L^p estimates for degenerate Ornstein-Uhlenbeck operators

Joint work with G. Cupini, E. Lanconelli, E. Priola

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Problem and main result

Let us consider the class of degenerate Ornstein-Uhlenbeck operators in $\mathbb{R}^N$:

$$\mathcal{A} = div\,(A\nabla) + \langle x, B\nabla\rangle = \sum_{i,j=1}^{N} a_{ij}\partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j},$$

where A and B are constant $N \times N$ matrices, A is symmetric and positive semidefinite. The evolution operator corresponding to $\mathcal{A}$,

$$L = \mathcal{A} \pm \partial_t,$$

is a Kolmogorov-Fokker-Planck ultraparabolic operator.

Example

Kolmogorov, 1934. For $N = 2$, $(v, x, t) \in \mathbb{R}^3$,

$$\partial_{vv}^2 u + v\partial_x u - \partial_t u = 0; \text{ here } A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This ultraparabolic operator possesses a fundamental solution smooth outside the pole.

Example

Backword Kolmogorov equation for the probability density $p(\mathbf{v}, \mathbf{x}, t)$ of a particle of position $\mathbf{x}$, velocity $\mathbf{v}$, mass m, suspended in a fluid of viscosity β, temperature T, subject to Brownian motion (k =Boltzmann const.)

$$\frac{\partial p}{\partial t} + \langle \mathbf{v}, \nabla_{\mathbf{x}} p \rangle - \beta \langle \mathbf{v}, \nabla_{\mathbf{v}} p \rangle + \frac{\beta k T}{m} div_{\mathbf{v}} (\nabla_{\mathbf{v}} p) = 0;$$

$$\text{here } A = \begin{bmatrix} \frac{\beta k T}{m} I_3 & 0_3 \\ 0_3 & 0_3 \end{bmatrix}; B = \begin{bmatrix} I_3 & -\beta I_3 \\ 0_3 & 0_3 \end{bmatrix}.$$

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  - (Nonlinear) Boltzmann-Landau equation in the kinetic theory of gases

$$\sum_{j=1}^{n} x_j \partial_{x_{j+n}} u + \partial_t u = \sum_{i,j=1}^{n} \partial_{x_i} \left(a_{ij} \left(\cdot, u \right) \partial_{x_j} u + b_i \left(\cdot, u \right) \right).$$

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- The operator $\mathcal{A}$ can be seen as the infinitesimal generator of the Ornstein-Uhlenbeck semigroup; accordingly, it has been studied by several authors by a semigroup approach.
- It is also natural to study $\mathcal{A} = div \left(A \nabla \right) + \langle x, B \nabla \rangle$ where A has *variable* coefficients; on the other hand, the constance of the matrix B is a crucial feature of this class of operators.

If we define the matrix:

$$C(t) = \int_0^t E(s)\, A E^T(s)\, ds, \text{ where } E(s) = \exp\left(-sB^T\right)$$

then it can be proved (Lanconelli-Polidoro, 1994) the equivalence between the three conditions:

1. the operator $\mathcal{A}$ is hypoelliptic ($\mathcal{A}u \in C^\infty(\Omega) \Longrightarrow u \in C^\infty(\Omega)$, for any open $\Omega \subset \mathbb{R}^N$), and the same holds for L;

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2. $C(t) > 0$ for any $t > 0$;
3. the following Hörmander's condition holds:

$$\text{rank} \qquad \mathcal{L}(X_1, X_2, ..., X_N, Y_0) = N, \text{ at any } x \in \mathbb{R}^N, \quad \text{where}$$

$$Y_0 = \langle x, B\nabla \rangle \quad \text{and } X_i = \sum_{j=1}^N a_{ij} \partial_{x_j} \quad i = 1, 2, ..., N.$$

For instance, in Kolmogorov' example:

$$\mathcal{A}=\partial_{vv}^2 + v\partial_x$$

- Condition 1: $\exists$ a fundamental solution smooth outside the pole

$$\Gamma\left((v,x,t);(0,0,0)\right) = \frac{c}{t^2}\exp\left(-\frac{v^2t^2+3vxt+3x^2}{t^3}\right) \quad t>0$$

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- Condition 2:

$$B=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}; \exp\left(-sB^T\right)=\begin{bmatrix}1 & -s\\ -s & s^2\end{bmatrix}; C(t)=\begin{bmatrix}t & -\frac{t^2}{2}\\ -\frac{t^2}{2} & \frac{t^3}{3}\end{bmatrix}$$

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- Condition 3:

$$Y_0 = v\partial_x; X_1 = \partial_v$$
$$[X_1, Y_0] \equiv X_1Y_0 - Y_0X_1 = \partial_x;$$

therefore $Y_0, X_1, [X_1, Y_0]$ span $\mathbb{R}^2$, that is $\text{rank}\mathcal{L}(X_1, Y_0) = 2$.

Under one of these conditions it is proved by Lanconelli-Polidoro (1994) that, for some basis of $\mathbb{R}^N$, the matrices A, B take the following form:

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

with $A_0 = (a_{ij})_{i,j=1}^{p_0}$ $p_0 \times p_0$ constant matrix ($p_0 \le N$), symmetric and positive definite:

$$\nu \left|\xi\right|^2 \le \sum_{i,j=1}^{p_0} a_{ij}\xi_i\xi_j \le \frac{1}{\nu}\left|\xi\right|^2 \quad \forall \xi \in \mathbb{R}^{p_0}, \text{some positive constant } \nu;$$

$$B = \begin{bmatrix} * & B_1 & 0 & \dots & 0 \\ * & * & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & B_r \\ * & * & * & \dots & * \end{bmatrix} \tag{2}$$

where B_j is a $p_{j-1} \times p_j$ block with rank p_j, $j = 1, 2, ..., r$, $p_0 \ge p_1 \ge ... \ge p_r \ge 1$ and $p_0 + p_1 + ... + p_r = N$.

Here we consider hypoelliptic degenerate Ornstein-Uhlenbeck operators, with the matrices A, B already written as above.

$$\mathcal{A} \equiv \sum_{i,j=1}^{p_0} a_{ij} \partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j}$$

with (a_{ij}) positive on $\mathbb{R}^{p_0}$. For this class of operators, we will prove the following global L^p estimates:

Theorem

For any $p \in (1, \infty)$ there exists a constant $c > 0$, depending on p, N, p_0, the matrix B and the number ν such that for any $u \in C_0^\infty(\mathbb{R}^N)$ one has:

$$\left\| \partial^2_{x_i x_j} u \right\|_{L^p(\mathbb{R}^N)} \leq c \left\{ \| \mathcal{A} u \|_{L^p(\mathbb{R}^N)} + \| u \|_{L^p(\mathbb{R}^N)} \right\} \text{ for } i, j = 1, 2, \ldots, p_0 \quad (3)$$

$$\| Y_0 u \|_{L^p(\mathbb{R}^N)} \leq c \left\{ \| \mathcal{A} u \|_{L^p(\mathbb{R}^N)} + \| u \|_{L^p(\mathbb{R}^N)} \right\}. \quad (4)$$

Also, an analogous weak $(1, 1)$ estimates hold.

Comparison with the existing literature

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- L^p estimates in the nondegenerate case $p_0 = N$ have been proved by Metafune, Prüss, Rhandi and Schnaubelt (2002, Ann. Sc. Norm. Sup. Pisa) by a semigroup approach.
- Note that, even in the nondegenerate case, global estimates in L^p or Hölder spaces are not straightforward, due to the unboundedness of the first order coefficients.

Relation with the evolution operator

We will deduce global estimates for $\mathcal{A}$ from an analogous estimate for $L = \mathcal{A} - \partial_t$ on the strip

$$S \equiv \mathbb{R}^N \times [-1, 1],$$

which can be of independent interest:

Theorem

For any $p \in (1, \infty)$ there exists a constant $c > 0$ such that

$$\left\| \partial^2_{x_i x_j} u \right\|_{L^p(S)} \leq c \, \| Lu \|_{L^p(S)} \quad \text{for } i, j = 1, 2, ..., p_0, \tag{5}$$

for any $u \in C_0^\infty(S)$.

Namely, let

$$\psi \in C_0^\infty(\mathbb{R})$$

be a cutoff function fixed once and for all, $\operatorname{sprt}\psi \subset [-1, 1]$, $\int_{-1}^{1} \psi(t)\, dt > 0$. If $u : \mathbb{R}^N \to \mathbb{R}$ is a C_0^∞ solution to the equation

$$\mathcal{A}u = f \text{ in } \mathbb{R}^N,$$

for some $f \in L^p(\mathbb{R}^N)$, let

$$U(x, t) = u(x)\, \psi(t);$$

then

$$LU(x, t) = f(x)\, \psi(t) - u(x)\, \psi'(t) \equiv F(x, t).$$

Therefore (5) applied to U gives

$$\left\| \partial^2_{x_i x_j} U \right\|_{L^p(S)} \leq c \, \| F \|_{L^p(S)} \quad \text{for } i, j = 1, 2, ..., p_0 \tag{6}$$

hence

$$\left\| \partial^2_{x_i x_j} u \right\|_{L^p(\mathbb{R}^N)} \leq c \left\{ \| f \|_{L^p(\mathbb{R}^N)} + \| u \|_{L^p(\mathbb{R}^N)} \right\}$$

with c also depending on ψ.

The geometric framework: the group of translations

Lanconelli-Polidoro (1994) proved that the operator L is left invariant with respect to the Lie-group translation

$$
\begin{aligned}
(x,t)\circ(\xi,\tau) &= (\xi + E(\tau)\,x, t+\tau)\,;\\
(\xi,\tau)^{-1} &= (-E(-\tau)\,\xi, -\tau)\,,\ \text{where}\\
E(\tau) &= \exp\left(-\tau B^T\right). \qquad (\text{Recall } \mathcal{A} = div\,(A\nabla) + \langle x, B\nabla\rangle\,)
\end{aligned}
$$

The geometric framework: dilations and principal part operator

Let us consider our operator L, with the matrices A, B written in the form (1), (2). Let B_0 the matrix obtained by annihilating every $*$ block in B:

$$B_0 = \begin{bmatrix} 0 & B_1 & 0 & \dots & 0 \\ 0 & 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_r \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{7}$$

By *principal part* of L we mean the operator

$$L_0 = div\,(A\nabla) + \langle x, B_0\nabla\rangle - \partial_t. \tag{8}$$

For any $\lambda > 0$, let us define the matrix of *dilations on* $\mathbb{R}^N$ (depending on B_0)

$$D(\lambda) = diag\left(\lambda I_{p_0}, \lambda^3 I_{p_1}, \dots, \lambda^{2r+1} I_{p_r}\right)$$

and the matrix of *dilations on* $\mathbb{R}^{N+1}$,

$$\delta(\lambda) = diag\left(\lambda I_{p_0}, \lambda^3 I_{p_1}, ..., \lambda^{2r+1} I_{p_r}, \lambda^2\right). \text{ Note: } \det(\delta(\lambda)) = \lambda^{Q+2}$$
$$\text{where } Q+2 = p_0 + 3p_1 + ... + (2r+1)\,p_r + 2$$

is called *homogeneous dimension of* $\mathbb{R}^{N+1}$.
A remarkable fact proved by Lanconelli-Polidoro (1994) is that the operator L_0 is homogeneous of degree two with respect to the dilations $\delta(\lambda)$:

$$L_0\,(u\,(\delta(\lambda)\,z)) = \lambda^2\,(L_0 u)\,(\delta(\lambda)\,z) \quad \forall u \in C_0^\infty\left(\mathbb{R}^{N+1}\right), z \in \mathbb{R}^{N+1}, \lambda > 0.$$

The operator L_0 is also left invariant with respect to the translations induced by the matrix B_0 (not B!):

$$(x,t) \odot (\xi,\tau) = (\xi + E_0(\tau)\,x, t+\tau) \text{ where } E_0(s) = \exp\left(-sB_0^T\right)$$
$$(\xi,\tau)^{-1} = (-E_0(-\tau)\,\xi, -\tau).$$

Moreover, the dilations $z \mapsto \delta(\lambda)\,z$ are automorphisms for the group $\left(\mathbb{R}^{N+1}, \odot\right)$.

An example of the previous facts

$$Lu = \partial_{xx}^2 u + x\partial_y u + \boxed{x\partial_x u} - \partial_t u;$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; B = \begin{bmatrix} \boxed{1} & 1 \\ 0 & 0 \end{bmatrix}; E(s) = I + \left(e^{-s} - 1\right) B^T;$$

$$(x, y, t) \circ (x', y', t') =$$

$$\left(x + x' + \left(e^{-t'} - 1\right) x, y + y' + \left(e^{-t'} - 1\right) x, t + t'\right)$$

$$L_0 u = \partial_{xx}^2 u + x\partial_y u - \partial_t u = 0;$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; B_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; E_0(s) = I - sB^T;$$

$$(x, y, t) \odot (x', y', t') = (x + x', y + y' - t'x, t + t')$$

$$\delta(\lambda)(x, y, t) = (\lambda x, \lambda^3 y, \lambda^2 t)$$

Known results for the principal part operator

- The principal part operator L, left invariant w.r.t. a group of translation and homogeneous of degree 2 w.r.t. a family of dilations, has been extensively studied in the last decades.

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- In this general case there is no reasonable hope of proving global L^p estimates on $\mathbb{R}^{N+1}$ for the evolution operator; the best one can hope (and what we actually do) is to prove L^p estimates on a strip $\mathbb{R}^N \times [-1, 1]$ for L, and to deduce global estimates on $\mathbb{R}^N$ for the stationary operator.

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- Acually, our result seems to be the first case of global L^p estimates for $\mathcal{A}$, proved without an underlying homogeneous group.

Fundamental solution

Theorem

Under the previous assumptions, the operator L possesses a fundamental solution

$$\Gamma(z,\zeta) = \gamma\left(\zeta^{-1} \circ z\right) \ \textit{for } z,\zeta \in \mathbb{R}^{N+1},$$

with

$$\gamma(z) = \begin{cases} 0 \ \textit{for } t \leq 0 \\ \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\left\langle C^{-1}(t)\,x, x\right\rangle - t\,TrB\right) & \textit{for } t > 0 \end{cases}$$

where $z = (x,t)$. *Recall that*

$$C(t) = \int_0^t E(s)\,AE^T(s)\,ds, \ \textit{where } E(s) = \exp\left(-sB^T\right)$$

and $C(t) > 0$ *for any* $t > 0$*; hence* $\gamma \in C^\infty\left(\mathbb{R}^{N+1} \setminus \{0\}\right)$.

Theorem (continued)

The following representation formulas hold:

$$u(z) = -(\gamma * Lu)(z) = -\int_{\mathbb{R}^{N+1}} \gamma\left(\zeta^{-1} \circ z\right) Lu(\zeta)\, d\zeta; \qquad (9)$$

$$\partial^2_{x_i x_j} u(z) = -PV\left(\partial^2_{x_i x_j}\gamma * Lu\right)(z) + c_{ij} Lu(z) \qquad (10)$$

for any $u \in C_0^\infty\left(\mathbb{R}^{N+1}\right)$, $i, j = 1, 2, ..., p_0$, *for suitable constants* c_{ij}. *The "principal value" in (10) must be understood as*

$$PV\left(\partial^2_{x_i x_j}\gamma * Lu\right)(z) \equiv \lim_{\varepsilon \to 0} \int_{d(z,\zeta)>\varepsilon} \left(\partial^2_{x_i x_j}\gamma\right)\left(\zeta^{-1} \circ z\right) Lu(\zeta)\, d\zeta.$$

The above theorem is proved by Hörmander (1967) (see also Lanconelli-Polidoro (1994)), apart from (10) which is proved in Di Francesco-Polidoro (2006). (*I will define later this* d).

The fundamental solution $\Gamma_0(z,\zeta) = \gamma_0(\zeta^{-1} \circ z)$ of the principal part operator L_0 enjoys special properties:

- for $t > 0$

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- Some relations link L to L_0: for $t \to 0$,

$$\langle C(t)\,x,x\rangle = \langle C_0(t)\,x,x\rangle\,(1+O(t));$$
$$\left\langle C^{-1}(t)\,x,x\right\rangle = \left\langle C_0^{-1}(t)\,x,x\right\rangle(1+O(t));$$
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$$\det C (t) = \det C_0 (t) (1 + O (t)).$$

- All the estimates on γ that we will need in the following are proved exploiting these relations between γ and γ_0, since γ_0 is easier to handle.

Estimate on the nonsingular part of the integral

We now localize the singular kernel $\partial^2_{x_i x_j}\gamma$ introducing a cutoff function

$$\eta \in C_0^\infty\left(\mathbb{R}^{N+1}\right) \text{ such that}$$
$$\eta(z) = 1 \text{ for } d(z,0) \le \rho_0/2;$$
$$\eta(z) = 0 \text{ for } d(z,0) \ge \rho_0,$$

where $\rho_0 \le 1$ will be fixed later.
Let us rewrite the representation formula as:

$$\begin{aligned}\partial^2_{x_i x_j} u &= -PV\left(\left(\eta\partial^2_{x_i x_j}\gamma\right) * Lu\right) - \left((1-\eta)\,\partial^2_{x_i x_j}\gamma * Lu\right) + c_{ij} Lu \\ &\equiv -PV\left(k_0 * Lu\right) - \left(k_\infty * Lu\right) + c_{ij} Lu\end{aligned} \quad (11)$$

having set:

$$\begin{aligned} k_0 &= \eta\partial^2_{x_i x_j}\gamma \\ k_\infty &= (1-\eta)\,\partial^2_{x_i x_j}\gamma \end{aligned} \quad (12)$$

for any $i, j = 1, 2, ..., p_0$.

Since in k_∞ the singularity of $\partial^2_{x_i x_j}\gamma$ has been removed and $\partial^2_{x_i x_j}\gamma$ has a fast decay as $x \longrightarrow \infty$, we can prove the following:

Theorem

For any $\rho_0 > 0$ *there exists* $c = c(\rho_0) > 0$ *such that for any* $z \in S$

$$\int_S \left| k_\infty \left(\zeta^{-1} \circ z\right)\right| d\zeta \leq c \tag{13}$$

$$\int_S \left| k_\infty \left(z^{-1} \circ \zeta\right)\right| d\zeta \leq c. \tag{14}$$

This immediately implies the following:

Corollary

For any $p \in [1, \infty]$ *there exists a constant* $c > 0$ *only depending on* p, N, p_0, ν *and the matrix* B *such that:*

$$\left\| -\left(k_\infty * Lu\right) + c_{ij} Lu \right\|_{L^p(S)} \leq c \left\| Lu \right\|_{L^p(S)} \text{ for any } u \in C_0^\infty\left(S\right), \tag{15}$$

any $i, j = 1, ..., p_0$.

Estimates on the singular kernel

By the representation formula and the last Corollary, our final goal will be achieved as soon as we will prove that

$$\|PV(k_0 * Lu)\|_{L^p(S)} \leq c\,\|Lu\|_{L^p(S)} \tag{16}$$

for any $u \in C_0^\infty(S)$, $i, j = 1, ..., p_0, 1 < p < \infty$.

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- Next steps are the following:
- We will prove that our singular kernel $k_0 = \eta \cdot \partial^2_{x_i x_j}\Gamma$, satisfies "standard estimates" (in the language of singular integrals theory) with respect to a suitable "quasidistance" d, which is a key geometrical object in our study.

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- We will study this object d, to understand which kind of abstract result about singular integrals can be applied, to prove (16)

The geometric framework: the "hybrid" quasidistance

- There is a natural homogeneous norm in $\mathbb{R}^{N+1}$, induced by the dilations $\delta(\lambda)$ associated to L_0:

$$\|(x,t)\| = \sum_{j=1}^{N} |x_j|^{1/q_j} + |t|^{1/2}$$

where q_j are the integers in $\delta(\lambda) = diag\left(\lambda^{q_1}, ..., \lambda^{q_N}, \lambda^2\right)$, and

$$\|\delta(\lambda) z\| = \lambda \|z\| \quad \text{for any } \lambda > 0, z \in \mathbb{R}^{N+1}.$$

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  - $\|\cdot\|$ is the homogeneous norm related to the principal part operator L_0 (i.e. to the matrix B_0)
  - recall that L *is not homogeneous* w.r.t. a family of dilations, and therefore does not have a natural homogeneous norm.

Estimates on the fundamental solution

Exploiting the relations between Γ (fund. sol. of L) and Γ_0 (fund. sol. of L_0), one can prove:

Theorem

(Di Francesco-Polidoro) The following "standard estimates" hold for Γ in terms of d: there exist $c > 0$ and $M > 1$ such that

$$\left|\partial^2_{x_i x_j}\Gamma(z,\zeta)\right| \leq \frac{c}{d(z,\zeta)^{Q+2}} \quad \forall z,\zeta \in S$$

$$\left|\partial^2_{x_i x_j}\Gamma(\zeta,w) - \partial^2_{x_i x_j}\Gamma(z,w)\right| \leq c\frac{d(w,z)}{d(w,\zeta)^{Q+3}} \quad \forall z,\zeta,w \in S$$

with $Md(w,z) \leq d(w,\zeta) \leq 1$.

An easy computation shows that the previous estimates extend to the kernel $k_0 = \eta\partial^2_{x_i x_j}\gamma$.

We can also prove the following:

Lemma

There exists $c > 0$ such that

$$\left| \int_{r_1 < \left\| \zeta^{-1} \circ z \right\| < r_2} k_0 \left(\zeta^{-1} \circ z \right) d\zeta \right| \leq c$$

for any $z \in S, 0 < r_1 < r_2$. Moreover, for every $z \in S$, the limit

$$\lim_{\varepsilon \to 0^+} \int_{\left\| \zeta^{-1} \circ z \right\| > \varepsilon} k_0 \left(\zeta^{-1} \circ z \right) d\zeta$$

exists, is finite, and independent of z.

- Hence, our singular integral satisfies good properties w.r.t. d.

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- What about the properties of d?

"Locally quasisymmetric quasidistance"

Theorem

(Di Francesco-Polidoro). For any $z, w, \zeta \in S$ with $d(z,w) \leq 1, d(\zeta,w) \leq 1$,

$$d(z,w) \leq c\, d(w,z)$$
$$d(z,\zeta) \leq c\left\{d(z,w) + d(w,\zeta)\right\}$$

Let us define the d-balls:

$$B(z,\rho) = \left\{\zeta \in \mathbb{R}^{N+1} : d(z,\zeta) < \rho\right\}.$$

Then the d-balls are open with respect to the Euclidean topology. Moreover, the topology induced by this family of balls coincides with the Euclidean topology.

Volume of metric balls

We can easily prove the following facts, about the measure of d-balls:

Theorem

(i) The following dimensional bound holds the measure of d-balls:

$$|B(z,\rho)| \leq c\rho^{Q+2} \qquad \forall z \in S, 0 < \rho < 1.$$

(ii) The following doubling condition holds in S:

$$|B(z,2\rho) \cap S| \leq c\,|B(z,\rho) \cap S| \qquad \forall z \in S, 0 < \rho < 1.$$

- Summarizing, we have:

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- Summarizing, we have:
  - a function d which is locally a (quasisymmetric) quasidistance, and
  - the Lebesgue measure which is locally doubling.
- Can we say that, for a fixed bounded cylinder $Q = \Omega \times [-1,1] \subset \mathbb{R}^{N+1}$, $(Q, d, dxdt)$ is a space of homogeneous type?

What's going wrong...

- To assure that d is a (quasisymmetric) quasidistance we have to choose (as "universe")

$$X = \Omega \times [-1, 1]$$

for some bounded domain $\Omega \subset \mathbb{R}^N$. In this space X, the balls are:

$$\widetilde{B}(z, \rho) = \{\zeta \in X : d(z, \zeta) < \rho\} = B(z, \rho) \cap X.$$

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- *We are forced to avoid using the doubling condition.*
- We can only rely on the upper bound:

$$|B(z, \rho) \cap X| \leq c\rho^{Q+2}.$$

Singular integrals on nonhomogeneous spaces

- The context we have just described is similar to that of *nonhomogeneous spaces*, which have been studied since the late 1990's by Nazarov-Treil-Volberg (Internat. Math. Res. Notices 1998), Tolsa (J. Reine Angew. Math. 1998), and other authors.

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    1. the proof of L^2 continuity for an operator with kernel satisfying standard estimates plus some kind of cancellation property;
    2. the proof of weak $(1,1)$ continuity for an operator which is continuous on L^2, with kernel satisfying standard estimates.

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  2. for the step $L^2 \Longrightarrow L^p$ they consider a separable metric space (X, d).
- In both cases, the measure usually satisfies the dimensional bound

$$\mu(B_r(x)) \leq cr^n \tag{17}$$

for some positive constants c, n, but can be nondoubling. The cancellation property considered in step (1) is usually very weak, inspired to the theorems $T(1)$, $T(b)$ or variants of them.

- In contrast with this setting, in our application it is essential to prove L^2 (and L^p) estimates in a bounded space endowed with a *general quasidistance* and a possibly nondoubling measure satisfying (17);

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- Our idea is to get a new proof of L^2 (and L^p) continuity for a Calderón-Zygmund operator, with the aforementioned features.
- This has been performed in the paper:
  M. Bramanti: *Singular integrals in nonhomogeneous spaces: and continuity from Hölder estimates.* To appear on Revista Matemàtica Iberoamericana.

Definition

We will say that (X, d, μ, k) is a *nonhomogeneous space with Calderón-Zygmund kernel* k if:

1. (X, d) is a set endowed with a quasisymmetric quasidistance d, such that the d-balls are open with respect to the topology induced by d;
2. μ is a positive regular Borel measure on X, and $\exists A, n > 0$ such that:

$$\mu(B(x,\rho)) \leq A\rho^n \text{ for any } x \in X, \rho > 0; \tag{18}$$

3. $k(x,y) : X \times X \to \mathbb{R}$ is a measurable kernel, and $\exists \beta > 0$ such that:

$$\begin{aligned} |k(x,y)| &\leq \frac{A}{d(x,y)^n} \text{ for any } x, y \in X; \\ |k(x,y) - k(x_0,y)| &\leq A\frac{d(x_0,x)^\beta}{d(x_0,y)^{n+\beta}} \end{aligned} \tag{19}$$

for any $x_0, x, y \in X$ with $d(x_0,y) \geq Ad(x_0,x)$, where n, A are as in (18)

Theorem

Let (X, d, μ, k) *be a bounded and separable nonhomogeneous space with Calderón-Zygmund kernel* k. *Also, assume that*
(i) $k^*(x, y) \equiv k(y, x)$ *satisfies (19);*
(ii) $\exists B > 0$ *such that* $\forall \rho > 0, x \in X$

$$\left| \int_{d(x,y)>\rho} k(x, y)\, d\mu(y) \right| + \left| \int_{d(x,y)>\rho} k^*(x, y)\, d\mu(y) \right| \leqslant B; \qquad (20)$$

(iii) for a.e. $x \in X$, *the limits*

$$\lim_{\rho \to 0} \int_{d(x,y)>\rho} k(x, y)\, d\mu(y); \quad \lim_{\rho \to 0} \int_{d(x,y)>\rho} k^*(x, y)\, d\mu(y)$$

exist finite.

Theorem (continued)

Then the operator

$$Tf(x) \equiv \lim_{\varepsilon \to 0} T_\varepsilon f(x) \equiv \lim_{\varepsilon \to 0} \int_{d(x,y)>\varepsilon} k(x,y) f(y) \, d\mu(y)$$

is well defined for any $f \in L^1(X)$, *and*

$$\|Tf\|_{L^p(X)} \leq c_p \|f\|_{L^p(X)} \text{ for any } p \in (1, \infty);$$

moreover, T *is weakly* $(1,1)$ *continuous. The constant* c_p *only depends on all the constants implicitly involved in the assumptions:* p, c_d, A, B, n, β, *diam*(X).

Conclusion of the proof of Lp estimates on the strip

- Thanks to the abstract result proved for singular integrals in nonhomogeneous spaces, we are able to prove the following local estimate for our singular integral:

Corollary

Let k_0 be our singular kernel. $\exists R_0 > 0$ s.t., $\forall z_0 \in S$, $R \leq R_0$, if a, b are two cutoff functions in $C^\alpha\left(\mathbb{R}^{N+1}\right)$ for some $\alpha > 0$, with $\operatorname{sprt} a$, $\operatorname{sprt} b \subset B(z_0, R)$, and we set

$$
\begin{aligned}
k(z,\zeta) &= a(z)\, k_0\left(\zeta^{-1} \circ z\right) b(\zeta); \\
Tf(z) &= PV \int_{B(z_0,R)} k(z,\zeta)\, f(\zeta)\, d\zeta,
\end{aligned}
$$

then $\forall p \in (1,\infty)$ $\exists c > 0$ (independent of z_0 and R) such that

$$
\|Tf\|_{L^p(B(z_0,R))} \leq c\, \|f\|_{L^p(B(z_0,R))} \qquad \forall f \in L^p(B(z_0,R)).
$$

- To deduce from this local estimate our desired estimate on the whole strip, one can think to apply a standard covering argument and cutoff function. Actually this works, but it is not trivial:

Lemma

For every $r_0 > 0$ and $K > 1$ there exist $\rho \in (0, r_0)$, a positive integer M and a sequence of points $\{z_i\}_{i=1}^{\infty} \subset S$ such that:

$$S \subset \bigcup_{i=1}^{\infty} B(z_i, \rho); \quad \sum_{i=1}^{\infty} \chi_{B(z_i, K\rho)}(z) \leq M \quad \forall z \in S.$$

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For every $r_0 > 0$ and $K > 1$ there exist $\rho \in (0, r_0)$, a positive integer M and a sequence of points $\{z_i\}_{i=1}^{\infty} \subset S$ such that:

$$S \subset \bigcup_{i=1}^{\infty} B(z_i, \rho); \quad \sum_{i=1}^{\infty} \chi_{B(z_i, K\rho)}(z) \leq M \quad \forall z \in S.$$

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  - the Lebesgue measure is locally doubling.

Strategy used to prove the Lp singular integral estimate in nonhomogeneous spaces

Step 1. Thanks to the cancellation property we assume, it is not difficult to prove, also in the nondoubling context and for any quasidistance, that the singular integral operator (or a suitable variant of this) is continuous on Hölder spaces $C^{\alpha}(X)$, where X is our nonhomogeneous space.

Theorem

Let (X, d, μ, k) be a bounded nonhomogeneous space with Calderón-Zygmund kernel k. For any $f \in C^{\alpha}(X)$, let

$$\widehat{T}f(x) = \int k(x,y)\,[f(y) - f(x)]\,d\mu(y)$$

(a) Then the integral defining $\widehat{T}f(x)$ is absolutely convergent for any $f \in C^{\alpha}(X), \alpha > 0, x \in X$.

Theorem (continued)

(b) Assume there exists a constant $B > 0$ such that

$$\left| \int_{d(x,y)>r} k(x,y)\, d\mu(y) \right| \leqslant B \quad \forall r > 0, x \in X. \tag{21}$$

Then the operator $\widehat{T}$ is continuous on $C^{\alpha}(X)$ for any $\alpha < \beta$ (β being the exponent in the mean value inequality (19) of k). More precisely:

$$\left| \widehat{T} f \right|_{C^{\alpha}(X)} \leqslant c \left| f \right|_{C^{\alpha}(X)}$$

$$\left\| \widehat{T} f \right\|_{\infty} \leqslant c R^{\alpha} \left| f \right|_{C^{\alpha}(X)}$$

where $R = \text{diam}(X)$, and c depends on $A, B, c_d, n, \alpha, \beta$.

Step 2. An abstract argument originally due to Krein (1947) allows to deduce the continuity of the singular integral operator on $L^2(X)$ from that on $C^\alpha(X)$.

This idea has been applied, in the doubling context, by Wittman (1987) and Fabes-Mitrea-Mitrea (1999) but perhaps this approach is not widely known.

Theorem

(Krein) Let H be a (real, for simplicity) Hilbert space and Y a linear normed space for which the inclusion $i : Y \to H$ is well defined, continuous and with dense range. Let $T, T^* : Y \to Y$ be two linear continuous operators on Y such that

$$(Tx, y) = (x, T^* y) \text{ for any } x, y \in Y,$$

where $(,)$ denotes the scalar product in H. Then T and T^* extend to linear continuous operators on H, with

$$\|T\|_{H\to H}, \|T^*\|_{H\to H} \leq \|T\|_{Y\to Y}^{1/2} \cdot \|T^*\|_{Y\to Y}^{1/2}.$$

Applying the theorem to $H = L^2(X)$, $Y = C^\alpha(X)$, $\widehat{T}$, we get the L^2 continuity of $\widehat{T}$ and therefore of T, because:

$$T_\varepsilon f(x) \equiv \int_{d(x,y)>\varepsilon} k(x,y) f(y)\, d\mu(y) =$$
$$= \int_{d(x,y)>\varepsilon} k(x,y)[f(y) - f(x)]\, d\mu(y) + f(x) \int_{d(x,y)>\varepsilon} k(x,y)\, d\mu(y);$$

hence, there exists

$$Tf(x) \equiv \lim_{\varepsilon \to 0} T_\varepsilon f(x) = \widehat{T} f(x) + f(x) h(x)$$

and $h \in L^\infty(X)$. Hence

$$\|Tf\|_{L^2(X)} \leq \left\| \widehat{T} f \right\|_{L^2(X)} + \|h\|_{L^\infty(X)} \|f\|_{L^2(X)} \tag{22}$$

Step 3. Once the L^2 continuity is proved, the weak $(1,1)$ continuity result proved by Nazarov-Treil-Volberg can be applied, with some minor adaptation: one has to check that their arguments actually work for any quasidistance, and not necessarily in a metric space. This immediately implies the desired L^p estimate.