_{B_{j+1}})dy \right|$$

by the cancellation property of $\widetilde{k}$

$$\begin{aligned}
&\le c\sum_{j=1}^{n} \int_{B_{j+1}\setminus B_j} \frac{1}{d(x_0,y)^Q} |f(y)-f_{B_{j+1}}|dy \\
&\le c\sum_{j=1}^{n} \frac{1}{|B_{j+1}|} \int_{B_{j+1}} |f(y)-f_{B_{j+1}}|dy \\
&\le cn[f]_{BMO_{loc}(B(\overline{x},R;5R))} \le c\left(1+\log\left(\frac{4R}{r}\right)\right)[f]_{BMO_{loc}(B(\overline{x},R;5R))}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|B| &\le \int_{B(x_0,2^{n+1}r)\setminus B(\overline{x},6R)} \left| \widetilde{k}(x_0,y)(f(y)-f_{2B}) \right| dy \\
&\le \frac{c}{R^Q} \int_{B(x_0,2^{n+1}r)} |f(y)-f_{2B})|\, dy \le \frac{c}{|B_{n+1}|} \int_{B_{n+1}} |f(y)-f_{2B})|\, dy \quad (4.14) \\
&\le \frac{c}{|B_{n+1}|} \int_{B_{n+1}} \left| f(y)-f_{B_{n+1}} + f_{B_{n+1}} - f_{B_n} + \cdots - f_{2B} \right| dy \\
&\le \frac{c}{|B_{n+1}|} \int_{B_{n+1}} \left| f(y)-f_{B_{n+1}} \right| dy + \sum_{j=2}^{n} \left| f_{B_{j+1}} - f_{B_j} \right| \\
&\le cn[f]_{BMO_{loc}(B(\overline{x},R;5R))} \le c\left(1+\log\left(\frac{4R}{r}\right)\right)[f]_{BMO_{loc}(B(\overline{x},R;5R))},
\end{aligned}$$

hence (4.13) is proved. Therefore

$$\begin{aligned}
\frac{1}{|B|}\int_B |III_2|dx &\le c\log\left(\frac{4R}{r}\right)[f]_{BMO_{loc}(B(\overline{x},R;5R))} \frac{1}{|B|}\int_B |b(x)-b_B|dx \quad (4.15) \\
&\le c[f]_{BMO_{loc}(B(\overline{x},R;5R))}[b]_{LMO_{loc}(B(\overline{x},R;3R))}.
\end{aligned}$$

Now we want to bound, $\forall x, y \in B$,

$$\begin{aligned}
&\left| \widetilde{T}((b-b_B)f_3)(x) - \widetilde{T}((b-b_B)f_3)(y) \right| \quad (4.16) \\
&\le \int_{B(\overline{x},8R)\setminus B(x_0,2r)} |\widetilde{k}(x,z)-\widetilde{k}(y,z)||b(z)-b_B||f(z)-f_{2B}|\, dz \\
&\le c\int_{B(\overline{x},8R)\setminus B(x_0,2r)} \frac{d(x,y)}{d(x_0,z)^{Q+1}} |b(z)-b_B||f(z)-f_{2B}|dz \\
&\le cr\int_{B(\overline{x},8R)\setminus B(x_0,2r)} \frac{1}{d(x_0,z)^{Q+1}} |b(z)-b_B||f(z)-f_{2B}|dz.
\end{aligned}$$

Let $B_j = B\left(x_0, 2^j r\right)$, let n be such that $2^n r < 9R \le 2^{n+1} r$. Then

$$
\begin{aligned}
&\left|\widetilde{T}((b-b_B)f_3)(x)-\widetilde{T}((b-b_B)f_3)(y)\right| \\
&\leq cr\sum_{j=1}^{n}\int_{B_{j+1}\setminus B_j}\frac{|b(z)-b_B||f(z)-f_{2B}|}{d(x_0,z)^{Q+1}}dz \\
&\leq cr\sum_{j=1}^{n}\frac{1}{(2^j r)^{Q+1}}\int_{B_{j+1}}|b(z)-b_B||f(z)-f_{2B}|dz \\
&\leq c\sum_{j=1}^{n}\frac{1}{2^j|B_{j+1}|}\left(\int_{B_{j+1}}|b(z)-b_B|^2dz\right)^{1/2}\left(\int_{B_{j+1}}|f(z)-f_{2B}|^2dz\right)^{1/2} \\
&\leq c\sum_{j=1}^{n}\frac{1}{2^j}\left(\frac{1}{|B_{j+1}|}\int_{B_{j+1}}|b(z)-b_B|^2dz\right)^{1/2}\left(\frac{1}{|B_{j+1}|}\int_{B_{j+1}}|f(z)-f_{2B}|^2dz\right)^{1/2}.
\end{aligned}
$$

We observe that, reasoning like in (4.14)

$$
\begin{aligned}
&\left(\frac{1}{|B_{j+1}|}\int_{B_{j+1}}|f(z)-f_{2B}|^2dz\right)^{1/2} \qquad (4.17)\\
&\leq cj\left(\frac{1}{|B_{j+1}|}\int_{B_{j+1}}|f(z)-f_{B_{j+1}}|^2dz\right)^{1/2} \\
&\leq cj[f]_{BMO_{loc}(B(\overline{x},R;19R))},
\end{aligned}
$$

and in the same way

$$
\left(\frac{1}{|B_{j+1}|}\int_{B_{j+1}}|b(z)-b_B|^2dz\right)^{1/2}\leq c\,(j+1)\,[b]_{BMO_{loc}(B(\overline{x},R;19R))}. \qquad (4.18)
$$

Then, by (4.17) and (4.18), (4.16) gives

$$
\begin{aligned}
&\left|\widetilde{T}((b-b_B)f_3)(x)-\widetilde{T}((b-b_B)f_3)(y)\right| \\
&\leq c\sum_{j=1}^{\infty}\frac{1}{2^j}j\,(j+1)\,[b]_{BMO_{loc}(B(\overline{x},R;19R))}[f]_{BMO_{loc}(B(\overline{x},R;19R))} \\
&\leq c[b]_{BMO_{loc}(B(\overline{x},R;19R))}[f]_{BMO_{loc}(B(\overline{x},R;19R))}.
\end{aligned}
$$

Hence

$$
\frac{1}{|B|}\int_B|III_4|dx\leq c[b]_{BMO_{loc}(B(\overline{x},R;19R))}[f]_{BMO_{loc}(B(\overline{x},R;19R))}. \qquad (4.19)
$$

From (4.9), (4.10), (4.11), (4.12), (4.15), and (4.19) we have

$$
\begin{aligned}
&[\widetilde{T}_b f]_{BMO_{loc}(B(\overline{x},R;3R))} \\
&\leq c[b]_{LMO_{loc}(B(\overline{x},R;KR))} \left\{ [f]_{BMO_{loc}(B(\overline{x},R;13R))} + \frac{c}{|B(\overline{x},R)|^{1/p}} \|f\|_{L^p(B(\overline{x},R))} \right\} \\
&+ c[b]_{BMO_{loc}(B(\overline{x},4R;KR))} [f]_{BMO_{loc}(B(\overline{x},R;KR))} + \\
&+ c[b]_{LMO_{loc}(B(\overline{x},R;3R))} [f]_{BMO_{loc}(B(\overline{x},R;13R))} + \\
&+ \frac{c}{|B(\overline{x},R)|^{1/p}} \|f\|_{L^p(B(\overline{x},R))} [b]_{BMO_{loc}(B(\overline{x},R;3R))} + \\
&+ c[b]_{LMO_{loc}(B(\overline{x},R;3R))} [f]_{BMO_{loc}(B(\overline{x},R;5R))} + \\
&+ c[b]_{BMO_{loc}(B(\overline{x},R;19R))} [f]_{BMO_{loc}(B(\overline{x},R;19R))} \\
&\leq c[b]_{LMO_{loc}(B(\overline{x},4R;KR))} \left\{ [f]_{BMO_{loc}(B(\overline{x},R;KR))} + \frac{c}{|B(\overline{x},R)|^{1/p}} \|f\|_{L^p(B(\overline{x},R))} \right\}
\end{aligned}
$$

by Lemma 2.7

$$
\leq c[b]_{LMO_{loc}(B(\overline{x},4R;KR))} \cdot \left\{ [f]_{BMO_{loc}(B(\overline{x},R;3R))} + \frac{1}{|B(\overline{x},R)|^{1/p}} \|f\|_{L^p(B(\overline{x},R))} \right\}
$$

and (4.7) is proved. The last assertion in the statement of the theorem then follows by (2.3). ■

4.3 Estimates for singular integrals with variable kernels

Now we are ready to prove the two theorems stated at the beginning of this section, regarding singular integrals with variable kernels and their commutator.

Proof of Theorem 4.1. Let $f \in BMO_{loc}(B(\overline{x},R;3R))$ with $\mathrm{sprt} f \subset B(\overline{x},R)$, $B(\overline{x},3R) \subset \Omega$. By the expansion in spherical harmonics (3.8) and Theorem 4.3,

$$
\begin{aligned}
&[K_{ij} f]_{BMO_{loc}(B(\overline{x},R;3R))} \leq \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} \left[c_{ij}^{km}\, T_{km} f \right]_{BMO_{loc}(B(\overline{x},R;3R))} \\
&\leq C \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} \left(\left\| c_{ij}^{km} \right\|_{L^\infty(B(\overline{x},3R))} + \left[c_{ij}^{km} \right]_{LMO_{loc}(B(\overline{x},R;3R))} \right) \cdot \\
&\cdot \left([T_{km} f]_{BMO_{loc}(B(\overline{x},R,KR))} + \frac{\|T_{km} f\|_{L^p(B(\overline{x},KR))}}{|B(\overline{x},R)|^{1/p}} \right).
\end{aligned}
$$

We now apply Theorem 4.9 with Remark 4.10, getting

$$
\begin{aligned}
[T_{km} f]_{BMO_{loc}(B(\overline{x},R;KR))} &\leq c\,(p,G,k,m)\, [f]_{BMO_{loc}(B(\overline{x},R;K'R))} \\
\|T_{km} f\|_{L^p(B(\overline{x},KR))} &\leq c\,(p,G,k,m)\, \|f\|_{L^p(B(\overline{x},KR))}
\end{aligned}
$$

where the constant $c(p, G, k, m)$ depends on k, m only through the constants A, B appearing in (4.4), (4.5) when the kernel is H_{km}. In turn, these constants are bounded in terms of

$$\sup_{|u|=1} |H_{km}(u)| + \sup_{|u|=1} |\nabla H_{km}(u)| \leq c(G) \cdot m^{\frac{N}{2}}.$$

by (3.11). A standard linearity argument shows then that actually

$$c(p, G, k, m) \leq c(p, G) \cdot m^{\frac{N}{2}}.$$

Next, we use the bounds on the coefficients c_{ij}^{km} contained in (3.13) and Theorem 3.5. They assure that for any positive integer n there exists a constant $c(n, G, \Lambda)$ such that

$$\begin{aligned}\left\|c_{ij}^{km}\right\|_{L^\infty} + \left[c_{ij}^{km}\right]_{LMO_{loc}(B(\overline{x},R;3R))} &\leq c(n, G, \Lambda) \cdot m^{-2n} \left(1 + [A]_{LMO_{loc}(B(\overline{x},R;3R))}\right) \\ &\leq c(n, G, \Lambda) \cdot m^{-2n} \left(1 + [A]_{LMO_{loc}(\Omega)}\right)\end{aligned}$$

where $A = \{a_{ij}\}_{i,j=1}^q$. Recalling also the bound (3.10) on the number g_m, we get

$$\begin{aligned}&[K_{ij}f]_{BMO_{loc}(B(\overline{x},R,3R))} \leq C \sum_{m=1}^{\infty} c(N) m^{N-2} \cdot c(n, G, \Lambda) m^{-2n} \left(1 + [A]_{LMO_{loc}(\Omega)}\right) \cdot \\ &\cdot c(p, G) \cdot m^{\frac{N}{2}} \left([f]_{BMO_{loc}(B(\overline{x},R;K'R))} + \frac{\|f\|_{L^p(B(\overline{x},R))}}{|B(\overline{x},R)|^{1/p}}\right) \\ &= c(G) \left(1 + [A]_{LMO_{loc}(\Omega)}\right) \left([f]_{BMO_{loc}(B(\overline{x},R;K'R))} + \frac{\|f\|_{L^p(B(\overline{x},R))}}{|B(\overline{x},R)|^{1/p}}\right) \cdot \\ &\cdot \sum_{m=1}^{\infty} c(n, G, \Lambda) m^{N+\frac{N}{2}-2-2n} \\ &= c(p, G, \Lambda) \left(1 + [A]_{LMO_{loc}(\Omega)}\right) \left([f]_{BMO_{loc}(B(\overline{x},R;K'R))} + \frac{\|f\|_{L^p(B(\overline{x},R))}}{|B(\overline{x},R)|^{1/p}}\right)\end{aligned}$$

where we have finally fixed n large enough to make the series converge. So the theorem is proved. ■

Proof of Theorem 4.2. The proof is similar to the previous one. We have:

$$\begin{aligned}&[C[K_{ij}, b] f]_{BMO_{loc}(B(\overline{x},R;3R))} \leq \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} \left[c_{ij}^{km}\, C[T_{km}, b] f\right]_{BMO_{loc}(B(\overline{x},R;3R))} \\ &\leq C \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} \left(\left\|c_{ij}^{km}\right\|_{L^\infty(B(\overline{x},3R))} + \left[c_{ij}^{km}\right]_{LMO_{loc}(B(\overline{x},R;3R))}\right) \cdot \\ &\cdot \left([C[T_{km}, b] f]_{BMO_{loc}(B(\overline{x},R;KR))} + \frac{\|C[T_{km}, b] f\|_{L^p(B(\overline{x},KR))}}{|B(\overline{x},R)|^{1/p}}\right).\end{aligned}$$

Next, by Theorem 4.11 and Remark 4.12,

$$[C\left[T_{km},b\right]f]_{BMO_{loc}(B(\overline{x},R;KR))}\leq c(p,G)\cdot m^{\frac{N}{2}}[b]_{LMO_{loc}B(\overline{x},HR,K'R)}\cdot$$
$$\cdot\left([f]_{BMO_{loc}(B(\overline{x},R;KR))}+\frac{1}{\left|B\left(\overline{x},R\right)\right|^{1/p}}\|f\|_{L^{p}(B(\overline{x},R))}\right)$$

while, by the commutator theorem on L^p (see (4.8)),

$$\|C\left[T_{km},b\right]f\|_{L^{p}(B(\overline{x},KR))}\leq c(p,G)\cdot m^{\frac{N}{2}}[b]_{LMO_{loc}B(\overline{x},HR,K'R)}\|f\|_{L^{p}(B(\overline{x},R))}$$

Reasoning like in the previous proof we conclude

$$[C\left[K_{ij},b\right]f]_{BMO_{loc}(B(\overline{x},R;3R))}\leq c(p,G,\Lambda)\left(1+[A]_{LMO_{loc}(\Omega)}\right)[b]_{LMO_{loc}(B(\overline{x},HR;K'R))}\cdot$$
$$\cdot\left([f]_{BMO_{loc}(B(\overline{x},R;KR))}+\frac{1}{\left|B\left(\overline{x},R\right)\right|^{1/p}}\|f\|_{L^{p}(B(\overline{x},R))}\right),$$

which by Lemma 2.7 gives the result. ∎

5 Local BMO^p estimates for second order derivatives

In this section we will prove our main result. As already explained in the introduction, the proof will proceed in three steps corresponding to the three subsections.

5.1 Local estimates for functions with small compact support

Theorem 5.1 *For every $p\in(1,\infty)$ there exist positive constants C,R_0, depending on the group G, the numbers p and Λ, and the $VLMO_{loc}$ moduli of the coefficients a_{hk}, and an absolute constant K', such that for every $\overline{x}\in\Omega$ with $R\leq R_0$ $B\left(\overline{x},K'R\right)\subset\Omega$ and every $u\in S_{loc}^{2,p,*}\left(B\left(\overline{x},R;3R\right)\right)$ with sprtu $\subset B\left(\overline{x},R\right)$ we have, for $i,j=1,2,...,q$,*

$$\|X_iX_j\,u\|_{BMO^p_{loc}(B(\overline{x},R;3R))}\leq C\left(1+[A]_{LMO_{loc}(\Omega)}\right)\cdot$$
$$\cdot\left\{\|Lu\|_{BMO^p_{loc}(B(\overline{x},R;3R))}+\frac{\|Lu\|_{L^p(B(\overline{x},R))}}{|B(\overline{x},R)|^{1/p}}\right\}.$$

Proof. Let us start from (3.7) for $u\in S_{loc}^{2,p,*}\left(B\left(\overline{x},R;3R\right)\right)$ and take $BMO_{loc}\left(B\left(\overline{x},R;3R\right)\right)$ seminorms of both sides:

$$[X_iX_j\,u]_{BMO_{loc}(B(\overline{x},R;3R))}\leq\left[K_{ij}\left(Lu\right)\right]_{BMO_{loc}(B(\overline{x},R;3R))}+$$
$$+\sum_{h,k=1}^{q}\left[C\left[K_{ij},a_{hk}\right]\left(X_hX_k\,u\right)\right]_{BMO_{loc}(B(\overline{x},R;3R))}+\left[\alpha_{ij}\cdot Lu\right]_{BMO_{loc}(B(\overline{x},R;3R))}$$

by Theorems 4.1, 4.2, 4.3, Lemma 2.7, and the bounds (3.2), (3.14) on the α_{ij}'s,

$$\leq C\left(1+[A]_{LMO_{loc}(\Omega)}\right)\left\{[Lu]_{BMO_{loc}(B(\overline{x},R;3R))}+\frac{\|Lu\|_{L^p(B(\overline{x},R))}}{|B(\overline{x},R)|^{1/p}}\right.$$

$$+\sum_{h,k=1}^{q}[a_{hk}]_{LMO_{loc}(B(\overline{x},HR;KR))}\left([X_hX_k\,u]_{BMO_{loc}(B(\overline{x},R;3R))}\right.$$

$$\left.\left.+\frac{1}{|B(\overline{x},R)|^{1/p}}\|X_hX_k\,u\|_{L^p(B(\overline{x},R))}\right)\right\}.$$

We now exploit the assumption $a_{hk}\in VLMO_{loc}(\Omega)$: for any $\varepsilon>0$, by (2.3) there exists R_0 such that for any $R\leq R_0$

$$\sum_{h,k=1}^{q}[a_{hk}]_{LMO_{loc}(B(\overline{x},HR;KR))}\leq\varepsilon.$$

Choosing ε such that

$$C\left(1+[A]_{LMO_{loc}(\Omega)}\right)\varepsilon\leq\frac{1}{2}$$

we get, for $R\leq R_0$, R_0 depending on the $VLMO_{loc}$ moduli of the a_{hk}'s,

$$\sum_{i,j=1}^{q}[X_iX_j\,u]_{BMO_{loc}(B(\overline{x},R;3R))}\leq C\left(1+[A]_{LMO_{loc}(\Omega)}\right)\cdot$$

$$\cdot\left\{[Lu]_{BMO_{loc}(B(\overline{x},R;3R))}+\frac{1}{|B(\overline{x},R)|^{1/p}}\left[\|Lu\|_{L^p(B(\overline{x},R))}+\sum_{i,j=1}^{q}\|X_iX_j\,u\|_{L^p(B(\overline{x},R))}\right]\right\}.$$

Also exploiting the known L^p estimates on $X_hX_k\,u$ (see [6]) we conclude:

$$\sum_{i,j=1}^{q}\|X_iX_j\,u\|_{BMO^p_{loc}(B(\overline{x},R;3R))}\leq C\left(1+[A]_{LMO_{loc}(\Omega)}\right)\cdot$$

$$\cdot\left\{\|Lu\|_{BMO^p_{loc}(B(\overline{x},R;3R))}+\frac{\|Lu\|_{L^p(B(\overline{x},R))}}{|B(\overline{x},R)|^{1/p}}\right\}.$$

■

5.2 Local estimates for functions with noncompact support

To remove from our basic estimate in Theorem 5.1 the assumption of compact support of the function u we need to use, as usual, cutoff functions and interpolation inequalities to handle the norms of first order derivatives of u. However, it turns out that the local norms of type $BMO^p_{loc}(B(\overline{x},R;3R))$ that have been useful to prove Theorem 5.1 are not adequate to establish interpolation inequalities. Instead, we have to use to this aim the more standard $BMO^p(B(\overline{x},R))$ norms, which also allows us to apply directly some results proved in [8]. We have the following:

Theorem 5.2 *For every $p\in(1,\infty)$ there exist positive constant C,R_0, depending on the group G, the numbers p and Λ, and the $VLMO_{loc}$ moduli of the coefficients a_{hk}, such that for every $\overline{x}\in\Omega$ with $B(\overline{x},R_0)\subset\Omega$, every t,R' with $R_0/2\le t<R'\le R_0$, every $u\in S^{2,p,*}(B(\overline{x},R'))$*

$$\|u\|_{S^{2,p,*}(B(\overline{x},t))}\le C\left(\frac{1}{(R'-t)^{\gamma'}}+1\right)\left(\|u\|_{BMO^p(B(\overline{x},R'))}+\|Lu\|_{BMO^p(B(\overline{x},R'))}\right).$$

Proof. Fix four numbers $R'>R>s>t>0$ and a cutoff function $\phi\in C_0^\infty(B(\overline{x},R'))$ such that

$$B(\overline{x},t)\prec\phi\prec B(\overline{x},s).$$

Then for any function $u\in S^{2,p,*}(B(\overline{x},R'))$ we have, by [8, Lemma 4.4, (ii)]:

$$\|X_iX_ju\|_{BMO^p(B(\overline{x},t))}=\|X_iX_j(u\phi)\|_{BMO^p(B(\overline{x},t))}\le c\|X_iX_j(u\phi)\|_{BMO^p(B(\overline{x},s))}$$

by Proposition 2.8 (a) and applying Theorem 5.1 to ϕu on $B(\overline{x},s;3s)$:

$$\le c\|X_iX_j(u\phi)\|_{BMO^p_{loc}(B(\overline{x},s;3s))}\le C_A\left\{\|L(u\phi)\|_{BMO^p_{loc}(B(\overline{x},s;3s))}+\frac{\|L(u\phi)\|_{L^p(B(\overline{x},s))}}{s^{Q/p}}\right\}$$

by (2.4)

$$\le C_A\left\{[L(u\phi)]_{BMO(B(\overline{x},R))}+\left(\frac{1}{(R-s)^{Q/p}}+\frac{1}{s^{Q/p}}+1\right)\|L(u\phi)\|_{L^p(B(\overline{x},s))}\right\}.$$

Now, by our choice of ϕ and [8, Lemma 4.12]

$$[L(u\phi)]_{BMO(B(\overline{x},R))}$$
$$\le\frac{c}{s-t}[Lu]_{BMO(B(\overline{x},R))}+\frac{c}{(s-t)^2}[X_ju]_{BMO(B(\overline{x},R))}+\frac{c}{(s-t)^3}[u]_{BMO(B(\overline{x},R))}$$

Next, for $\frac{R_0}{2}\le t<R$, we pick $s=(t+R)/2$, hence

$$\|X_iX_ju\|_{BMO^p(B(\overline{x},t))}\le C_A\left\{\frac{c}{R-t}[Lu]_{BMO(B(\overline{x},R))}+\frac{c}{(R-t)^2}[X_ju]_{BMO(B(\overline{x},R))}\right.$$
$$\left.+\frac{c}{(R-t)^3}[u]_{BMO(B(\overline{x},R))}+\left(\frac{1}{(R-t)^{Q/p}}+\frac{1}{R_0^{Q/p}}+1\right)\|L(u\phi)\|_{L^p(B(\overline{x},R))}\right\}$$

so that, adding to both sides $\|X_j u\|_{BMO^p(B(\overline{x},t))} + \|u\|_{BMO^p(B(\overline{x},t))}$ and exploiting the estimates on $\|u\|_{S^{2,p}(B(\overline{x},R))}$ which are known by [6] we can write

$$\|u\|_{S^{2,p,*}(B(\overline{x},t))} \leq C \left\{ \frac{1}{R-t} [Lu]_{BMO(B(\overline{x},R))} + \frac{1}{(R-t)^2} \|X_j u\|_{BMO^p(B(\overline{x},R))} \right.$$
$$\left. + \frac{1}{(R-t)^3} \|u\|_{BMO^p(B(\overline{x},R))} + \left(\frac{1}{(R-t)^{2+Q/p}} + 1 \right) \left[\|Lu\|_{L^p(B(\overline{x},R))} + \|u\|_{L^p(B(\overline{x},R))} \right] \right\}.$$

We can then apply the interpolation inequality for BMO seminorms proved in [8, Thm 4.15] and the analogous interpolation inequality for L^p norms proved in [6, Thm. 21]: choosing $R' > R$ such that $R' - R = R - t$, for some $\alpha > 0$ and any $\delta > 0$ we have

$$\|X_j u\|_{BMO^p(B(\overline{x},R))} \leq \delta \|X_i X_j u\|_{BMO^p(B(\overline{x},R'))} + \frac{c}{\delta^\alpha (R'-R)^{2\alpha}} \|u\|_{BMO^p(B(\overline{x},R'))}.$$

Choosing $\delta = (R-t)^2 \varepsilon$, with ε to be chosen later, we get

$$\|u\|_{S^{2,p,*}(B(\overline{x},t))} \leq C \left\{ \frac{1}{R'-t} [Lu]_{BMO(B(\overline{x},R'))} + \varepsilon \|X_i X_j u\|_{BMO^p(B(\overline{x},R'))} \right.$$
$$+ \frac{c}{\varepsilon^\alpha (R'-t)^{2+4\alpha}} \|u\|_{BMO^p(B(\overline{x},R'))} + \frac{1}{(R-t)^3} \|u\|_{BMO^p(B(\overline{x},R))}$$
$$\left. + \left(\frac{1}{(R-t)^{2+Q/p}} + 1 \right) \left[\|Lu\|_{L^p(B(\overline{x},R))} + \|u\|_{L^p(B(\overline{x},R))} \right] \right\}.$$

For ε small enough we then get

$$\|u\|_{S^{2,p,*}(B(\overline{x},t))} \leq \frac{1}{3} \|u\|_{S^{2,p,*}(B(\overline{x},R'))}$$
$$+ C \left(\frac{1}{(R'-t)^\gamma} + 1 \right) \left(\|u\|_{BMO^p(B(\overline{x},R'))} + \|Lu\|_{BMO^p(B(\overline{x},R'))} \right)$$

for some $\gamma > 0$ and any $R' > t > R_0/2$. Applying [8, Lemma 4.14] we finally get

$$\|u\|_{S^{2,p,*}(B(\overline{x},t))} \leq C \left(\frac{1}{(R'-t)^\gamma} + 1 \right) \left(\|u\|_{BMO^p(B(\overline{x},R'))} + \|Lu\|_{BMO^p(B(\overline{x},R'))} \right).$$

■

5.3 Interior estimates in a domain

Theorem 5.2, by the same techniques in [8, proof of Thm 4.8] immediately gives the following:

Theorem 5.3 *For any $\Omega' \Subset \Omega$, with Ω, Ω' regular domains (see below), every $p \in (1, \infty)$ there exists a positive constant C depending on Ω, Ω', the group G,* *the numbers p and Λ, the $VLMO_{loc}$ moduli of the coefficients a_{hk}, such that for every $u \in S^{2,p,*}(\Omega)$*

$$\|u\|_{S^{2,p,*}(\Omega')} \leq C\left(\|u\|_{BMO^p(\Omega)} + \|Lu\|_{BMO^p(\Omega)}\right).$$

We recall that in [8] a domain Ω is called *regular* if it satisfies the property

$$|B(x,r) \cap \Omega| \geq c\,|B(x,r)| \tag{5.1}$$

for any $x \in \Omega, 0 < r <$diamΩ. It is proved in [8, Lemma 4.2] that, in particular, a metric ball is regular.

We are also interested in deriving from Theorem 5.2 a version of the above estimates involving local BMO norms, which is our main result, stated in § 2.3, and *does not* require assumption (5.1). Namely, we now come to the

Proof of Theorem 2.10. For fixed $\Omega_1 \Subset \Omega_2 \Subset \Omega$, let $R_0 > R_2 > 0$ be two numbers such that for any $\overline{x} \in \Omega_1$ $B(\overline{x}, R_2) \subset \Omega_2$ and $B(\overline{x}, R_0) \subset \Omega$.

Pick $t < \min\left(\frac{R_0}{6}, \frac{R_2}{2}\right)$, then by Theorem 5.2 and Proposition 2.8 (a) we have

$$\begin{aligned}\|u\|_{S^{2,p,*}(B(\overline{x},t))} &\leq C\left(\|u\|_{BMO^p(B(\overline{x},2t))} + \|Lu\|_{BMO^p(B(\overline{x},2t))}\right)\\ &\leq C\left(\|u\|_{BMO^p_{loc}(B(\overline{x},2t,6t))} + \|Lu\|_{BMO^p_{loc}(B(\overline{x},2t,6t))}\right)\\ &\leq C\left(\|u\|_{BMO^p_{loc}(\Omega_2,\Omega)} + \|Lu\|_{BMO^p_{loc}(\Omega_2,\Omega)}\right).\end{aligned} \tag{5.2}$$

Recall that the constant C in the above estimate depends on the domains but not on $\overline{x}$. Clearly, the norm $\|u\|_{S^{2,p}(\Omega_2)}$ is bounded by a finite sum of N terms of the kind $\|u\|_{S^{2,p}(B(\overline{x}_i,t))}$, and then is bounded by N times the right hand side of (5.2). Next, to bound the terms

$$[X_iX_ju]_{BMO_{loc}(\Omega_1,\Omega_2)} + [X_ju]_{BMO_{loc}(\Omega_1,\Omega_2)} + [u]_{BMO_{loc}(\Omega_1,\Omega_2)},$$

take a ball $B(\overline{x}, r)$ centered at some $\overline{x} \in \Omega_1$ and contained in Ω_2. If $r \leq t$ (with t as in (5.2)) then

$$\begin{aligned}&\frac{1}{|B_r(\overline{x})|}\int_{B_r(\overline{x})} \left|X_iX_ju(y) - X_iX_ju_{B_r(\overline{x})}\right| dy \leq \|u\|_{S^{2,p,*}(B(\overline{x},t))}\\ &\leq C\left(\|u\|_{BMO^p_{loc}(\Omega_2,\Omega)} + \|Lu\|_{BMO^p_{loc}(\Omega_2,\Omega)}\right).\end{aligned}$$

If $r > t$ (but $B(\overline{x}, r) \subset \Omega_2$) then, exploiting the local $S^{2,p}$ estimates of [6],

$$\begin{aligned}&\frac{1}{|B_r(\overline{x})|}\int_{B_r(\overline{x})} \left|X_iX_ju(y) - (X_iX_ju)_{B_r(\overline{x})}\right| dy \leq 2\frac{1}{|B_r(\overline{x})|}\int_{B_r(\overline{x})} |X_iX_ju(y)|\,dy\\ &\leq 2\left(\frac{1}{|B_r(\overline{x})|}\int_{B_r(\overline{x})} |X_iX_ju(y)|^p\,dy\right)^{1/p} \leq \frac{c}{t^{Q/p}}\|X_iX_ju\|_{L^p(\Omega_2)}\\ &\leq \frac{c}{t^{Q/p}}\left(\|u\|_{L^p(\Omega)} + \|Lu\|_{L^p(\Omega)}\right) \leq \frac{c}{t^{Q/p}}\left(\|u\|_{BMO^p_{loc}(\Omega_2,\Omega)} + \|Lu\|_{BMO^p_{loc}(\Omega_2,\Omega)}\right).\end{aligned}$$

In any case

$$[X_iX_ju]_{BMO_{loc}(\Omega_1,\Omega_2)} \le C\left(\|u\|_{BMO^p_{loc}(\Omega_2,\Omega)} + \|Lu\|_{BMO^p_{loc}(\Omega_2,\Omega)}\right)$$

Analogously we can bound $[X_ju]_{BMO_{loc}(\Omega_1,\Omega_2)} + [u]_{BMO_{loc}(\Omega_1,\Omega_2)}$. Exploiting again the local $S^{2,p}$ estimates we conclude

$$\|u\|_{S^{2,p,*}_{loc}(\Omega_1,\Omega_2)} \le C\left(\|u\|_{BMO^p_{loc}(\Omega_2,\Omega)} + \|Lu\|_{BMO^p_{loc}(\Omega_2,\Omega)}\right)$$

and we are done. ■

6 Appendix. Uniform bounds on the oscillation of the fundamental solution

The aim of this section is to prove Theorem 3.3, that we have exploited in § 3 to prove LMO uniform bounds on the coefficients c^{km}_{ij} which appear in the spherical harmonics expansion of Γ_{ij}. Let us recall its statement:

Theorem 3.4 *For any nonnegative integer* p*, there exists a constant* $c_{\Lambda,p}$ *such that for any* $x_1, x_2, y \in R^N$ *we have:*

$$\left|X_{i_1}X_{i_2}...X_{i_p}\Gamma(x_1,y) - X_{i_1}X_{i_2}...X_{i_p}\Gamma(x_2,y)\right| \le c_{\Lambda,p}\|A(x_1) - A(x_2)\|\,\|y\|^{2-Q-p} \tag{6.1}$$

where the differential operators X_{i_j} *act on the* y*-variable.*

Let us recall again that $\|y\|$ stands for the homogeneous norm in G while $\|A(x_1) - A(x_2)\|$ stands for the usual matrix norm in $\mathbb{R}^{2q}$. Actually, the proof of this theorem amounts to revising some results and techniques contained in [2], [3] and [4]. Namely, the following fact is proved in [4, Corollary 7.13]:

Theorem 6.1 *For any nonnegative integer* p*, there exists a positive constant* $c_{\Lambda,p}$ *such that*

$$\begin{aligned}&\left|X_{i_1}X_{i_2}...X_{i_p}\Gamma(x_1,y) - X_{i_1}X_{i_2}...X_{i_p}\Gamma(x_2,y)\right| \\ &\le c_{\Lambda,p}\|A(x_1) - A(x_2)\|^{1/s}\|y\|^{2-Q-p}\end{aligned} \tag{6.2}$$

for every $i_1, ..., i_p \in \{1, 2, ..., q\}$, *and for every* $y \in \mathbb{R}^N \setminus \{0\}$, *where the differential operators* X_{i_j} *act on the* y*-variable.*

Recall that s is the maximum lenght of commutators required to span $\mathbb{R}^N$.

Comparing (6.2) with (6.1), one sees that what we need is to replace the exponent $1/s$ of the matrix norm on the right hand side of (6.2) with the exponent 1.

In order to prove Theorem 3.3, we will now revise the proof of Theorem 6.1 given in [4], which proceeds in four steps:

Step 1. The Authors assume G to a be a free Carnot group. Under this assumption, they consider the *evolution operator*

$$H_A = \partial_t - \sum a_{ij}X_iX_j$$

where $A = \{a_{ij}\}$ is a symmetric positive, constant, matrix, in a fixed "ellipticity class" $\mathcal{M}_\Lambda$:

$$\Lambda|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}\xi_i\xi_j \leq \Lambda^{-1}|\xi|^2 \ \forall \xi \in \mathbb{R}^q.$$

For this operator H_A they consider the fundamental solution ("heat kernel") $h_A(x,t)$ and prove the following estimate (see [3, Theorem 7.5]):

$$|h_{A_1}(x,t) - h_{A_2}(x,t)| \leq c_\lambda \|A_1 - A_2\|^{1/s} t^{-Q/2} \exp\left(-\frac{\|x\|^2}{c_\Lambda t}\right) \tag{6.3}$$

for $x \in \mathbb{R}^N, t > 0$, any $A_1, A_2 \in \mathcal{M}_\Lambda$.

Step 2. Under the same assumptions, the Authors extend the previous bound to the derivatives of h_A, proving (see [3, Theorem 7.7]): for any nonnegative integers p, m, there exist positive constants c_λ and $c_{\lambda,p,m}$ such that

$$\begin{aligned} &\left|X_{i_1}...X_{i_p}(\partial_t)^m h_{A_1}(x,t) - X_{i_1}...X_{i_p}(\partial_t)^m h_{A_2}(x,t)\right| \leq \\ &\leq c_{\Lambda,p,m}\|A_1 - A_2\|^{1/s} t^{-(Q+p+2m)/2} \exp\left(-\frac{\|x\|^2}{c_\Lambda t}\right) \end{aligned} \tag{6.4}$$

for $x \in \mathbb{R}^N, t > 0$, any $i_1, i_2, ..., i_p \in \{1, 2, ..., q\}$, and any $A_1, A_2 \in \mathcal{M}_\Lambda$.

Step 3. The assumption that G is free in now removed, by a suitable "lifting result" proved in [2] (see also [3, Theorem 8.3]), so that (6.4) is established for *any* Carnot group.

Step 4. The Authors prove (see [3, Theorem 3.9]) that the function

$$\Gamma_A(x) = \int_0^{+\infty} h_A(x,t)\, dt$$

is the fundamental solution to the stationary operator

$$L_A = -\sum a_{ij} X_i X_j.$$

Then, integrating in the t variable the estimate (6.4) they get

$$\left|X_{i_1}...X_{i_p}\Gamma_{A_1}(x,t) - X_{i_1}...X_{i_p}\Gamma_{A_2}(x,t)\right| \leq c_{\Lambda,p,q}\|A_1 - A_2\|^{1/s}\|y\|^{2-Q-p}$$

which is essentially (6.2).

We are going to show that the following refinement of (6.3) can be proved:

Proposition 6.2 *Under the same assumptions and with the same notation of Step 1, we have*

$$|h_{A_1}(x,t) - h_{A_2}(x,t)| \leq c_\Lambda \|A_1 - A_2\| \cdot t^{-Q/2} \exp\left(-\frac{\|x\|^2}{c_\Lambda t}\right)$$

for any $x \in G, t > 0$.

(The improvement consists in the exponent 1 instead of $1/s$, for $\|A_1 - A_2\|$). Before proving this proposition, we collect in the following theorem a number of results proved in [2], that we will need.

Theorem 6.3 *Let G be a free homogeneous Carnot group and let $A \in \mathcal{M}_\Lambda$. Then there exists a Lie group automorphism T_A of G, commuting with the dilations of G, such that*

$$h_A(x,t;\xi,\tau) = J_A(x) \cdot h_G(T_{A_1}(x), t; T_{A_1}(\xi), \tau) \tag{6.5}$$

where h_G is the heat kernel for $\partial_t - \sum X_i^2$ on G , which is actually a convolution kernel:

$$h_G(T_{A_1}(x), t; T_{A_1}(\xi), \tau) = h_G\left(T_{A_1}(\xi)^{-1} \circ T_{A_1}(x), t-\tau\right) \tag{6.6}$$

and has the following homogeneity

$$h_G\left(D(\lambda)x, \lambda^2 t\right) = t^{-Q} h_G(x,t) \text{ for any } \lambda > 0, \tag{6.7}$$

while $J_A(x) = |\det \mathcal{J}_{T_A}(x)|$, where $\mathcal{J}_{T_A}$ is the Jacobian of T_A. Moreover $J_A(x)$ turns out to be constant in x, and

$$(c_\Lambda)^{-1} \leq J_A \leq c_\Lambda \tag{6.8}$$

$$|J_{A_1} - J_{A_2}| \leq c_\Lambda \|A_1 - A_2\| \tag{6.9}$$

$$(c_\Lambda)^{-1} \|x\| \leq \|T_A(x)\| \leq c_\Lambda \|x\| \tag{6.10}$$

$$|T_{A_1}(x) - T_{A_2}(x)| \leq c_\Lambda \|A_1 - A_2\|, \tag{6.11}$$

for any $x \in G$, any $A, A_1, A_2 \in \mathcal{M}_\Lambda$.

Relations (6.5), (6.8), (6.9), (6.10) are (1.5), (2.19), (2.20), (2.21) in [2], respectively (note that our symbol h corresponds to Γ in [2]); for (6.11), see the last line in [2]; (6.6) and (6.7) are known properties of the heat kernel on Carnot groups, see also [3].

Proof of Proposition 6.2. Here we somewhat revise the proof of [3, Theorem 7.5].

$$\begin{aligned} &|h_{A_1}(x,t) - h_{A_2}(x,t)| = |J_{A_1} h_G(T_{A_1}(x), t) - J_{A_2} h_G(T_{A_2}(x), t)| \leq \\ &\leq |J_{A_1} - J_{A_2}| \, |h_G(T_{A_1}(x), t)| + J_{A_2} |h_G(T_{A_1}(x), t) - h_G(T_{A_2}(x), t)| \equiv I + II. \end{aligned} \tag{6.12}$$

By (6.9), the Gaussian estimate for h_G and (6.10):

$$I \leq c_\Lambda \|A_1 - A_2\| \, t^{-Q/2} \exp\left(-\frac{\|T_{A_1}(x)\|}{ct}\right) \leq c_\Lambda \|A_1 - A_2\| \, t^{-Q/2} \exp\left(-\frac{\|x\|}{ct}\right), \tag{6.13}$$

while by (6.8)

$$II \leq c_\Lambda |h_G(T_{A_1}(x), t) - h_G(T_{A_2}(x), t)|. \tag{6.14}$$

We now need the following

Claim 6.4 *There exists $k>0$ such that if $\|x\|=1$, then*

$$|h_G(T_{A_1}(x),t)-h_G(T_{A_2}(x),t)|\le c_\Lambda\|A_1-A_2\|\,t^{-Q/2-k}\exp\left(-\frac{1}{ct}\right). \tag{6.15}$$

Let us first show how we can conclude the proof using the claim, then we will prove (6.15).

For any $x\in G$, let $x=D_{\|x\|}(x')$ with $\|x'\|=1$, and let us apply (6.15), keeping in mind (6.7) and the fact that T_A commutes with dilations:

$$T_{A_i}(x)=T_{A_i}\left(D_{\|x\|}(x')\right)=D_{\|x\|}\left(T_{A_i}(x')\right)\text{ for } i=1,2, \text{ hence}$$

$$\begin{aligned}
&|h_G(T_{A_1}(x),t)-h_G(T_{A_2}(x),t)|=\\
&=\left|h_G\left(D_{\|x\|}(T_{A_1}(x')),\|x\|^2\frac{t}{\|x\|^2}\right)-h_G\left(D_{\|x\|}(T_{A_2}(x')),\|x\|^2\frac{t}{\|x\|^2}\right)\right|=\\
&=\|x\|^{-Q}\left|h_G\left(T_{A_1}(x'),\frac{t}{\|x\|^2}\right)-h_G\left(T_{A_2}(x'),\frac{t}{\|x\|^2}\right)\right|\le\\
&\le\|x\|^{-Q}c_\Lambda\|A_1-A_2\|\left(\frac{t}{\|x\|^2}\right)^{-Q/2-k}\exp\left(-\frac{\|x\|^2}{ct}\right)=\\
&=c_\Lambda\|A_1-A_2\|\,t^{-Q/2}\left(\frac{\|x\|^2}{t}\right)^{k}\exp\left(-\frac{\|x\|^2}{ct}\right)\le\\
&\le c_\Lambda\|A_1-A_2\|\,t^{-Q/2}\exp\left(-\frac{\|x\|^2}{ct}\right),
\end{aligned}$$

possibly changing the constant c inside the exp. By (6.12), (6.13), (6.14) this implies the result.

Let us now prove the Claim. We will bound the left hand side of (6.15) applying Lagrange theorem (in the standard form, instead of the Lagrange theorem for vector fields which is applied in the proof of [3, Theorem 7.5]).

$$|h_G(T_{A_1}(x),t)-h_G(T_{A_2}(x),t)|\le|T_{A_1}(x)-T_{A_2}(x)|\sup_{y\in\left[T_{A_1}(x),T_{A_2}(x)\right]}|\nabla_y h_G(y,t)|. \tag{6.16}$$

Recalling (6.10), we now note that for some constants $\delta_0,c_0\in(0,1)$ we can say that

$$\text{if } \|x\|=1, y\in[T_{A_1}(x),T_{A_2}(x)] \text{ and } |T_{A_1}(x)-T_{A_2}(x)|\le\delta_0, \text{ then } \|y\|\ge c_0.$$

We then distinguish two cases.

Case 1. $|T_{A_1}(x) - T_{A_2}(x)| \leq \delta_0$. Then we proceed from (6.16), expressing Euclidean derivatives of h_G in terms of the vector fields X_i's and their commutators, and exploiting Gaussian bounds for h_G proved in [3, Theorem 5.3]

$$\sup_{y\in\left[T_{A_1}(x),T_{A_2}(x)\right]} |\nabla_y h_G(y,t)| \leq \tag{6.17}$$

$$\leq ct^{-Q/2-k} \cdot \sup_{y\in\left[T_{A_1}(x),T_{A_2}(x)\right]} \exp\left(-\frac{\|y\|}{c_\Lambda t}\right) \leq ct^{-Q/2-k}\exp\left(-\frac{1}{c_\Lambda t}\right).$$

By (6.16), (6.17) and (6.11) we get (6.15).

Case 2. $|T_{A_1}(x) - T_{A_2}(x)| > \delta_0$. We then apply (6.3) (that is the result already proved in [3]):

$$|h_G(T_{A_1}(x),t) - h_G(T_{A_2}(x),t)| \leq c_\Lambda \|A_1 - A_2\|^{1/s} t^{-Q/2}\exp\left(-\frac{1}{ct}\right) \leq$$

$$\leq \frac{c}{\delta_0}|T_{A_1}(x) - T_{A_2}(x)| \|A_1 - A_2\|^{1/s} t^{-Q/2}\exp\left(-\frac{1}{ct}\right) \leq$$

by (6.11)

$$\leq c\|A_1 - A_2\|^{1+1/s} t^{-Q/2}\exp\left(-\frac{1}{ct}\right) \leq c\|A_1 - A_2\| t^{-Q/2}\exp\left(-\frac{1}{ct}\right)$$

since the matrix norm $\|A_1 - A_2\|$ is always bounded in $\mathcal{M}_\Lambda$. Finally, the last expression can be bound by

$$c\|A_1 - A_2\| t^{-Q/2-k}\exp\left(-\frac{1}{ct}\right),$$

possibly changing the exponent inside the exp.

So the Claim is proved and the proof of the Proposition is complete. ■

Starting from the previous Proposition one can now proceed following word by word the arguments of Steps 2, 3 and 4 in [3], concluding the proof of Theorem 3.3.

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