

Commutators of Singular Integrals on Homogeneous Spaces.

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Sunto. – Sia (X, d, μ) uno spazio omogeneo, K un operatore di integrale singolare, continuo da $\mathcal{L}^p(X)$ a $\mathcal{L}^p(X)$ per ogni $p \in (1, \infty)$ e tale che il suo nucleo soddisfi opportune disuguaglianze puntuali. Sia $a \in BMO(X)$; si dimostra che il commutatore

$$C[K, a](f) = aKf - K(af)$$

è continuo su \mathcal{L}^p per ogni $p \in (1, \infty)$ e soddisfa la seguente stima:

$$\|C[K, a](f)\|_p \leq c(p, K, X) \|a\|_* \|f\|_p$$

dove $\|\cdot\|_*$ è la seminorma BMO. Si mostra anche come applicare questo risultato ad alcuni esempi concreti di operatori di integrale singolare su spazi omogenei di misura finita o infinita, fornendo condizioni sul nucleo di K , sufficienti affinché l'operatore di integrale singolare K sia ben definito e continuo su \mathcal{L}^p .

Introduction.

A well-known result of 1976 by Coifman-Rochberg-Weiss (see [CRW]) is that, if K is a Calderón-Zygmund operator on $\mathcal{L}^p(\mathbb{R}^n)$ and $a \in BMO$, then the commutator

$$C[K, a](f) = aKf - K(af)$$

is a well defined linear continuous operator from \mathcal{L}^p into \mathcal{L}^p , for every $p \in (1, \infty)$ and

$$(0.1) \quad \|C[K, a](f)\|_p \leq c(p, n, K) \|a\|_* \|f\|_p.$$

Over the years several extensions of this result have been proved. For instance, Chanillo [Cha] in '82 proved an analogous result for fractional integral operators; Bloom [Bl] in '85 proved a weighted estimate for the commutator of the conjugate operator; Segovia and

Torrea [ST1] in '89 generalized the above result to vector valued functions and more recently (see [ST2]) they considered higher order commutators. In this paper we consider singular integral operators of Calderón-Zygmund type, defined on a homogeneous space, and prove the Coifman-Rochberg-Weiss' theorem in this context. In another one [BC2] we deal with commutators of *fractional* integrals on homogeneous spaces.

The motivation for this paper arose when, proving \mathcal{L}^p estimates for parabolic equations with *VMO* coefficients, we needed to extend [CRW]'s result to parabolic Calderón-Zygmund operators (see [BC1]). That of homogeneous spaces seems a natural setting for this problem since the basic real analysis tools, as maximal function, sharp function and *BMO*, can be defined in a natural way in this context; also, the related results, analogous to the Euclidean case, have been proved (see [CW1], [Bu]). Moreover, the present extension of (0.1) to homogeneous spaces turns out to be the suitable tool to prove \mathcal{L}^p -estimates for some ultraparabolic operators with *VMO* coefficients. This problem is considered by Bramanti-Cerutti-Manfredini in [BCM]. See also Remark 4.13 for other notes about applications of our results to problems arising from P.D.E.'s.

In § 1-2 we discuss definitions of homogeneous space and of Calderón-Zygmund type operator, as well as recall known results. In § 2-3 we prove our main results, namely the \mathcal{L}^p estimate (0.1) for the commutator, respectively in the case of an infinite measure space and in a finite measure one. It turns out that, while the infinite measure case is fairly straightforward, the finite measure one is significantly different. We want to stress that our definition of Calderón-Zygmund operator requires \mathcal{L}^p continuity: in other words, our results say that, under suitable assumptions on the kernel and the space, the \mathcal{L}^p estimate on the operator K implies (0.1) for its commutator. So, if we are given a concrete singular kernel on a homogeneous space, to apply this theory we need to know that an \mathcal{L}^p continuous operator is actually defined. This problem is discussed in § 4. Namely we first discuss some general sufficient conditions on the kernel under which \mathcal{L}^p -continuity of the singular integral can be assured, and then give specific examples arising in the applications in which our theorem (both in the infinite and finite measure case) can be applied.

Finally, we wish to point out that our results are obtained under quite general assumptions on the space X , namely X does

not need to be endowed with a group structure neither the measure of a ball in X needs to behave like a fixed power of the radius.

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1. – Some basic facts about homogeneous spaces.

Several different definitions of homogeneous spaces can be found in the literature. Here we refer particularly to [CW1,2], [MS], [Bu].

DEFINITION 1.1. – Let X be a set. A function $d: X \times X \rightarrow [0, \infty)$ is called *quasidistance* if:

- i) $d(x, y) = 0 \Leftrightarrow x = y$;
- ii) $d(x, y) = d(y, x)$;
- iii) there exists a constant $c_d \geq 1$ such that for every $x, y, z \in X$

$$(1.1) \quad d(x, y) \leq c_d [d(x, z) + d(z, y)].$$

In Section 4 we will show that, for the results contained in this paper, condition ii) can be weakened to require only «quasisymmetry», instead of symmetry. (See Remark 4.6).

If (X, d) is a set endowed with a quasidistance, the *balls* $B_r(x) \equiv B(x, r) = \{y \in X: d(x, y) < r\}$ (for $x \in X$ and $r > 0$) form a base for a complete system of neighborhoods of X , so that X is a Hausdorff space. Note that the balls are *not* in general open sets; if they are, then they form a base for the topology of X .

DEFINITION 1.2. – We say that (X, d, μ) is a *homogeneous space* if:

- i) X is a set endowed with a quasidistance d , such that the balls are open sets in the topology induced by d ,
- ii) μ is a positive Borel measure on X , satisfying the *doubling condition*:

$$(1.2) \quad 0 < \mu(B_{2r}(x)) \leq c_\mu \cdot \mu(B_r(x)) < \infty$$

for every $x \in X$, $r > 0$, some constant $c_\mu > 1$.

The numbers c_d, c_μ in (1.1)-(1.2) will be called *constants of X*

and we will write $c(X)$ for a constant depending on the constants of X .

We note that every homogeneous space is separable. This follows, for instance, from Theorem (1.2) of [CW1].

A fundamental result about quasidistances is the following one:

THEOREM 1.3 (Macias-Segovia, [MS]). – *Let d be a quasidistance on a set X . Then there exists a quasidistance δ on X such that:*

(i) *d and δ are equivalent, that is there exist positive constants c_1, c_2 such that for every $x, y \in X$:*

$$c_1 \delta(x, y) \leq d(x, y) \leq c_2 \delta(x, y);$$

(ii) *δ is «locally Hölder continuous»; more precisely there exist $\alpha \in (0, 1]$ and $c > 0$ such that for every $x, y, z \in X$:*

$$|\delta(x, z) - \delta(y, z)| \leq c \delta(x, y)^\alpha [\delta(x, z) + \delta(y, z)]^{1-\alpha}.$$

If d is a distance, d is Lipschitz, and the previous result is trivial.

We say that $f \in \mathcal{C}_0^\eta$ if f has bounded support and

$$\|f\|_\eta \equiv \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\eta} < \infty.$$

If (X, d, μ) is a homogeneous space with μ a regular measure, then the space \mathcal{C}_0^η is dense in \mathcal{L}^p for every $\eta \in (0, 1]$, $p \in [1, \infty)$. (This follows from Thm. 1.3).

We will use also the following space of test functions

$$\mathcal{D} = \{f \in \mathcal{L}^\infty(X) \mid \text{the support of } f \text{ is bounded}\}.$$

Note that \mathcal{D} is dense in \mathcal{L}^p for every $p \in [1, \infty)$.

The definition of standard real analysis tools, such as the *maximal function* Mf , the *sharp function* $f^\#$, the *BMO seminorm* $\|f\|_*$ and the *BMO space*, naturally carries over to this context, namely:

$$Mf(x) = \sup_{x \in B} \int_B |f(y)| d\mu(y);$$

where the *sup* is taken over all balls containing x and $\int_B \dots = 1/\mu(B) \int_B \dots$;

$$f^\#(x) = \sup_{x \in B} \int_B |f(y) - f_B| d\mu(y)$$

where $f_B = \int_B f(x) d\mu(x)$;

$$\|f\|_* = \sup_x f^\#(x) = \sup_B \int_B |f(y) - f_B| d\mu(y);$$

$$BMO \equiv \{f \in \mathcal{L}_{loc}^1(X) : \|f\|_* < \infty\}.$$

We recall three results related to these concepts, which have been proved in the context of homogeneous spaces in [CW1], [Bu].

THEOREM 1.4 (Maximal Inequality). – *For every $p \in (1, \infty]$ there exists a constant $c(X, p)$ such that for every $f \in \mathcal{L}^p$:*

$$\|Mf\|_p \leq c \|f\|_p.$$

THEOREM 1.5 (John-Nirenberg Lemma). – *For every $p \in [1, \infty)$ there exists a constant $c(X, p)$ such that for every $f \in BMO$, every ball B :*

$$\left(\int_B |f(x) - f_B|^p d\mu(x) \right)^{1/p} \leq c \|f\|_*.$$

THEOREM 1.6 (Sharp Inequality). – *For every $p \in [1, \infty)$ there exists a constant $c(X, p)$ such that for every $f \in \mathcal{L}^p$:*

if $\mu(X) = \infty$: $\|f\|_p \leq c \|f^\#\|_p$

if $\mu(X) < \infty$: $\left\| f - \int_X f \right\|_p \leq c \|f^\#\|_p.$

Moreover, the following two lemmas about *BMO* functions hold:

LEMMA 1.7. – *Let $a \in BMO$ and $M > 1$. Then for every ball $B_r(x)$ and every positive integer j :*

$$|a_{B_M^j r} - a_{B_r}| \leq c \cdot j \|a\|_*$$

where c depends on M and on the doubling constant c_μ .

PROOF.

$$|a_{B_{Mr}} - a_{B_r}| =$$

$$\left| \int_{B_r} (a(x) - a_{B_{Mr}}) d\mu(x) \right| \leq \frac{c}{\mu(B_{Mr})_{B_{Mr}}} \int |a(x) - a_{B_{Mr}}| d\mu(x) \leq c \|a\|_*.$$

Hence:

$$\begin{aligned} |a_{B_{M^j r}} - a_{B_r}| &\leq |a_{B_r} - a_{B_{Mr}}| + |a_{B_{Mr}} - a_{B_{M^2 r}}| + \\ &\dots + |a_{B_{M^{j-1} r}} - a_{B_{M^j r}}| \leq c \cdot j \|a\|_*. \quad \blacksquare \end{aligned}$$

LEMMA 1.8. - Let $f \in BMO$, and let:

$$f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq n, \\ n & \text{if } f(x) \geq n, \\ -n & \text{if } f(x) \leq -n. \end{cases}$$

Then $f_n \in BMO$ and:

$$\|f_n\|_* \leq c \|f\|_*$$

with c an absolute constant.

Finally, we recall the following result about homogeneous spaces of finite measure. For sake of completeness we include the proof, having been unable to find a written reference.

LEMMA 1.9. - Let (X, d, μ) be a homogeneous space. Then $\mu(X) < \infty$ if and only if X is bounded.

PROOF. - We have to prove that if $\mu(X) < \infty$ then X is bounded (the reverse follows from the definition of doubling condition).

By contradiction. Let x_0 be a fixed point of X and x_k a sequence of points such that $r_k \equiv d(x_0, x_k) \uparrow \infty$. By the triangle inequality we can prove that:

$$(1.3) \quad B\left(x_0, \frac{r_k}{2c_d}\right) \cap B\left(x_k, \frac{r_k}{2c_d}\right) = \emptyset \quad \text{and} \quad B\left(x_0, \frac{r_k}{2c_d}\right) \subseteq B(x_k, Hr_k)$$

with $H = c_d(1 + 1/4c_d)$.

Then $\mu(B(x_0, r_k/2c_d)) \uparrow \mu(X)$ so that, by (1.3), $B(x_k, r_k/2c_d) \rightarrow 0$.

But then, by doubling and (1.3),

$$\mu\left(B\left(x_0, \frac{r_k}{2c_d}\right)\right) \leq \mu(B(x_k, Hr_k)) \leq c(X) \cdot \mu\left(B\left(x_k, \frac{r_k}{2c_d}\right)\right) \rightarrow 0$$

so that $\mu(X) = 0$, contradiction. ■

2. – Commutators of Calderón-Zygmund operators on homogeneous spaces.

DEFINITION 2.1. – We say that K is a Calderón-Zygmund operator (CZ operator) on the homogeneous space (X, d, μ) if:

i) $K: \mathcal{L}^p(X) \rightarrow \mathcal{L}^p(X)$ is linear continuous for every $p \in (1, \infty)$;

ii) there exists a measurable function $k: X \times X \rightarrow \mathbb{R}$ such that for every $f \in \mathcal{D}$, for a.e. $x \notin \text{sppt } f$:

$$Kf(x) = \int_X k(x, y)f(y) d\mu(y),$$

iii) the kernel k satisfies the following *pointwise Hörmander condition*: there exist constants $c_K > 0$, $\beta > 0$, $M > 1$ such that for every $x_0 \in X$, $r > 0$, $x \in B_r(x_0)$, $y \notin B_{Mr}(x_0)$,

$$(H1) \quad |k(x_0, y) - k(x, y)| \leq \frac{c_K}{\mu(B(x_0; y))} \frac{d(x_0, x)^\beta}{d(x_0, y)^\beta}$$

where, $B(x; y)$ is the ball with center x and radius $2d(x, y)$. We will write $c(p, K)$ for a constant depending on the number $p \in (1, \infty)$, the \mathcal{L}^p norm of K and the constants in (H1). The following holds:

PROPOSITION 2.2. – *Condition (H1) implies the following integral Hörmander condition: there exist constants $A > 1$, $B > 0$ such that for every $x_0, x \in X$:*

$$(H0) \quad \int_{d(x_0, y) \geq Ad(x_0, x)} |k(x_0, y) - k(x, y)| d\mu(y) \leq B.$$

(Condition (H0) is assumed in [CG], [CW1] to prove the fundamental result about \mathcal{L}^p continuity of an integral operator which is continuous on \mathcal{L}^2 . This result will be recalled and applied in § 4).

PROOF. - Fix $x_0, x \in X$ and apply (H1) with $r = 2d(x_0, x)$:

$$\int_{d(x_0, y) \geq 2Md(x_0, x)} |k(x_0, y) - k(x, y)| d\mu(y) \leq$$

$$c_K \cdot d(x_0, x)^\beta \int_{d(x_0, y) \geq 2Md(x_0, x)} \frac{d\mu(y)}{\mu(B(x_0; y)) \cdot d(x_0, y)^\beta} =$$

$$c_K \cdot d(x_0, x)^\beta \sum_{j=1}^{\infty} \int_{2^j Md(x_0, x) \leq d(x_0, y) \leq 2^{j+1} Md(x_0, x)} \frac{d\mu(y)}{\mu(B(x_0; y)) \cdot d(x_0, y)^\beta} \leq$$

$$c_K \cdot d(x_0, x)^\beta \sum_{j=1}^{\infty} \frac{1}{[2^j Md(x_0, x)]^\beta} = c_K \cdot c(M, \beta). \quad \blacksquare$$

Examples of CZ operators on homogeneous spaces will be discussed in detail in § 4. For a CZ operator K , consider the commutator

$$C[K, a](f) = aKf - K(af)$$

where $a \in BMO$. (When K is implicitly understood, we shall write more briefly: $T_a f = C[K, a](f)$). Our goal is to prove the following estimate:

$$\|T_a f\|_p \leq c(p, K, X) \|a\|_* \|f\|_p$$

for every $p \in (1, \infty)$. The above definition of $C[K, a](f)$ is, up to now, purely formal: if $f \in \mathcal{L}^p$ and $a \in BMO$, $K(af)$ may not be well defined. However, if $a \in BMO$ by the John-Nirenberg lemma (Theorem 1.5), $a \in \mathcal{L}_{loc}^p$ for every $p \in [1, \infty)$. Therefore if $f \in \mathcal{O}$ and $a \in BMO$, $T_a f$ is well defined and belongs to \mathcal{L}_{loc}^p for every $p \in [1, \infty)$; if, moreover, $a \in \mathcal{L}^\infty$, then $T_a f \in \mathcal{L}^p$ for every $p \in (1, \infty)$. The following proposition is a consequence of the above remark, the density of \mathcal{O} in \mathcal{L}^p and Lemma 1.8.

PROPOSITION 2.3. - *If (2.1) holds for every $a \in \mathcal{L}^\infty$ and every $f \in \mathcal{O}$, then for every $a \in BMO$ the operator T_a can be continuously extended to \mathcal{L}^p for every $p \in (1, \infty)$, and (2.1) holds for every $a \in BMO$, $f \in \mathcal{L}^p$.*

Let us now state the first main result.

THEOREM 2.4. - *Let K be a CZ operator; then for every $p \in (1, \infty)$*

there exists a constant $c = c(p, K, X)$ such that for every $a \in \mathcal{L}^\infty$, every $f \in \mathcal{O}$:

$$\|(T_a f)^\# \|_p \leq c \|a\|_* \|f\|_p.$$

PROOF. — We shall prove the following inequality: for every $r \in (1, \infty)$ there exist a constant $c(r, K, X)$ such that

$$(2.2) \quad |(T_a f)^\#(\bar{x})| \leq c \cdot \|a\|_* \{M(|Kf|^{1/r})(\bar{x}) + (M(|f|^r)(\bar{x}))^{1/r}\}$$

for every $\bar{x} \in X$.

From (2.2) the result follows by the continuity of K and the maximal inequality (Theorem 1.4). To prove (2.2), let us fix a ball $B_\delta = B_\delta(x_0)$ containing \bar{x} , and let x be any point of B_δ . Let us write:

$$(2.3) \quad T_a f(x) = K(af)(x) - a(x)K(f)(x) = \\ K((a - a_{B_\delta})f\chi_{B_{M\delta}})(x) + K((a - a_{B_\delta})f\chi_{X \setminus B_{M\delta}})(x) - (a(x) - a_{B_\delta}) \cdot K(f)(x) \equiv \\ A(x) + B(x) + C(x).$$

Following [To], p. 418, we get, by Theorem 1.5, Lemma 1.7 and the \mathcal{L}^p estimate on K :

$$(2.4) \quad \int_{B_\delta} |C(x) - C_{B_\delta}| d\mu(x) \leq c(X, r) \|a\|_* \cdot (M(|K(f)|^r)(\bar{x}))^{1/r};$$

$$(2.5) \quad \int_{B_\delta} |A(x) - A_{B_\delta}| d\mu(x) \leq c(r, K, X) \|a\|_* (M(|f|^r)(\bar{x}))^{1/r}.$$

To bound the middle term, observe that

$$(2.6) \quad \int_{B_\delta} |B(x) - B_{B_\delta}| d\mu(x) \leq 2 \int_{B_\delta} |B(x) - B(x_0)| d\mu(x)$$

and that, by (H1)

$$(2.7) \quad |B(x) - B(x_0)| \leq$$

$$\int_{X \setminus B_{M\delta}(x_0)} |k(x, y) - k(x_0, y)| |a(y) - a_{B_\delta}| |f(y)| d\mu(y) \leq$$

$$\begin{aligned}
& c_K \cdot d(x_0, x)^\beta \int_{X \setminus B_{M\delta}(x_0)} \frac{|a(y) - a_{B_\delta}| |f(y)|}{\mu(B(x_0; y)) \cdot d(x_0, y)^\beta} d\mu(y) \leq \\
& c_K \cdot \delta^\beta \left(\int_{X \setminus B_{M\delta}(x_0)} \frac{|a(y) - a_{B_\delta}|^{r'}}{\mu(B(x_0; y)) \cdot d(x_0, y)^\beta} d\mu(y) \right)^{1/r'} \cdot \\
& \left(\int_{X \setminus B_{M\delta}(x_0)} \frac{|f(y)|^r}{\mu(B(x_0; y)) \cdot d(x_0, y)^\beta} d\mu(y) \right)^{1/r}.
\end{aligned}$$

Moreover:

$$\begin{aligned}
(2.8) \quad & \int_{X \setminus B_{M\delta}(x_0)} \frac{|f(y)|^r}{\mu(B(x_0; y)) \cdot d(x_0, y)^\beta} d\mu(y) = \\
& \sum_{j=1}^{\infty} \int_{M^j \delta \leq d(x_0, y) < M^{j+1} \delta} \frac{|f(y)|^r}{\mu(B(x_0; y)) \cdot d(x_0, y)^\beta} d\mu(y) \leq \\
& \leq \sum_{j=1}^{\infty} \frac{1}{[M^j \delta]^\beta} \cdot \frac{1}{\mu(B(x_0, 2 \cdot M^j \delta))} \int_{B(x_0, M^{j+1} \delta)} |f(y)|^r d\mu(y) \leq \\
& \text{(by the doubling condition)} \leq c(c_\mu, M, \beta) \cdot \frac{1}{\delta^\beta} \cdot M(|f|^r)(\bar{x}).
\end{aligned}$$

Similarly:

$$\begin{aligned}
(2.9) \quad & \int_{X \setminus B_{M\delta}(x_0)} \frac{|a(y) - a_{B_\delta}|^{r'}}{\mu(B(x_0; y)) \cdot d(x_0, y)^\beta} d\mu(y) \leq \\
& c(c_\mu) \cdot \sum_{j=1}^{\infty} \frac{1}{[M^j \delta]^\beta} \cdot \int_{B(x_0, M^{j+1} \delta)} |a(y) - a_{B_\delta}|^{r'} d\mu(y) \leq
\end{aligned}$$

(by Theorem 1.5 and Lemma 1.7)

$$\frac{1}{\delta^\beta} \sum_{j=1}^{\infty} c(r, K, X) \|a\|^{r'} * \left(\frac{1 + j^{r'}}{M^{\beta j}} \right) = c(r, K, X) \cdot \frac{1}{\delta^\beta} \cdot \|a\|^{r'} * .$$

Finally (2.2) follows from (2.3)-(2.9), and the theorem is proved. ■

As an immediate consequence of Theorems 2.4, 2.3 and 1.6, we obtain the \mathcal{L}^p -estimate (2.1) in the case $\mu(X) = \infty$:

THEOREM 2.5. – *Let X be a homogeneous space with $\mu(X) = \infty$, K a CZ operator, $a \in BMO$. Then for every $p \in (1, \infty)$ the operator T_a can be continuously extended to \mathcal{L}^p , and*

$$\|T_a f\|_p \leq c(p, K, X) \|a\|_* \|f\|_p.$$

Let us note that in the case $\mu(X) < \infty$, applying Theorem 1.6 to Theorem 2.4 we get:

$$(2.10) \quad \left\| T_a f - \int_X T_a f \right\|_p \leq c(p, K, X) \|a\|_* \|f\|_p$$

for every $a \in \mathcal{L}^\infty$, $f \in \mathcal{O}$, $p \in (1, \infty)$. In the next section we will show how to get the desired estimate from this result.

3. – Estimate on the commutator of Calderón-Zygmund operators on finite measure homogeneous spaces.

Throughout this section X will be a finite measure homogeneous space, unless otherwise stated. Let us state the main result we are going to prove:

THEOREM 3.1. – *Let (X, d, μ) be a homogeneous space and K a CZ operator on X such that:*

- i) $\mu(X) < \infty$ and μ is regular;
- ii) the kernels k and k^* satisfy condition (H1) (where $k^*(x, y) = k(y, x)$);
- iii) k satisfies the growth condition:

$$(3.1) \quad |k(x, y)| \leq \frac{c}{\mu(B(x; y))} \quad \text{for every } x, y \in X, \text{ some constant } c;$$

Then for every $a \in BMO$, $p \in (1, \infty)$, the operator T_a can be continuously extended to \mathcal{L}^p and

$$\|T_a f\|_p \leq c \|a\|_* \|f\|_p$$

with c depending on p , X , the \mathcal{L}^p norm of K and the constants of k and k^* appearing in conditions (H1), (3.1).

Let us sketch the line of the proof of Theorem 3.1. By (2.10), to get the result we need to estimate

$$\int_X (T_a f)(x) d\mu(x).$$

Since the function $T_a f$ cannot be written explicitly using our «abstract» definition of CZ operator, we shall consider suitable «approximating» integral operators K_ε , and estimate the mean value of the commutators $C[K_\varepsilon, a](f)$. Finally, we shall prove that K_ε converges, in an appropriate weak sense, to some operator \tilde{K} which is not necessarily K , but has the same commutator.

The first step is the following:

THEOREM 3.2. - *Suppose that k and k^* satisfy condition (H1), $\int_X |k(x, y)| d\mu(y)$ is a bounded function and the same is true for k^* .*

Set, for $f \in \mathcal{O}$:

$$Kf(x) = \int_X k(x, y) f(y) d\mu(y)$$

$$K^*f(x) = \int_X k^*(x, y) f(y) d\mu(y).$$

Let $a \in \mathcal{L}^\infty$, T_a and T_a^* be the commutators of K and K^* , respectively. Then, for every $p \in (1, \infty)$ there exists a positive constant c , depending on p , X , the \mathcal{L}^p norms of K and K^* , and the constants appearing in condition (H1) for k and k^* , such that

$$\|T_a f\|_p \leq c(p, K, X) \|a\|_* \|f\|_p.$$

PROOF. - By (2.10) we have that for every $a \in \mathcal{L}^\infty$, $f \in \mathcal{O}$:

$$(3.2) \quad \|T_a f\|_p \leq c \|a\|_* \|f\|_p + \mu(X)^{1/p-1} \left| \int_X (T_a f)(x) d\mu(x) \right|.$$

Therefore, let us consider:

$$\int_X (T_a f)(x) d\mu(x) = \int_X d\mu(x) \int_X k(x, y) [a(x) - a(y)] f(y) d\mu(y).$$

Since a, f are bounded functions and $\mu(X) < \infty$, the integral on the

right hand side of the last equation converges absolutely and equals

$$-\int_X f(y) d\mu(y) \int_X k(x, y)[a(y) - a(x)] d\mu(y) = -\int_X f(y) T_a^* 1(y) d\mu(y).$$

Hence:

$$\left| \int_X (T_a f)(x) d\mu(x) \right| \leq \|f\|_p \|T_a^* 1\|_{p'}.$$

Since also k^* satisfies (H1), (2.10) holds for T_a^* , too, and

$$\|T_a^* 1\|_{p'} \leq c \|a\|_* \mu(X)^{1/p'} + \mu(X)^{-1/p} \left| \int_X (T_a^* 1)(x) d\mu(x) \right|.$$

Again by Fubini's theorem, we get:

$$(3.5) \quad \int_X (T_a^* 1)(x) d\mu(x) = \int_X (Ka)(y) d\mu(y) - \int_X (K^* a)(x) d\mu(x).$$

By Hölder and the \mathcal{L}^p estimate on K :

$$(3.6) \quad \left| \int_X (Ka)(y) d\mu(y) \right| \leq \|Ka\|_p \cdot \mu(X)^{1/p'} \leq c \|a\|_p \cdot \mu(X)^{1/p'}.$$

Without loss of generality, we can assume

$$\int_X a(x) d\mu(x) = 0$$

since the commutator we are estimating is not affected by adding a constant to a . We claim that

$$\|a\|_p \leq c_p \|a\|_* \mu(X)^{1/p}.$$

To see this, recall that by Lemma 1.9, X is bounded, so it coincides with some ball B . Then, by Theorem 1.5,

$$\int_X |a(x)|^p d\mu(x) = \mu(X) \cdot \int_B |a(x) - a_B|^p d\mu(x) \leq c \mu(X) \|a\|_*^p.$$

Therefore from (3.6) we get:

$$(3.7) \quad \left| \int_X (Ka)(y) d\mu(y) \right| \leq c \|a\|_* \mu(X).$$

An analogous estimate holds for the second term in (3.5) and from (3.2)-(3.7) the result follows. ■

The second step is the construction of suitable «truncated operators» to which we will apply the previous theorem. Since we will need this construction also in § 4, while considering general homogeneous spaces (not necessarily of finite measure) we drop for a moment the assumption $\mu(X) < \infty$. By Theorem 1.3, we can endow X with a quasidistance δ equivalent to d and locally Hölder of exponent α . Consider the functions $\psi_i: [0, \infty) \rightarrow [0, 1]$ ($i = 1, 2$) defined by:

$$\psi_1(t) = \begin{cases} 0 & \text{for } t \leq 1, \\ t - 1 & \text{for } 1 < t < 2, \\ 1 & \text{for } t \geq 2, \end{cases}$$

$$\psi_2(t) = \begin{cases} 1 & \text{for } t \leq 2, \\ 3 - t & \text{for } 2 < t < 3, \\ 0 & \text{for } t \geq 3, \end{cases}$$

For any $\varepsilon \in (0, 1)$ set

$$\psi_\varepsilon(t) = \begin{cases} \psi_1\left(\frac{t}{\varepsilon}\right) & \text{for } 0 \leq t \leq 2, \\ \psi_2(\varepsilon t) & \text{for } t \geq 2, \end{cases}$$

$$(3.8) \quad k_\varepsilon(x, y) = k(x, y) \cdot \psi_\varepsilon(\delta(x, y)).$$

LEMMA 3.3. — Suppose $\alpha \in (0, 1]$ is a number for which Theorem 1.3 (ii) holds for X , and K satisfies condition (H1) with exponent β ; moreover, suppose k satisfies the growth condition (3.1). Then k_ε defined above satisfies condition (H1), with constants independent of ε . More precisely, the exponent appearing in (H1) is $\gamma = \min(\alpha, \beta)$.

REMARK 3.4. — Observe that if K satisfies (H1) and (3.1) with respect to d and μ , then it satisfies similar conditions with respect to δ and μ (for any quasidistance δ equivalent to d); in particular, (H1) holds with the same exponent β . Therefore to prove Lemma 3.3, we don't need to distinguish d and δ : henceforth d will be the quasidistance of X that satisfies a Hölder condition of exponent α (recall Theorem 1.3).

PROOF OF LEMMA 3.3. – Let us start by noting that the triangle inequality implies that if $x \in B_r(x_0)$ and $y \notin B_{Mr}(x_0)$, there exist two positive constants c_1, c_2 depending on c_d, M such that

$$(3.9) \quad c_1 d(x, y) \leq d(x_0, y) \leq c_2 d(x, y)$$

(assuming $M > c_d$).

Now, let $x_0 \in X, r > 0, x \in B_r(x_0), y \notin B_{Mr}(x_0)$.

$$|k_\varepsilon(x_0, y) - k_\varepsilon(x, y)| \leq |\psi_\varepsilon(d(x_0, y))[k(x_0, y) - k(x, y)]| + \\ |k(x, y)[\psi_\varepsilon(d(x_0, y)) - \psi_\varepsilon(d(x, y))]| = I + II.$$

By definition of ψ_ε and (H1)

$$I \leq \frac{c_K}{\mu(B(x_0, y))} \cdot \frac{d(x_0, x)^\beta}{d(x_0, y)^\beta}.$$

Also, observe that the quantity

$$|\psi_\varepsilon(d(x_0, y)) - \psi_\varepsilon(d(x, y))| \equiv A$$

can be nonzero only if one the following cases occurs:

1st case: one of the numbers $d(x_0, y)$ and $d(x, y)$ is in $(0, 2\varepsilon)$.

2nd case: one of the numbers $d(x_0, y)$ and $d(x, y)$ is in $(2\varepsilon, 3/\varepsilon)$, the other is $> 2\varepsilon$.

1st case.

$$A \leq \frac{|d(x_0, y) - d(x, y)|}{\varepsilon} \leq (\text{by Theorem 1.3}) \\ \leq c \cdot d(x_0, x)^\alpha \cdot \frac{[d(x_0, y) + d(x, y)]^{1-\alpha}}{\varepsilon} \leq (\text{by (3.9)}) \\ \leq c \cdot \frac{d(x_0, x)^\alpha}{d(x_0, y)^\alpha}.$$

2nd case.

$$A \leq c \cdot \varepsilon d(x_0, x)^\alpha [d(x_0, y)^{1-\alpha} + d(x, x_0)^{1-\alpha}] \leq (\text{by (3.9)}) \\ \leq c \cdot \frac{d(x_0, x)^\alpha}{d(x_0, y)^\alpha}.$$

Therefore

$$II \leq \frac{c}{\mu(B(x_0; y))} \cdot \frac{d(x_0, x)^\alpha}{d(x_0, y)^\alpha},$$

and the theorem is proved. ■

Let's now consider the operators:

$$(3.10) \quad K_\varepsilon f(x) = \int_X k_\varepsilon(x, y) f(y) d\mu(y).$$

Note that condition (3.1) and the doubling condition imply

$$\int_X |k_\varepsilon(x, y)| d\mu(y) \leq \int_{\varepsilon < d(x, y) < 3/\varepsilon} \frac{d\mu(y)}{\mu(B(x; y))} \leq c(\varepsilon) \text{ uniformly in } x$$

and the same is true for k^* . In particular, for $f \in \mathcal{O}$, $K_\varepsilon f$ and $K_\varepsilon^* f$ are well defined and bounded.

The following \mathcal{L}^p continuity estimate on K_ε with constants independent of ε , may be obtained adapting to our context the proof of a known result by M. Cotlar, regarding Calderón-Zygmund operators on \mathbb{R}^n (see for instance [Me], p. 241).

THEOREM 3.5. – *Let K be a CZ operator; let the kernel k satisfy (3.1); let K_ε be defined as in (3.8), (3.10). Then for every $q \in (1, \infty)$ there exists a constant c independent of ε such that:*

$$|K_\varepsilon f(x)| \leq c \{M(Kf)(x) + (M(|f|^q(x)))^{1/q} + Mf(x)\}$$

for every $x \in X$, $f \in \mathcal{L}^q$.

COROLLARY 3.6. – *Under the same assumptions of Theorem 3.5, let*

$$K^* f(x) = \sup_{\varepsilon > 0} |K_\varepsilon f(x)|.$$

Then K^ is continuous from \mathcal{L}^p to \mathcal{L}^p , for every $p \in (1, \infty)$.*

The following proposition is now a consequence of Theorem 3.2, Lemma 3.3 and Corollary 3.6.

PROPOSITION 3.7. – *Let K be a CZ operator; let the kernels k and k^* satisfy condition (H1); moreover, suppose k satisfies (3.1). Then*

the following estimate holds

$$\|C[K_\varepsilon, a](f)\|_p \leq c \|a\|_* \|f\|_p$$

for every $p \in (1, \infty)$, $a \in \mathcal{L}^\infty$, $f \in \mathcal{O}$, with c depending on p and the constants involved in the assumptions, but not on ε .

Next, we need to specify in what sense the operator K_ε approximate K .

THEOREM 3.8. – *Let K and K_ε be as in Proposition 3.7, and assume that the measure μ is regular. Then there exist a sequence $\varepsilon_n \rightarrow 0$ and a function $b \in \mathcal{L}^\infty$ such that for every $p \in (1, \infty)$, every $f, g \in \mathcal{O}$ we have:*

$$\int_X f K_{\varepsilon_n} g \rightarrow \int_X f K g + \int_X b f g \quad \text{for } n \rightarrow \infty .$$

A similar result has been proved for Calderón-Zygmund operators in \mathbb{R}^n (see for instance [Me], p. 227). The proof of this result, thought, uses tools characteristic of \mathbb{R}^n (Fourier transform, dyadic decomposition etc.) and therefore it cannot be repeated in our context. Our proof, on the other hand, relies entirely on topological and measure theoretical properties.

PROOF. – We know that:

$$(3.11) \quad \|K_\varepsilon f\|_p \leq c_p \|f\|_p$$

for every $p \in (1, \infty)$, $f \in \mathcal{O}$ and so for every $f \in \mathcal{L}^p$, with c_p independent of ε .

Observe that the space \mathcal{L}^p is separable. To see this, note that since X is separable (as we noted after Definition 1.2) and the balls form a base for the topology, X has a countable base. Also, the measure is regular, hence Borel sets can be approximated, in measure, with open sets, so that the Borel σ -algebra is separable, and \mathcal{L}^p is separable for every $p \in [1, \infty)$.

From (3.11), by compactness, we can construct a sequence $\varepsilon_n \rightarrow 0$ such that $K_{\varepsilon_n} f$ converges weakly in \mathcal{L}^p , for $n \rightarrow \infty$, for every f belonging to some dense subset of \mathcal{L}^p . But then, again by (3.11),

$$(3.12) \quad \int_X g K_{\varepsilon_n} f \rightarrow \int_X g \tilde{K} f$$

for every $f \in \mathcal{L}^p$, $g \in \mathcal{L}^{p'}$, in particular for every $f, g \in \mathcal{O}$, with $\tilde{K}: \mathcal{L}^p \rightarrow \mathcal{L}^p$ linear continuous.

In order to study the relationship between \tilde{K} and K , let $f, g \in \mathcal{O}$ such that $\text{dist}(\text{sprt } f, \text{sprt } g) \geq \varepsilon_0 > 0$. Then:

$$\begin{aligned} \int_X (fK_{\varepsilon_n}g)(x) d\mu(x) &= \int_{\text{sprt } f} f(x) d\mu(x) \int_{\text{sprt } g} k_{\varepsilon_n}(x, y) g(y) d\mu(y) = \\ &\left(\text{for every } \varepsilon_n < \frac{\varepsilon_0}{2} \right) \\ &= \int_X (fKg)(x) d\mu(x). \end{aligned}$$

Taking the limit for $n \rightarrow \infty$, we have

$$(3.13) \quad \int_X f\tilde{K}g = \int_X fKg.$$

Now fix $g \in \mathcal{O}$, let C be its support. Set $C_n = \{x \in X: \text{dist}(x, C) \geq 1/n\}$; then $\bigcup_n C_n = X \setminus C$. If $E \subseteq C_n$, E a Borel set, taking $f = \chi_E$ in (3.13) we get

$$\int_E \tilde{K}g = \int_E Kg,$$

i.e. $\tilde{K}g = Kg$ a.e. in $X \setminus C$. Therefore the operator

$$K_1 \equiv \tilde{K} - K,$$

is local, that is

$$\text{for every } g \in \mathcal{O}, \quad K_1 g(x) = 0 \text{ for a.e. } x \notin \text{sprt } g.$$

Let us now show that K_1 is the multiplication by a bounded function. Let E be a Borel set, C_n an increasing sequence of closed sets and A_n a decreasing sequence of open sets such that $\mu(A_n \setminus C_n) \rightarrow 0$, $C_n \subseteq E \subseteq A_n$. Set:

$$(3.14) \quad b(x) \equiv (K_1 1)(x) = (K_1 \chi_{C_n})(x) + (K_1 \chi_{A_n \setminus C_n})(x) + (K_1 \chi_{A_n^c})(x)$$

and take limits in (3.14) for $n \rightarrow \infty$. Since $\mu(A_n \setminus C_n) \rightarrow 0$, $K_1 \chi_{A_n \setminus C_n} \rightarrow 0$ in \mathcal{L}^p , and a subsequence converges a.e. Since $\chi_{C_n} \rightarrow \chi_E$ in \mathcal{L}^p , $K_1 \chi_{C_n} \rightarrow K_1 \chi_E$ in \mathcal{L}^p and a subsequence converges a.e. Moreover

$(K_1\chi_{A_n^\varepsilon})(x) = 0$ for a.e. $x \in E$, since K_1 is local. Then for a.e. $x \in E$ we have:

$$(K_1\chi_E)(x) = b(x).$$

On the other hand $(K_1\chi_{C_n})(x) = 0$ for a.e. $x \notin E$, and therefore $(K_1\chi_E)(x) = 0$ for a.e. $x \notin E$. This implies that for every Borel set E

$$(K_1\chi_E)(x) = b(x) \cdot \chi_E(x)$$

and, by linearity, and analogous identity holds for every simple function and so, by density, for every function in \mathcal{O} :

$$(3.15) \quad (\tilde{K}f - Kf)(x) = (K_1f)(x) = b(x) \cdot f(x) \quad \text{for every } f \in \mathcal{O},$$

Moreover $b \in \mathcal{L}^\infty$ since for every $f, g \in \mathcal{O}$

$$\left| \int_X (bfg)(x) d\mu(x) \right| = \left| \int_X (gK_1f)(x) d\mu(x) \right| \leq c \|f\|_p \|g\|_{p'},$$

and, taking $h \in \mathcal{O}$, $h \geq 0$, and letting $f = h^{1/p}$, $g = h^{1/p'}$ we have:

$$(3.16) \quad \left| \int_X (bh)(x) d\mu(x) \right| \leq c \|h\|_1$$

From (3.12), (3.15), (3.16) the theorem follows. ■

The previous theorem can also be read as a «representation theorem for CZ operators»: if K is a CZ operator, there exist a sequence $\varepsilon_n \rightarrow 0$ and a function $b \in \mathcal{L}^\infty$ such that for every $f, g \in \mathcal{O}$ we have:

$$\int_X (fKg)(x) d\mu(x) = \lim_n \int_X f(x) d\mu(x) \int_X k_{\varepsilon_n}(x, y) g(y) d\mu(y) + \int_X (bfg)(x) d\mu(x).$$

In other words, the «abstract» operator K , allows the above «concrete» representation; note, however, that if we only know that K satisfies Definition 2.1 with kernel k , the function b remains unknown and K is *not* really determined; on the other hand, the *commutator* of K is uniquely determined by the kernel.

PROOF OF THEOREM 3.1. - Let $a \in \mathcal{L}^\infty$, $f, g \in \mathcal{O}$. By Theorem 3.8 we have:

$$\begin{aligned} \int_X gC[K_{\varepsilon_n}, a](f) &= \\ &= \int_X (gaK_{\varepsilon_n} f - gK_{\varepsilon_n}(af)) \rightarrow \int_X \{ga(Kf + bf) - g(K(af) + baf)\} = \int_X gT_a f. \end{aligned}$$

By Proposition 3.7 we have

$$\left| \int_X gT_a f \right| = \lim_n \left| \int_X gC[K_{\varepsilon_n}, a](f) \right| \leq c \|g\|_p \cdot \|a\|_* \|f\|_p.$$

Hence, by the density of \mathcal{O} in \mathcal{L}^p :

$$\|T_a f\|_p \leq c \|a\|_* \|f\|_p$$

for every $a \in \mathcal{L}^\infty$, $f \in \mathcal{O}$. By Proposition 2.3, the theorem follows. ■

4. - \mathcal{L}^p continuity of singular integral operators. Examples and applications.

In this section we want to show how our results about the commutator of a CZ operator we have proved so far (i.e. Theorem 2.5 in the case $\mu(X) = \infty$, Theorem 3.1 in the case $\mu(X) < \infty$) can be applied to given concrete singular integral operators. The problem here is to assure that the kernel k actually defines a singular integral operator, continuous on \mathcal{L}^p . Our goal is to prove this fact under some natural assumptions on the kernel. This is possible basically through two known theorems, which will be recalled later. Our main results are contained in Theorems 4.1, 4.2, 4.3, 4.5.

THEOREM 4.1. - *Let (X, d, μ) be a homogeneous space, μ a regular measure. Let $k: X \times X \setminus \{x = y\} \rightarrow \mathbb{R}$ a kernel satisfying the following conditions:*

- (G) *the growth condition (3.1);*
- (H) *the Hörmander condition with respect to both variables, i.e. both k and k^* satisfy (H1);*
- (C) *the cancellation property with respect to both variables:*

there exists $C > 0$ such that for every $r, R, 0 < r < R < \infty$, a.e. x

$$\left| \int_{r < d(x,y) < R} k(x,y) d\mu(y) \right| \leq C,$$

and the same is true for k^* .

Let k_ε be the truncated kernel defined as in § 3, and for any $f \in \mathcal{C}_0^0$, set

$$K_\varepsilon f(x) = \int_X k_\varepsilon(x,y) f(y) d\mu(y).$$

Then K_ε can be extended to a linear continuous operator from \mathcal{L}^p into \mathcal{L}^p for every $p \in (1, \infty)$, and

$$\|K_\varepsilon f\|_p \leq c \|f\|_p \quad \text{for every } f \in \mathcal{L}^p,$$

where the constant c depends on X, p and all the constants involved in the assumptions, but not on ε .

THEOREM 4.2. – Suppose all the assumptions of Theorem 4.1 hold. Then there exist a sequence $\varepsilon_n \rightarrow 0$ and a CZ operator K such that for every $f, g \in \mathcal{D}$

$$\int_X f \cdot K_{\varepsilon_n} g \rightarrow \int_X f \cdot K g \quad \text{for } \varepsilon_n \rightarrow 0.$$

Moreover, if K_1, K_2 are two operators obtained in this way by two different sequences ε_n there exists a function $b \in \mathcal{L}^\infty$ such that

$$K_1 f - K_2 f = b \cdot f$$

for every $f \in \mathcal{L}^p$, and therefore K_1 and K_2 have the same commutator (for which the results of sections 2-3 hold).

In the particular case of antisymmetric kernels, the previous result can be improved:

THEOREM 4.3. – Suppose all the assumptions of Thm. 4.1 hold. Moreover let

$$k(x,y) = -k(y,x) \quad \text{for every } x, y \in X, x \neq y.$$

Then for any $f, g \in \mathcal{C}_0^0$,

$$(4.1) \quad \langle K_\varepsilon f, g \rangle \rightarrow \langle Kf, f \rangle \equiv$$

$$\frac{1}{2} \int_X \int_X k(x, y) [f(y)g(x) - f(x)g(y)] d\mu(x) d\mu(y)$$

for $\varepsilon \rightarrow 0$. In particular, the CZ limit operator is unique.

For a kernel which is not necessarily antisymmetric, we are interested in getting a better result about the convergence of K_ε , that is uniqueness of the limit and strong convergence in \mathcal{L}^p . These facts can be proved under stronger assumptions on k_ε . Before stating the result, we need one more

DEFINITION 4.4. - Let (X, d, μ) be an unbounded homogeneous space. For every $x \in X$, define

$$r_x = \sup \{r > 0 : B_r(x) = \{x\}\}$$

(with the convention $\sup \emptyset = 0$). We say that (X, d, μ) satisfies a *reverse doubling condition* if there exist $c'_\mu > 1$, $M > 1$ such that for every $x \in X$, $r > r_x$:

$$(4.2) \quad \mu(B_{Mr}(x)) \geq c'_\mu \cdot \mu(B_r(x)).$$

It can be proved that, under some general additional geometric assumptions on the space (X, d) , (4.2) is actually a consequence of the doubling condition on μ . (See [BC2]).

THEOREM 4.5. - Let (X, d, μ) be a homogeneous space such that μ is regular and, if X is unbounded, the reverse doubling condition (4.2) holds. Let k be a kernel satisfying all the assumptions of Theorem 4.1. Moreover assume that for a.e. $x \in X$ there exists

$$(L) \quad \lim_{\varepsilon \rightarrow 0} \int_{d(x,y) < 1} k_\varepsilon(x, y) d\mu(y) \equiv b(x).$$

Then for every $f \in \mathcal{L}^p$, $p \in (1, \infty)$, $K_\varepsilon f$ converges (strongly) in \mathcal{L}^p for $\varepsilon \rightarrow 0$.

REMARK 4.6 (Homogeneous spaces with a quasisymmetric quasidistance). - Although, in the definition of homogeneous space, we have required the symmetry of d , the above results still hold if d is

only *quasisymmetric*, that is satisfies:

$$(4.3) \quad d(y, x) \leq c \cdot d(x, y)$$

for some positive constant c , every $x, y \in X$. (See Examples B and C in section 4.b for applications in which the natural quasidistance is quasisymmetric).

Note that, if d satisfies (i), (iii) of Definition 1.1 and (4.3), setting

$$d'(x, y) = d(x, y) + d(y, x)$$

we get a symmetric quasidistance d' , equivalent to d . Therefore if $k(x, y)$ is a kernel satisfying condition (G) or (H) with respect to (X, d, μ) , it satisfies the same condition (possibly with other constants) with respect to (X, d', μ) , which is a homogeneous space. Moreover, if it satisfies (C) and (G) with respect to (X, d, μ) , let, for $0 < r < R < \infty$:

$$C_1 = \{y: r < d'(x, y) < R\};$$

$$C_2 = \{y: r < d(x, y) < R/(1 + c)\};$$

$$C_3 = \{y: r/(1 + c) \leq d(x, y) \leq r\};$$

$$C_4 = \{y: R/(1 + c) \leq d(x, y) \leq (1 + c)R\};$$

with c as in (4.3). Then by (C), (G) and the doubling condition,

$$\begin{aligned} \left| \int_{C_1} k(x, y) d\mu(y) \right| &= \left| \int_{C_2} k(x, y) d\mu(y) + \int_{C_1 \setminus C_2} k(x, y) d\mu(y) \right| \leq \\ &\leq c + \int_{C_3} |k(x, y)| d\mu(y) + \int_{C_4} |k(x, y)| d\mu(y) \leq \text{const.} \end{aligned}$$

So k satisfies (C) with respect to (X, d', μ) . ■

Analogously, we can see that if k satisfies (L) and (G) with respect to (X, d, μ) , then it satisfies (L) with respect to (X, d', μ) .

As a consequence, Theorems 4.1, 4.2, 4.3, 4.5 still hold if in the definition of homogeneous space we weaken the assumptions, asking d to be quasisymmetric, instead of symmetric; moreover, the same is true for the commutator estimate (Theorems 2.5-3.1).

All the theorems we have stated so far depend on the following two known results.

THEOREM 4.7, by Coifman-de Guzmàn. (See [CG]. A detailed

proof is exposed in [CW1]). - Let (X, d, μ) be a homogeneous space, $k: X \times X \rightarrow \mathbb{R}$ a measurable function satisfying an integral Hörmander condition (H0) (see Proposition 2.2). Assume that the operator

$$Kf(x) = \int_X k(x, y) f(y) d\mu(y)$$

is continuous from \mathcal{L}^2 to \mathcal{L}^2 . Then for every $p \in [1, 2]$ there exists a constant c_p depending on p , the \mathcal{L}^2 norm of K and the constants appearing in (H0), such that for every $f \in \mathcal{L}^2 \cap \mathcal{L}^p$:

$$\|Kf\|_p \leq c_p \|f\|_p \quad \text{if } 1 < p \leq 2,$$

$$\mu\{x \in X: |Kf(x)| > t\} \leq c_1 \frac{\|f\|}{t} \quad \text{for every } t > 0, \text{ if } p = 1.$$

If also k^* satisfies condition (H0), then, by duality, we get that K is also \mathcal{L}^p continuous for $p \in (2, \infty)$.

Note that the theorem cannot be directly applied to a *singular* kernel, that is with the growth allowed by condition (G), because the integral of such a kernel with an \mathcal{L}^2 function does not converge.

The second result we need is an extension to homogeneous spaces of the so called « $T(1)$ theorem» by David-Journé, [DJ]. The extension we use is due to Christ, [Chr]. (A less general extension was given by David-Journé-Semmes in [DJS]). To state the theorem we need some more definitions.

Let $\eta \in (0, 1]$ such that $\mathcal{C}_0^\eta(X)$ is dense in $\mathcal{L}^p(X)$ for every $p \in [1, \infty)$, and let $(\mathcal{C}_0^\eta)'$ be the dual space of \mathcal{C}_0^η .

DEFINITION 4.8. - We say that a linear continuous operator $T: \mathcal{C}_0^\eta \rightarrow (\mathcal{C}_0^\eta)'$, continuous in the natural topologies, satisfies the *weak boundedness property* (WBP) if there exists a constant $c > 0$ such that for any $x_0 \in X$, $r > 0$, $f, g \in \mathcal{C}_0^\eta$ with support contained in $B_r(x_0)$, $\|f\|_\infty, \|g\|_\infty \leq 1$, $\|f\|_\eta, \|g\|_\eta \leq r^{-\eta}$:

$$(WBP) \quad |\langle Tf, g \rangle| \leq c \cdot \mu(B_r(x_0)).$$

THEOREM 4.9, by Christ (see [Chr]). - Let (X, d, μ) be a homogeneous space, $T: \mathcal{C}_0^\eta \rightarrow (\mathcal{C}_0^\eta)'$ a linear continuous operator satisfying (WBP). Suppose there exists a kernel $k: X \times X \setminus \{x = y\} \rightarrow \mathbb{R}$, satisfy-

ing (G) and (H) (see Thm. 4.1) such that for any $f, g \in \mathcal{C}_0^\infty$ with disjoint supports

$$\langle Tf, g \rangle = \int_X \int_X k(x, y) f(y) g(x) d\mu(y) d\mu(x).$$

Moreover suppose that $T(1)$ and $T^*(1)$ belong to BMO . Then T can be extended to a continuous operator from \mathcal{L}^2 into \mathcal{L}^2 .

REMARK 4.10. – The definition of $T(1)$ and $T^*(1)$ is not in general trivial. However we will apply this theorem to the truncated operators K_ε , so that the definition of $K_\varepsilon 1$ and $K_\varepsilon^* 1$ will be clear.

4.a. – Proof of Theorems 4.1, 4.2, 4.3, 4.5.

PROOF OF THEOREM 4.1. – Let us consider the truncated kernels k_ε . We are going to check that all assumptions of Thm. 4.9 are fulfilled, with constants independent of ε . Conditions (C) and (G) imply that $K_\varepsilon 1$ and $K_\varepsilon^* 1$ are bounded functions, uniformly in ε , and therefore have BMO norms bounded uniformly in ε . By (G) and the doubling condition,

$$\int_X |k_\varepsilon(x, y)| d\mu(y) \leq c(\varepsilon) \quad \text{uniformly in } x,$$

so that $K_\varepsilon f(x)$ is well defined for any $f \in \mathcal{C}_0^\infty$; if also $g \in \mathcal{C}_0^\infty$ we can write

$$\langle K_\varepsilon f, g \rangle = \int_X \int_X k_\varepsilon(x, y) f(y) g(x) d\mu(y) d\mu(x)$$

where the integral is absolutely convergent. To estimate $\langle K_\varepsilon f, g \rangle$, let $f, g \in \mathcal{C}_0^\infty$, $\text{sppt } f, \text{sppt } g \subset B_r(x_0)$ for some $x_0 \in X$, $r > 0$. Then

$$\begin{aligned} K_\varepsilon f(x) &= \int_{d(x,y) < r} k_\varepsilon(x, y) [f(y) - f(x)] d\mu(y) + f(x) \cdot \int_{d(x,y) < r} k_\varepsilon(x, y) d\mu(y) + \\ &+ \int_{d(x,y) \geq r} k_\varepsilon(x, y) f(y) d\mu(y) = I + II + III; \end{aligned}$$

$$|I| \leq (\text{by (G)}) c \|f\|_\eta \int_{0 < d(x,y) < r} \frac{d(x, y)^\eta}{\mu(B(x; y))} d\mu(y) \leq$$

(expanding the integral as $\sum_{n=1}^\infty \int_{r/2^{n+1} < d(x,y) < r/2^n} \dots$ and using the dou-

bling property)

$$\leq c \|f\|_\eta \cdot r^\eta;$$

$$|II| \leq |f(x)| \left| \int_{d(x,y) < r} k_\varepsilon(x,y) d\mu(y) \right| \leq c |f(x)|$$

since:

$$\begin{aligned} & \int_{d(x,y) < r} k_\varepsilon(x,y) d\mu(y) = \\ &= \int_{2\varepsilon < d(x,y) < r} k(x,y) d\mu(y) + \int_{\varepsilon < d(x,y) \leq 2\varepsilon} k_\varepsilon(x,y) d\mu(y) \equiv A + B; \end{aligned}$$

$$|A| \leq c \text{ by (C).}$$

Finally, $|B| \leq \int_{\varepsilon < d(x,y) \leq 2\varepsilon} |k(x,y)| d\mu(y) \leq c$ by (G) and the doubling condition.

$$|III| \leq c \int_{d(x,y) \geq r} \frac{|f(y)|}{\mu(B(x;y))} d\mu(y).$$

Therefore:

$$\begin{aligned} & \left| \int_X g(x) K_\varepsilon f(x) d\mu(x) \right| \leq \\ & c \int_{B_r(x_0)} |g(x)| \left(\|f\|_\eta \cdot r^\eta + |f(x)| + \int_{d(x,y) \geq r} \frac{|f(y)|}{\mu(B(x;y))} d\mu(y) \right) d\mu(x) \leq \\ & c \|g\|_\infty \cdot \mu(B_r(x_0)) \left(\|f\|_\eta \cdot r^\eta + \|f\|_\infty + \|f\|_\infty \cdot \sup_{x, r \leq d(x,y) \leq c \cdot r} \int \frac{d\mu(y)}{\mu(B(x;y))} \right). \end{aligned}$$

where we used the assumption on the supports of f and g . Since the last integral in brackets is bounded uniformly in r , by the doubling condition, we get:

$$|\langle K_\varepsilon f, g \rangle| \leq c \cdot \mu(B_r(x_0)) \|g\|_\infty (\|f\|_\eta \cdot r^\eta + \|f\|_\infty)$$

with c an absolute constant. This proves that $K_\varepsilon: \mathcal{C}_0 \rightarrow (\mathcal{C}_0)'$ is continuous, and satisfies (WBC), with constants independent of ε . Then by Thm. 4.9, K_ε can be extended to a continuous operator from \mathcal{L}^2 into

\mathcal{L}^2 , with norm uniformly bounded in ε . From this fact and condition (H), by Thm. 4.7, we also get

$$(4.4) \quad \|K_\varepsilon f\|_p \leq c_p \|f\|_p$$

for every $p \in (1, 2]$. Since the same reasoning can be applied to K_ε^* , by duality we get that (4.4) actually holds for every $p \in (1, \infty)$. This proves Thm. 4.1. ■

For sake of brevity, we skip the proof of Thm. 4.2, since it is similar to that of Thm. 3.8, with minor changes in the case $\mu(X) = \infty$.

To prove Thms. 4.3 and 4.5 we need the following technical lemma, which replaces in homogeneous spaces the use of Young inequality to estimate some «convolution» integrals.

LEMMA 4.11. – Let $f \in \mathcal{L}^1$, f boundedly supported, and set, for every $r > 0$

$$g_r(x) = \int_{d(x,y) > r} \frac{|f(y)|}{\mu(B(x; y))} d\mu(y).$$

Then, for every $p \in (1, \infty)$ and $r > 0$, $g_r \in \mathcal{L}^p$; moreover $g_r \rightarrow 0$ in \mathcal{L}^p for $r \rightarrow +\infty$.

PROOF. – Suppose first that X is unbounded. Note that, by the doubling condition, $\mu(B(x; y))$ is comparable with $\mu(B(y; x))$:

$$(4.5) \quad \mu(B(x; y)) \leq c(c_\mu, c_d) \cdot \mu(B(y; x)).$$

Let $h \in \mathcal{L}^{p'}$

$$\left| \int_X (hg_r)(x) d\mu(x) \right| \leq \int_X |f(y)| d\mu(y) \int_{d(x,y) > r} \frac{|h(x)|}{\mu(B(x; y))} d\mu(x) \leq$$

(by (4.5) and Hölder)

$$\leq c \|h\|_{p'} \int_X |f(y)| \left(\int_{d(x,y) > r} \frac{d\mu(x)}{\mu(B(y; x))^p} \right)^{1/p} d\mu(y).$$

Now

$$\begin{aligned} \int_{d(x,y) > r} \frac{d\mu(x)}{\mu(B(y;x))^p} &= \sum_{n=0}^{\infty} \int_{2^n r < d(x,y) \leq 2^{n+1} r} \frac{d\mu(x)}{\mu(B(y;x))^p} \leq \\ &\leq \sum_{n=n_0}^{\infty} \frac{1}{\mu(B(y, 2^{n+1} r))^{p-1}} \leq \end{aligned}$$

(by (4.2), with $n_0 = \max\{n: 2^n r < r_y\}$ if $r_y > r$, $n_0 = 0$ otherwise)

$$\leq \left(\sum_{n=0}^{\infty} \frac{1}{(c'_\mu)^{(p-1)(n+1)}} \right) \cdot \frac{1}{\mu(B(y,r))^{p-1}}.$$

Hence:

$$(4.6) \quad \left| \int_X (hg_r)(x) d\mu(x) \right| \leq c(p, X) \|h\|_{p'} \int \frac{|f(y)|}{\mu(B(y,r))^{1/p'}} d\mu(y).$$

Recall that f is boundedly supported, let $\text{sprt } f \subseteq B_R(x_0)$. Let us show that for any fixed $r > 0$

$$(4.7) \quad \inf_{y \in B_R(x_0)} \mu(B(y,r)) = c > 0.$$

To see this, observe that, for every $y \in B_R(x_0)$, $B_R(x_0) \subseteq B_{2c_d R}(y)$, so that, by the doubling condition,

$$0 < \mu(B_R(x_0)) \leq c(X, r, R) \cdot \mu(B_r(y)).$$

From (4.6)-(4.7) we have

$$\left| \int_X (hg_r)(x) d\mu(x) \right| \leq c(p, X) \|h\|_{p'} \cdot c(X, r, R) \|f\|_1 = c(p, X, r, f) \|h\|_{p'}$$

hence $g_r \in \mathcal{L}^p$. To show that $g_r \rightarrow 0$ in \mathcal{L}^p for $r \rightarrow \infty$ observe that for every $r \geq 1$, $|g_r(x)| \leq |g_1(x)| \in \mathcal{L}^p$. Moreover $g_r(x) \rightarrow 0$ a.e. for $r \rightarrow \infty$, as we can see applying Lebesgue's theorem to

$$g_r(x) = \int_X \chi_{X \setminus B_r(x)}(y) \frac{|f(y)|}{\mu(B(x;y))} d\mu(y).$$

Therefore, again by Lebesgue's theorem $g_r \rightarrow 0$ in \mathcal{L}^p for $r \rightarrow \infty$.

If X is bounded, by (4.7) g_r is bounded for every r , and for r sufficiently large $g_r \equiv 0$. So the lemma is proved. ■

PROOF OF THEOREM 4.3. - Since for the K_ε 's Theorem 4.1 holds, it

is enough to check the weak convergence (4.1). By antisymmetry and with a change of variables, for any fixed $f, g \in \mathcal{C}_0^j$ with support contained in some ball B :

$$\langle K_\varepsilon f, g \rangle = \frac{1}{2} \int_X \int_X k_\varepsilon(x, y) [f(y)g(x) - f(x)g(y)] d\mu(x) d\mu(y).$$

Now:

$$\begin{aligned} |\langle K_\varepsilon f, g \rangle - \langle Kf, g \rangle| &\leq c \cdot \varepsilon^\eta \int_B d\mu(x) \int_{\varepsilon < d(x, y) < 2\varepsilon} \frac{d\mu(y)}{\mu(B(x; y))} + \\ &\int_B d\mu(x) \int_{2/\varepsilon < d(x, y) < 3/\varepsilon} \frac{|f(y)g(x)| + |f(x)g(y)|}{\mu(B(x; y))} d\mu(y) = I + II. \end{aligned}$$

Now

$$|I| \leq c \cdot \varepsilon^\eta \mu(B) \rightarrow 0$$

while $|II| \rightarrow 0$ by Lemma 4.11. ■

PROOF OF THEOREM 4.5. - For $f \in \mathcal{C}_0^j$ let us write

$$K_\varepsilon f(x) =$$

$$\begin{aligned} \int_{\varepsilon < d(x, y) < 1} k_\varepsilon(x, y) [f(y) - f(x)] d\mu(y) + f(x) \cdot \int_{\varepsilon < d(x, y) < 1} k_\varepsilon(x, y) d\mu(y) + \\ \int_{1 \leq d(x, y) < 3/\varepsilon} k_\varepsilon(x, y) f(y) d\mu(y) = I_\varepsilon + II_\varepsilon + III_\varepsilon. \end{aligned}$$

$$\begin{aligned} I_\varepsilon = \int_{\varepsilon < d(x, y) < 2\varepsilon} k_\varepsilon(x, y) [f(y) - f(x)] d\mu(y) + \\ \int_{2\varepsilon < d(x, y) < 1} k(x, y) [f(y) - f(x)] d\mu(y) = A_\varepsilon + B_\varepsilon. \end{aligned}$$

With an estimate similar to that in the proof of Thm. 4.1, $|A_\varepsilon| \leq c \|f\|_\eta \cdot \varepsilon^\eta \rightarrow 0$ for $\varepsilon \rightarrow 0$. The function $B_\varepsilon(x)$ is boundedly supported and converges uniformly, as $\varepsilon \rightarrow 0$, to

$$(4.8) \quad \int_{0 < d(x, y) < 1} k(x, y) [f(y) - f(x)] d\mu(y)$$

which is bounded by $c\|f\|_\eta$, as a similar estimate shows. So B_ε converges to (4.8) in \mathcal{L}^p .

Since f is compactly supported, our assumption (L) implies that $II_\varepsilon \rightarrow b(x)f(x)$ uniformly, and therefore in \mathcal{L}^p . Here we used the fact that

$$\int_{\varepsilon < d(x,y) < 1} k_\varepsilon(x,y) d\mu(y)$$

has limit for $\varepsilon \rightarrow 0$. This follows from (L), as can be seen for instance approximating the cutoff function ψ_ε , which enters the definition of k_ε , by piecewise constant functions.

To prove that III_ε converges in \mathcal{L}^p , let us show that $\|III_\varepsilon - III_\eta\|_p \rightarrow 0$ for $\varepsilon, \eta \rightarrow 0$. Suppose for instance $0 < \varepsilon < \eta$

$$(4.9) \quad III_\varepsilon - III_\eta = \int_{2/\eta < d(x,y) \leq 2/\varepsilon} k(x,y)f(y) d\mu(y) + \int_{2/\varepsilon < d(x,y) < 3/\varepsilon} k_\varepsilon(x,y)f(y) d\mu(y) - \int_{2/\eta < d(x,y) < 3/\eta} k_\eta(x,y)f(y) d\mu(y).$$

Each term in (4.9) is majorized by

$$c \int_{d(x,y) \geq 2/\eta} \frac{|f(y)|}{\mu(B(x;y))} d\mu(y)$$

which converges to zero in \mathcal{L}^p , as $\eta \rightarrow 0$, by Lemma 4.11. This completes the proof of Thm. 4.5. ■

4.b. – Examples of singular integral operators on homogeneous spaces.

We start with the following

REMARK 4.12. – Whenever, in a homogeneous space, the following holds

$$(4.10) \quad c_1 r^N \leq \mu(B_r(x)) \leq c_2 r^N$$

for every $r > 0$, $x \in X$, some positive constants c_1, c_2, N , condition (H1) can be rewritten as: there exist $c_K > 0, \beta > 0, M > 1$ such that for

every $x_0 \in X$, $r > 0$, $x \in B_r(x_0)$, $y \notin B_{Mr}(x_0)$:

$$(H2) \quad |k(x_0, y) - k(x, y)| \leq c_K \cdot \frac{d(x_0, x)^\beta}{d(x_0, y)^{\beta+N}}.$$

EXAMPLE A. - *Singular integrals with mixed homogeneities*, studied by Fabes and Rivière in [FR] and applications to parabolic equations.

Let $X = \mathbb{R}^N$, μ the Lebesgue measure, d the distance defined as follows. Let $\alpha_1, \dots, \alpha_N$ be N real numbers, with $1 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_N$; the equation

$$\sum_{j=1}^N \frac{x_j^2}{\varrho^{2\alpha_j}} = 1$$

allows for every $x \in \mathbb{R}^N \setminus \{0\}$ a unique positive solution $\varrho(x)$. Set $\varrho(0) = 0$ and define $d(x, y) = \varrho(x - y)$. It can be proved (see [FR]) that d is actually a distance. Introducing the polar type change of variables

$$\begin{cases} x_1 = \varrho^{\alpha_1} \cos \varphi_1 \dots \cos \varphi_{N-2} \cos \varphi_{N-1} \\ x_2 = \varrho^{\alpha_1} \cos \varphi_1 \dots \sin \varphi_{N-1} \\ \dots \\ x_N = \varrho^{\alpha_N} \sin \varphi_1, \end{cases}$$

we find $dx = \varrho^{\alpha-1} d\varrho d\sigma$, with $\alpha = \sum_{j=1}^N \alpha_j$ and $d\sigma$ the surface measure on $\Sigma_{N-1} = \{|x| = 1\}$. Therefore we can compute:

$$(4.11) \quad \mu(B_r(x)) = c(N) \cdot r^\alpha \quad \text{for every } r > 0.$$

In particular (4.10) holds, and (\mathbb{R}^N, d, dx) is a homogeneous space. The unit ball with respect to d is the euclidean unit ball. Let

$$(4.12) \quad k(x, y) = \frac{\Omega(x - y)}{\varrho(x - y)^\alpha},$$

where the function Ω has the following mixed homogeneity of degree zero

$$\Omega(t^{\alpha_1} x_1, t^{\alpha_2} x_2, \dots, t^{\alpha_N} x_N) = \Omega(x_1, \dots, x_N), \quad \text{for every } t > 0, x \in \mathbb{R}^N$$

and satisfies the Hölder condition: there exist $c > 0$, $\beta \in (0, 1]$ such that for every $x, y \in \Sigma_{N-1}$:

$$(4.14) \quad |\Omega(x) - \Omega(y)| \leq c|x - y|^\beta.$$

It is easy to check that the following condition (H2) is fulfilled: if α, β are as above, there exists a constant c such that for every $x_0 \in \mathbb{R}^N$, $x \in B_r(x_0)$, $y \notin B_{2r}(x_0)$:

$$(4.15) \quad |k(x_0, y) - k(x, y)| \leq c \cdot \frac{d(x_0, x)^\beta}{d(x_0, y)^{\beta+\alpha}}.$$

The growth condition (G) follows from (4.11) and the definition of k . Moreover, if we assume that Ω satisfies the vanishing property

$$(4.16) \quad \int_{\Sigma_{N-1}} \Omega(x) d\sigma(x) = 0,$$

then conditions (C) and (L) are trivial. Therefore, by Thms. 4.1, 4.5, for every $p \in (1, \infty)$ it is well defined

$$Kf(x) = \lim_{\varepsilon \rightarrow 0} \int k_\varepsilon(x, y) f(y) dy,$$

where the limit exists in \mathcal{L}^p sense, and K is continuous. (This result was proved with a direct approach in [FR]). For this operator the commutator estimate of Thm. 2.5 holds.

Kernels of the above kind arise for instance considering parabolic operators with constant coefficients:

$$L = \sum_{i,j=1}^n a_{ij} \partial_{\xi_i \xi_j}^2 - \partial_t,$$

where a_{ij} is a symmetric and positive matrix. Here $x \equiv (\xi, t) \in \mathbb{R}^{n+1} \equiv \mathbb{R}^N$ (we set $N = n + 1$). Let Γ^0 be the fundamental solution for L with pole at the origin, and let

$$k(x, y) = (\partial_{\xi_i \xi_j}^2 \Gamma^0)(x - y), \quad \text{for any } i, j = 1, \dots, n.$$

Then k is a kernel as in (4.12)-(4.13), with $\alpha_1 = \dots = \alpha_n = 1$, $\alpha_{n+1} = 2$. For any test function u we can write:

$$u(x) = \int_{\mathbb{R}^{n+1}} \Gamma^0(x - y) Lu(y) dy$$

and, differentiating twice the above formula with respect to ξ we find

$$\partial_{\xi_i \xi_j}^2 u(x) = P.V. \int_{\mathbb{R}^{n+1}} \partial_{\xi_i \xi_j}^2 \Gamma^0(x - y) Lu(y) dy + c(n) \cdot Lu(x).$$

Therefore, the \mathcal{L}^p -continuity of the singular integral implies \mathcal{L}^p -estimates for the (spacial) second derivatives of u , in terms of Lu and u . ■

EXAMPLE B. – *Kolmogorov-type operators*, studied by Lanconelli-Polidoro in [LP].

Let us consider the following linear partial differential operator:

$$L \equiv \sum_{i,j=1}^q a_{ij} \partial_{x_i x_j} + \sum_{i,j=1}^N b_{ij} x_i \partial_{x_j} - \partial_t \equiv \sum_{i,j=1}^q a_{ij} \partial_{x_i x_j} + \langle x, BD \rangle - \partial_t,$$

$$D = (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_N})$$

where $(x, t) \in \mathbb{R}^{N+1}$, $\{a_{ij}\}$ is a constant, symmetric, positive $q \times q$ matrix (i.e. the principal part is uniformly elliptic on \mathbb{R}^q , $q < N$), and B is an $N \times N$ constant matrix with the following structure:

$$B \equiv \begin{bmatrix} 0 & B_1 & 0 & \dots & 0 \\ 0 & 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_r \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

where for every $k = 1, \dots, r$, B_k is a matrix $p_{k-1} \times p_k$ with rank p_k and $q = p_0 \geq p_1 \geq \dots \geq p_r$, $p_0 + p_1 + \dots + p_r = N$. Under these assumptions, the operator L is hypoelliptic, and the matrix B induces in the space \mathbb{R}^{N+1} a (noncommutative) group of translations

$$(x, t) \circ (y, \tau) = (y + E(\tau)x, t + \tau), \quad \text{where } E(\tau) = \exp(-\tau B^T),$$

such that L is invariant for left-translations. Moreover, there exists a group of dilations on \mathbb{R}^{N+1} , which we denote by $\{D(\lambda)\}_{\lambda > 0}$, where every $D(\lambda)$ is a group homomorphism, acting as follows:

$$(4.17) \quad D(\lambda)(x, t) = (\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_N} x_N, \lambda^2 t)$$

where α_i are suitable integers ≥ 1 . The operator L is homogeneous of degree 2 with respect to the group of dilations:

$$L(u(D(\lambda)z)) = \lambda^2 \cdot (Lu)(D(\lambda)z),$$

with $z = (x, t)$. For $N = 2$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have the simplest example:

$$L = \partial_{x_1}^2 + x_1 \partial_{x_2} - \partial_t.$$

In this case the groups of translations and dilations are, respectively:

$$(x_1, x_2, t) \circ (y_1, y_2, \tau) = (x_1 + y_1, x_2 + y_2 - \tau x_1, t + \tau);$$

$$D(\lambda)(x_1, x_2, t) = (\lambda x_1, \lambda^3 x_2, \lambda^2 t).$$

We can define a *homogeneous norm*, setting, for any $z \in \mathbb{R}^{N+1}$, $z \neq 0$,

$$\|z\| = \varrho \Leftrightarrow \left| D\left(\frac{1}{\varrho}\right)z \right| = 1,$$

with $|\cdot| =$ euclidean norm; also, let $\|0\| = 0$. Then:

- i) $\|D(\lambda)z\| = \lambda\|z\|$ for every $z \in \mathbb{R}^{N+1}$, $\lambda > 0$;
- ii) the set $\{z \in \mathbb{R}^{N+1} : \|z\| = 1\}$ is the euclidean sphere Σ_{N+1} ;
- iii) for every $z, \zeta \in \mathbb{R}^{N+1}$

$$\|z \circ \zeta\| \leq c(\|z\| + \|\zeta\|) \quad \text{for some constant } c = c(B) \geq 1.$$

Defining:

$$d(z, \zeta) = \|\zeta^{-1} \circ z\|,$$

we have that d is a quasisymmetric quasidistance (see Remark 4.6). Since the Lebesgue's measure is left and right-invariant under the group law, and

$$\det D(\lambda) = \lambda^{Q+2},$$

with $Q = \alpha_1 + \dots + \alpha_N$, α_i as in (4.17) we have that $|\mathcal{B}(z, r)| = |\mathcal{B}(0, r)| = |\mathcal{B}(0, 1)|r^{Q+2}$, for every $z \in \mathbb{R}^{N+1}$ and $r > 0$. Therefore dz is a *doubling measure* with respect to the quasidistance d , and $(\mathbb{R}^{N+1}, dz, d)$ is a homogeneous space.

Let Γ^0 be the fundamental solution for L , with pole at the origin, which is homogeneous of degree $-Q$ with respect to $D(\lambda)$, and let

$$k(z, \zeta) = (\partial_{x_i x_j}^2 \Gamma^0)(\zeta^{-1} \circ z), \quad \text{for any } i, j = 1, \dots, q.$$

Then k is a *CZ* kernel on the above homogeneous space, for which the \mathcal{L}^p -estimate of Thms. 4.1, 4.5 and the commutator estimate of Thm. 2.5 hold. ■

REMARK 4.13. – *Applications of the commutator theorem to get a priori estimates for partial differential operators.* In the above examples, \mathcal{L}^p -continuity of singular integral operators implies \mathcal{L}^p -estimates for (suitable) second derivatives of u , in terms of Lu and u . The differential operator L which is considered has constant coefficients or, more generally, is translation invariant, with respect to a suitable group. The *commutator* estimate can also be used to get \mathcal{L}^p -estimates, for suitable classes of P.D.O.'s with *variable* coefficients. This technique was used first by Chiarenza-Frasca-Longo [CFL1, 2], who applied Coifman-Rochberg-Weiss' Theorem to get \mathcal{L}^p estimates for uniformly elliptic operators in nondivergence form, with *VMO* coefficients. The class *VMO* («vanishing mean oscillation», see [Sa]) is contained in *BMO*, and properly contains the space of uniformly continuous functions, so the \mathcal{L}^p theory for elliptic operators with *VMO* coefficients is an interesting extension of the classical one. This kind of result has been extended to the (uniformly) parabolic case by Bramanti-Cerutti [BC1], who proved a parabolic version of CRW Theorem. In another paper, Bramanti-Cerutti-Manfredini [BCM], using the commutator theorem in homogeneous spaces, prove \mathcal{L}^p estimates for a special class of ultraparabolic equations, of Kolmogorov-Fokker-Planck type (see Example B for the translation invariant case). It should be noted that, although CRW-type Theorem provides a very powerful tool, the \mathcal{L}^p -estimates for second derivatives for operators with variable coefficients are not a direct consequence of it. Infact, in these cases, the representation formula for the derivatives of u is obtained by «unfreezing» the coefficients of L after having written the representation formula for the operator with constant coefficients. Therefore the kernels one obtains are of the kind $k(x, x - y)$ and are *not* themselves *CZ* kernels; instead, one has to expand k as a series in terms of kernels that turn out to be *CZ* kernels. This technique requires a deep analysis of the properties of the singular kernel $k(x, x - y)$.

EXAMPLE C. – *The Kohn-Laplacian on the Heisenberg group.* (see [St3] for the proofs of the facts listed below).

Let us consider $\mathbb{C}^n \times \mathbb{R}$ endowed with the group law:

$$(\xi, t) \circ (\eta, s) = (\xi + \eta, t + s + 2 \operatorname{Im}(\xi \cdot \bar{\eta})),$$

$$(\xi, t)^{-1} = (-\xi, -t).$$

$(\mathbb{C}^n \times \mathbb{R}, \circ)$, or, equivalently, $(\mathbb{R}^{2n+1}, \circ)$, is a Lie group, for which the Lebesgue measure in \mathbb{R}^{2n+1} is the Haar measure. Moreover the dilations

$$D(\lambda)(\xi, t) = (\lambda\xi, \lambda^2 t) \quad (\lambda > 0)$$

are group automorphisms. We can define in \mathbb{R}^{2n+1} a *homogeneous norm*

$$|(\xi, t)| = (|\xi|^4 + t^2)^{1/4}$$

and a *quasidistance*

$$d((\xi, t), (\eta, s)) = |(\eta, s)^{-1} \circ (\xi, t)|.$$

Then \mathbb{R}^{2n+1} , with this quasidistance and the Lebesgue's measure, is a homogeneous space, such that the measure of a ball of radius r equals $\text{constant} \cdot r^Q$, with $Q = 2n + 2$. (Therefore Q is called the *homogeneous dimension* of \mathbb{R}^{2n+1}).

Consider now the Lie algebra canonically associated to the Lie group. To describe it, let us write, from now on, $(\xi, t) = (x, y, t)$, with $x, y \in \mathbb{R}^n$; let X_j, Y_j, T , respectively, be the left-invariant vector fields on \mathbb{R}^{2n+1} that equal $\partial / \partial x_j, \partial / \partial y_j, \partial / \partial t$ at the origin. Then one can compute:

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t};$$

$$Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t};$$

$$T = \frac{\partial}{\partial t}.$$

Moreover,

$$[X_j, Y_j] = -4T \quad \text{for } j = 1, \dots, n$$

and all the other commutators vanish. The operator

$$\mathcal{L} = - \sum_{j=1}^n (X_j^2 + Y_j^2)$$

is the Kohn-Laplacian on the Heisenberg group; it is a hypoelliptic operator, homogeneous of degree two with respect to the dilations $D(\lambda)$, and it has a fundamental solution Γ^0 , with pole at the origin, $D(\lambda)$ -homogeneous of degree $-(Q - 2)$; moreover, Γ^0 is explicitly known:

$$\Gamma^0(\xi, t) = c(n) / (|\xi|^4 + t^2)^{n/2} .$$

For any test function u we can write:

$$u(z) = \int_{\mathbb{R}^{2n+1}} \Gamma^0(\xi^{-1} \circ z) \mathcal{L}u(\xi) d\xi$$

and, differentiating twice the above formula with respect to the vector fields X_j or Y_j we find, for example,

$$X_i X_j u(z) = P.V. \int_{\mathbb{R}^{2n+1}} X_i X_j \Gamma^0(\xi^{-1} \circ z) \mathcal{L}u(\xi) d\xi + c(n) \cdot \mathcal{L}u(z) .$$

The kernel $k(z, \xi) = X_i X_j \Gamma^0(\xi^{-1} \circ z)$ turns out to be a CZ kernel on the homogeneous space described above; for the corresponding operator the \mathcal{L}^p estimates and the commutator estimate hold.

We expect that this fact could be a starting point to get \mathcal{L}^p estimates for the «second derivatives» $X_i X_j u$ for a class of operators with variable coefficients, modeled on the Kohn-Laplacian. (See Remark 4.13). ■

The next two examples are taken from [CW1], pp. 76-77.

EXAMPLE D. – *Singular kernels on euclidean spaces with weighted measures.* (This example is related to some weighted singular integrals in \mathbb{R}^n studied by Stein in [St1]).

Let $X = \mathbb{R}^n$, $d\mu = |x|^{-\gamma} dx$ with $0 \leq \gamma < n$, d the euclidean distance. It is well known (see [GR], p. 407) that the function $|x|^\alpha$ is an A_p -weight for $-n < \alpha < n(p - 1)$, hence $|x|^{-\gamma}$ with $0 \leq \gamma < n$ is an A_p -weight for every $p \in [1, \infty)$. In particular $d\mu$ is doubling, so that X is a homogeneous space; one can prove that $d\mu$ satisfies also a reverse doubling condition. Note that $d\mu$ is *not* translation invariant. Now, let

$$(4.18) \quad k(x, y) = \frac{\Omega(x - y)}{|x - y|^n} (|x|^\gamma - |y|^\gamma)$$

where Ω is a homogeneous function of degree zero (in the usual

sense) satisfying (4.14), (4.15). In this case (4.10) does not hold, but the following inequalities can be easily verified:

$$(4.19) \quad \mu(B(x; y)) \leq \begin{cases} c|x-y|^n|x|^{-\gamma} & \text{if } |x-y| < |x|/2, \\ c|x-y|^{n-\gamma} & \text{if } |x-y| \geq |x|/2. \end{cases}$$

By (4.18) and (4.19), we can see that k satisfies condition (G). Moreover, one can prove that k satisfies (H) with exponent $\beta' = \min(\beta, \gamma)$. (β is the same number appearing in (4.15)). Finally, also (C) and (L) can be proved, the last one following from the analogous result for classical CZ integrals on \mathbb{R}^n . So also in this case, by Thms. 4.1, 4.5, the kernel k defines a unique CZ operator, for which the commutator estimate holds.

EXAMPLE E. - *singular integrals on the sphere*. (The following operators, discussed in [CW1], are related to the Riesz transform on the sphere, studied in [KV2]).

Let $X = \Sigma_{n-1}$ the surface of the unit ball in \mathbb{R}^n , d the euclidean distance (in \mathbb{R}^n) and μ the surface measure. We know that

$$c_1 r^{n-1} \leq \mu(B_r(x)) \leq c_2 r^{n-1}$$

for every $x \in X$, $0 < r < 2$, with $0 < c_1 < c_2$.

Let us consider the kernels:

$$k_{ij}(x, y) = \frac{x_i y_j - x_j y_i}{|x - y|^n}$$

where i, j ($i \neq j$) are two fixed indexes between 1 and n . (In the following we will suppose these indexes fixed once and for all, and we will drop them from k). One can check that $k(x, y) \leq c \cdot |x - y|^{n-1}$, so that (G) holds for k . Moreover, a condition (H2) is satisfied: there exists $c > 0$ such that for every $x_0 \in \Sigma_{n-1}$, $r > 0$, $x \in B_r(x_0)$, $y \notin B_{2r}(x_0)$:

$$|k(x_0, y) - k(x, y)| \leq c \cdot \frac{|x_0 - x|}{|x_0 - y|^n}.$$

From this condition (H) follows. Note that, for every $r, R > 0$,

$$\int_{r < |x-y| < R} k(x, y) d\sigma(y) = 0$$

so that (C) and (L) hold. Again, a unique CZ operator is well defined, for which the commutator estimate holds, by Thm. 3.1.

EXAMPLE F. – This example exhibits a kernel which defines infinitely many CZ operators, all of them having the same commutator. In other words, Thm. 4.1 and 4.2 apply, but Thm. 4.5 does not.

In \mathbb{R}^n , with the usual distance and measure, define the convolution kernel:

$$k(x, y) = k(x - y) \quad \text{with } k(x) = \frac{1}{|x|^n} \cos \log |x| .$$

Then:

$$|k(x)| \leq \frac{1}{|x|^n} \quad \text{that is (G) holds;}$$

$$|\nabla k(x)| \leq \frac{c}{|x|^{n+1}} , \quad \text{which implies (H);}$$

$$\int_{r < |x| < R} k(x) dx = c_n \cdot [\sin \log R - \sin \log r] ,$$

so that (C) holds, but

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |x| < 1} k(x) dx \quad \text{does not exist .}$$

By Thm. 4.1 we know that

$$\|K_\varepsilon f\|_p \leq c_p \|f\|_p \quad \text{for every } p \in (1, \infty)$$

with c_p independent of ε . By Thm. 4.2 we also know that the commutator $C[K_\varepsilon, a]$ converges weakly to $C[K, a]$, where K is any CZ operator of an equivalence class. For this commutator the estimate of Thm. 2.5 holds. Nevertheless, $K_\varepsilon f$ does not converge in any sense to a CZ operator. We can actually compute, for any sequence $\varepsilon_k \rightarrow 0$,

$$\begin{aligned} \lim_{\varepsilon_k \rightarrow 0} K_{\varepsilon_k} f(x) = & \int_{|x-y| < 1} k(x-y)[f(y) - f(x)] dy + \\ & \int_{|x-y| > 1} k(x-y)f(y) dy - c_n f(x) \cdot \left[\lim_{\varepsilon_k \rightarrow 0} \sin \log \varepsilon_k \right] . \end{aligned}$$

Hence in general $K_{\varepsilon_k} f$ does not converge, and there are sequences for which $K_{\varepsilon_k} f \rightarrow Kf + b \cdot f$, where K is a fixed operator and b is any number in the interval $[-c_n, c_n]$. ■

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