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Interior $HW^{1,p}$ estimates for divergence degenerate elliptic systems in Carnot groups☆

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ABSTRACT

Let $X_1, \ldots, X_q$ be the basis of the space of horizontal vector fields on a homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^n, \circ)$ $(q < n)$. We consider the following divergence degenerate elliptic system

$$\sum_{\beta=1}^{N} \sum_{i,j=1}^{q} X_i \left(a_{\alpha\beta}^{ij}(x) X_j u^\beta \right) = \sum_{i=1}^{q} X_i f_\alpha^i, \quad \alpha = 1, 2, \ldots, N$$

where the coefficients $a_{\alpha\beta}^{ij}$ are real valued bounded measurable functions defined in $\Omega \subset \mathbb{G}$, satisfying the strong Legendre condition and belonging to the space $VMO_{loc}(\Omega)$ (defined by the Carnot–Carathéodory distance induced by the X_i's). We prove interior $HW^{1,p}$ estimates $(2 \le p < \infty)$ for weak solutions to the system.

1. Introduction

Let

$$X_i = \sum_{j=1}^{n} b_{ij}(x)\, \partial_{x_j}, \quad i = 1, 2, \ldots, q,$$

be a family of real smooth vector fields defined in some bounded domain $\Omega \subset \mathbb{R}^n$ $(q < n)$ and satisfying Hörmander's condition: the Lie algebra generated by $X_1, \ldots, X_q$ spans $\mathbb{R}^n$ at any point of Ω. Since Hörmander's famous paper [24], there has been tremendous work on the geometric properties of Hörmander's vector fields; see [28,25,20–22,26,27], and references therein. Meanwhile, regularity for linear degenerate elliptic equations involving vector fields has been investigated and many results have been proved; see for instance [19,30,2–7,33,26,27] and references therein; as for subelliptic systems structured on Hörmander's vector fields, we can quote [17,35,31].

In this paper we consider divergence degenerate elliptic systems structured on Hörmander's vector fields in Carnot groups. Namely (here we briefly state our assumptions and result; precise definitions and assumptions will be given in Section 2.1), let $X_1, \ldots, X_q$ be the canonical basis of the space of horizontal vector fields in a homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^n, \circ)$; we consider the system

$$X_i \left(a_{\alpha\beta}^{ij}(x) X_j u^\beta \right) = X_i f_\alpha^i \tag{1.1}$$

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in some domain $\Omega \subset \mathbb{R}^n$ where $\alpha, \beta = 1, \ldots, N$, $i, j = 1, 2, \ldots, q$, $\mathbf{F} = \left(f_\alpha^i\right) \in L^p\left(\Omega; \mathbb{M}^{N\times q}\right)$ $(2 \leq p < \infty)$ is a given $N \times q$ matrix. In (1.1) and throughout the paper, the summation is understood for repeated indices. If the tensor $\left\{a_{\alpha\beta}^{ij}(x)\right\}$ satisfies the strong Legendre condition (see (2.7)), by the Lax–Milgram theorem the natural functional framework for solutions to (1.1) is the Sobolev space $HW_{loc}^{1,2}\left(\Omega; \mathbb{R}^N\right)$, so the regularity problem for (1.1) amounts to asking: if $\mathbf{F} \in L^p\left(\Omega; \mathbb{M}^{N\times q}\right)$ for some $p > 2$, can we say that $u \in HW^{1,p}$, at least locally? We will prove an affirmative answer to this question (see Theorem 2.12), under the assumption that the coefficients $a_{\alpha\beta}^{ij}$ belong to the space $VMO_{loc}(\Omega)$, with respect to the Carnot–Carathéodory distance induced by the vector fields. Under this respect, this result is in the same spirit as the L^p regularity results proved for nonvariational elliptic equations by Chiarenza et al. [14,15], for elliptic systems by Chiarenza et al. [13] (see also [12]), and for nondivergence equations structured on Hörmander's vector fields by Bramanti and Brandolini [2,3], while analogous regularity estimates in Morrey spaces have been proved for instance by Di Fazio et al. in [18], and by Palagachev–Softova in [29]. However, the technique of the proof in the present case is completely different. Namely, while in all the aforementioned papers L^p or Morrey estimates are proved by exploiting representation formulas for solutions and singular integral estimates, in the case of subelliptic *systems*, even on Carnot groups, no result about representation formulas by means of homogeneous fundamental solutions seems to be known. Hence we have to make use of a different technique, which has been designed and exploited in a series of papers by Byun–Wang to deal with elliptic equations and systems, also in very rough domains; see [34,8,9] and references therein. Namely, the key technical point is a series of local estimates involving the *maximal function* of $|Xu|^2$ (Sections 4 and 5) which hold under an assumption of smallness of the mean oscillation of the coefficients. One of the tools used to prove these local estimates is the possibility of approximating, locally, the solution to a system with small datum and small oscillation of the coefficients by the solution to a different system, with *constant* coefficients (Section 3). In turn, the solution to a constant coefficients system on a Carnot group is known to satisfy an L^∞ gradient bound (see Theorem 2.10) which turns out to be a key tool in our proof. This result about systems with constant coefficients in Carnot groups has been proved by Shores [31], and represents one of the main reasons why we have restricted ourselves to the case of Carnot groups instead of considering general Hörmander's vector fields.

This paper represents the first case of study of L^p estimates on the "subelliptic gradient" Xu for subelliptic systems. Di Fazio and Fanciullo in [17] have deduced interior Morrey regularity in spaces $L^{2,\lambda}$ for weak solutions to the system (1.1) under the assumption that the coefficients $a_{\alpha\beta}^{ij}$ belong to the class $VMO_X \cap L^\infty$, while Schauder-type estimates have been proved for subelliptic systems by Xu and Zuily [35].

This paper is organized as follows. In Section 2 we recall some basic facts about Carnot groups and state precisely our assumptions and main results; in Section 3 we prove the approximation result for local solutions to the original system by means of solutions to a system with constant coefficients; in Section 4 we prove some local estimates on the Hardy–Littlewood maximal function of $|Xu|^2$, and in Section 5 we come to the proof of our main result.

2. Preliminaries and statement of the results

2.1. Background on Carnot groups

We are going to recall here a few facts about Carnot groups that we will need in the following. For the proofs, more properties, and examples, we refer the reader to the paper [19], the books [1] and [32, Chapters XII–XIII].

Definition 2.1 (*Homogeneous Carnot Groups*)**.** A *homogeneous group* $\mathbb{G}$ is the set $\mathbb{R}^n$ endowed with a Lie group operation $\circ$ ("translation"), where the origin is the group identity, and a family $\{D(\lambda)\}_{\lambda>0}$ of group automorphisms ("dilations"), acting as follows:

$$D(\lambda)(x_1, x_2, \ldots, x_n) = (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \ldots, \lambda^{\alpha_n} x_n) \quad \forall \lambda > 0$$

for some fixed exponents $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$. The number $Q = \sum_{j=1}^n \alpha_j$ is called the *homogeneous dimension* of $\mathbb{G}$.

We say that a vector field $X = \sum_{j=1}^n b_j(x)\,\partial_{x_j}$ is left invariant if for any smooth function f one has

$$X^x(f(y \circ x)) = (Xf)(y \circ x) \quad \forall x, y \in \mathbb{G};$$

we say that X is k-homogeneous if for any smooth function f one has

$$X(f(D(\lambda)x)) = \lambda^k (Xf)(D(\lambda)x) \quad \forall \lambda > 0, x \in \mathbb{G}.$$

Let X_i $(i = 1, 2, \ldots, n)$ be the unique left invariant vector field on $\mathbb{G}$ which at the origin coincides with ∂_{x_i}. We assume that for some integer $q < n$ the vector fields $X_1, X_2, \ldots, X_q$ are 1-homogeneous and satisfy Hörmander's condition in $\mathbb{R}^n$: the Lie algebra generated by the X_i's at any point has dimension n. Under these assumptions we say that $\mathbb{G}$ is a *homogeneous Carnot group* and that $\left\{X_1, X_2, \ldots, X_q\right\}$ is the *canonical basis* of the space of horizontal vector fields.

The properties required in the above definition have a number of consequences: the exponents α_i are actually positive integers, the Lie algebra of $\mathbb{G}$ is stratified, homogeneous and nilpotent; the vector fields X_i have polynomial coefficients. Moreover, the Lebesgue measure of $\mathbb{R}^n$ is the Haar measure in $\mathbb{G}$.

Like for any set of Hörmander's vector fields, it is possible to define the corresponding *Carnot–Carathéodory distance* d_X, as follows.

Definition 2.2 (*CC-distance*). For any $\delta > 0$, let C_δ be the set of absolutely continuous curves $\phi : [0, 1] \to \mathbb{R}^n$ such that

$$\phi'(t) = \sum_{i=1}^{q} a_i(t) X_i(\phi_i(t)) \quad \text{with } |a_i(t)| \le \delta \quad \text{for a.e. } t \in [0, 1].$$

Then

$$d_X(x, y) = \inf\{\delta > 0 : \exists \phi \in C_\delta \text{ with } \phi(0) = x, \phi(1) = y\}.$$

The function d_X turns out to be finite for any couple of points, and is actually a distance, called Carnot–Carathéodory distance; due to the structure of Carnot group, d_X is also left invariant and 1-homogeneous on $\mathbb{G}$. Let

$$B_r(x) = \{y \in \mathbb{G} : d_X(x, y) < r\}$$

be the metric ball of center x and radius r in $\mathbb{G}$. Since the Lebesgue measure in $\mathbb{R}^n$ is the Haar measure on $\mathbb{G}$, one has (writing $|A|$ for the measure of A)

$$|B_r(x)| = \omega_{\mathbb{G}} r^Q, \tag{2.1}$$

where Q is the homogeneous dimension of $\mathbb{G}$ and $\omega_{\mathbb{G}}$ is a positive constant.

Next, we need to define the function spaces we will use in the following.

Definition 2.3 (*Horizontal Sobolev Spaces*). For any $p \ge 1$ and domain $\Omega \subset \mathbb{G}$, let us define the *horizontal Sobolev space*:

$$HW^{1,p}(\Omega; \mathbb{R}^N) = \left\{u \in L^p(\Omega; \mathbb{R}^N) : \|u\|_{HW^{1,p}(\Omega;\mathbb{R}^N)} < \infty\right\}$$

with the norm

$$\|u\|_{HW^{1,p}(\Omega;\mathbb{R}^N)} = \|u\|_{L^p(\Omega;\mathbb{R}^N)} + \|Xu\|_{L^p(\Omega;\mathbb{R}^N)},$$

having set

$$\|u\|_{L^p(\Omega;\mathbb{R}^N)} = \||u|\|_{L^p(\Omega)}, \quad \text{with } |u| = \left(\sum_{\alpha=1}^{N} |u^\alpha|^2\right)^{1/2} \quad \text{and}$$

$$\|Xu\|_{L^p(\Omega;\mathbb{R}^N)} = \||Xu|\|_{L^p(\Omega)}, \quad \text{with } |Xu| = \left(\sum_{\alpha=1}^{N}\sum_{i=1}^{q} |X_i u^\alpha|^2\right)^{1/2}.$$

Also, we define the space $HW^{1,p}_{loc}(\Omega; \mathbb{R}^N)$ as the space of functions u such that $u\phi \in HW^{1,p}(\Omega; \mathbb{R}^N)$ for any $\phi \in C_0^\infty(\Omega)$ and the space $HW_0^{1,p}(\Omega; \mathbb{R}^N)$ as the closure of $C_0^\infty(\Omega; \mathbb{R}^N)$ in the norm $HW^{1,p}(\Omega; \mathbb{R}^N)$.

Definition 2.4 (*BMO-type Spaces*). For any $\Omega' \Subset \Omega$, let R_0 be a number such that $B_r(x) \Subset \Omega$ for any $x \in \Omega'$ and $r \le R_0$. For any $f \in L^1_{loc}(\Omega)$ and $r \le R_0$, let

$$\eta_{\Omega',R_0,f}(r) = \sup_{x_0\in\Omega', 0<\rho\le r} \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} \left|f(x) - f_{B_\rho(x_0)}\right|^2 dx,$$

where $f_{B_\rho(x_0)} = \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f(x)\,dx$.

We say that f is (δ, R)-vanishing in Ω' (for a couple of fixed positive numbers δ, R, with $R \le R_0$) if

$$\eta_{\Omega',R_0,f}(R) < \delta^2.$$

We say that $f \in VMO_{loc}(\Omega)$ if for any $\Omega' \Subset \Omega$ and R_0 such that $B_r(x) \Subset \Omega$ for any $r \le R_0$ and $x \in \Omega'$, we have

$$\eta_{\Omega',R_0,f}(r) \to 0 \quad \text{as } r \to 0.$$

The function $\eta_{\Omega',R_0,f}$ is called the local *VMO* modulus of f on Ω'.

We will use the following well-known result by Jerison (see [25, Theorem 2.1] for the case $p = 2$ and [25, Section 6] for $p \ne 2$).

Theorem 2.5 (*Poincaré's Inequality*). *For $1 \le p < \infty$ there exists a positive constant $c = c(\mathbb{G}, p)$, such that for any $u \in HW^{1,p}(B_R)$,*

$$\left\| u - u_{B_R} \right\|_{L^p(B_R)} \le cR \left\| Xu \right\|_{L^p(B_R)} . \tag{2.2}$$

If $u \in HW_0^{1,p}(B_R)$,

$$\left\| u \right\|_{L^p(B_R)} \le cR \left\| Xu \right\|_{L^p(B_R)} . \tag{2.3}$$

The previous theorem holds for a general system of Hörmander's vector fields; in that case, however, some restriction on the center and radius of the ball B_R applies (see [25, Theorem 2.1]); on a Carnot group, instead, due to the dilation invariance of the inequalities (2.2) and (2.3), these hold for any ball B_R and with an "absolute" constant c.

We will also make use of the following.

Definition 2.6 (*Space of Homogeneous Type, See [16]*). Let S be a set and $d : S \times S \to [0, \infty)$ a quasidistance, that is, for some constant $c \ge 1$ one has

$$\begin{aligned} & d(x, y) = 0 \Longleftrightarrow x = y \\ & d(x, y) = d(y, x) \\ & d(x, y) \le c\left[d(x, z) + d(z, y)\right] \end{aligned} \tag{2.4}$$

for all $x, y, z \in S$. The balls defined by d induce a topology in S; let us assume that the d-balls are open in this topology. Moreover, assume that there exists a regular Borel measure μ on S, such that the "doubling condition" is satisfied:

$$\mu\left(B_{2r}(x)\right) \le c\mu\left(B_r(x)\right), \tag{2.5}$$

for every $r > 0$, $x \in S$ and some positive constant c. Then we say that (S, d, μ) is a *space of homogeneous type*.

Remark 2.7. Note that in our context any Carnot–Carathéodory ball $B_R(x_0)$ is a d_X-regular domain (see for instance [4, Lemma 4.2]), that is there exists a positive constant c_d such that

$$\left|B_R(x_0) \cap B_r(x)\right| \ge c_d \left|B_r(x)\right| \quad \forall r > 0, \forall x \in B_R(x_0). \tag{2.6}$$

This implies that $(B_R(x_0), d_X, dx)$ is a space of homogeneous type. Moreover, a simple dilation argument shows that, in a Carnot group, the constant c_d, and therefore the doubling constant of $(B_R(x_0), d_X, dx)$, is independent of R.

2.2. Assumptions and known results about degenerate systems

The general assumptions which will be in force throughout the paper are collected in the following.

Assumption (H). We assume that $\mathbb{G}$ is a homogeneous Carnot group in $\mathbb{R}^n$ and $\left\{X_1, X_2, \ldots, X_q\right\}$ is the canonical basis of the space of horizontal vector fields in $\mathbb{G}$ (see Definition 2.1). We assume that the coefficients

$$\left\{a_{\alpha\beta}^{ij}\right\}_{\substack{i,j=1,\ldots,q \\ \alpha,\beta=1,\ldots,N}}$$

in (1.1) are real valued, bounded measurable functions defined in Ω and satisfying the *strong Legendre condition*: there exists a constant $\mu > 0$ such that

$$\mu|\xi|^2 \le a_{\alpha\beta}^{ij}(x)\xi_i^\alpha \xi_j^\beta \le \mu^{-1}|\xi|^2 \tag{2.7}$$

for any $\xi \in \mathbb{M}^{N\times q}$, a.e. $x \in \Omega$.

We recall the standard definition of weak solution.

Definition 2.8. We say that $u \in HW^{1,2}\left(\Omega; \mathbb{R}^N\right)$ is a weak solution to the system (1.1), if it satisfies

$$\int_\Omega a_{\alpha\beta}^{ij}(x) X_j u^\beta X_i \varphi^\alpha \, dx = \int_\Omega f_\alpha^i X_i \varphi^\alpha \, dx$$

for any $\varphi \in HW_0^{1,2}\left(\Omega; \mathbb{R}^N\right)$.

Recall that on a Carnot group the transpose of a vector field is just the opposite: $X_i^* = -X_i$. Hence the above definition of weak solution is consistent with the way the system (1.1) is written.

Remark 2.9. Let $B_R \subset \Omega$ be any metric ball. If $f_\alpha^i \in L^2(B_R)$ and $u_0 \in HW^{1,2}(B_R)$, then by assumption (2.7) and Poincaré's inequality (2.3) we can apply Lax–Milgram's theorem, and conclude that there exists a unique solution $u \in HW^{1,2}\left(B_R; \mathbb{R}^N\right)$ to system (1.1) such that $u - u_0 \in HW_0^{1,2}(B_R)$. Moreover, the following a priori estimate holds:

$$\|u\|_{HW^{1,2}(B_R;\mathbb{R}^N)} \le c\left(\|\mathbf{F}\|_{L^2(B_R;\mathbb{M}^{N\times q})} + \|u_0\|_{HW^{1,2}(B_R;\mathbb{R}^N)}\right) \tag{2.8}$$

for some constant c only depending on $\mathbb{G}$, μ, R (see [11, Chapter 8] for a proof of this fact in the elliptic case).

The next result is taken from [31, Corollary 19]. See also [23], where the analogous parabolic inequality is proved.

Theorem 2.10. *Let $v \in HW^{1,2}\left(B(x_0, KR); \mathbb{R}^N\right)$ be a solution to the system*

$$X_i\left(a_{\alpha\beta}^{ij} X_j v^\beta\right) = 0 \quad \text{in } B(x_0, KR)$$

with constant coefficients $a_{\alpha\beta}^{ij}$ satisfying (2.7) and some $K > 1$. Then $v \in C^\infty\left(B(x_0, KR); \mathbb{R}^N\right)$; moreover

$$\sup_{B_R(x_0)} |Xv|^2 \le cR^{-2}\frac{1}{|B_{KR}(x_0)|}\int_{B_{KR}(x_0)} |v|^2\, dx,$$

where the positive constant c depends on K, μ, $\mathbb{G}$, N but is independent of x_0, R and v.

The following result can be proved in a completely standard way by suitable cutoff functions (for the analogous elliptic version see for instance [11, Theorem 2.1 p.134]).

Theorem 2.11 (*Caccioppoli's Inequality*). *Let $u \in HW^{1,2}\left(B_R(\bar{x}); \mathbb{R}^N\right)$ be a weak solution to (1.1) in $B_R(\bar{x}) \subset \Omega$. There exists a constant $c > 0$ depending on $\mathbb{G}$, N, R such that for any $\rho \in (0, R)$,*

$$\int_{B_\rho(\bar{x})} |Xu(x)|^2\, dx \le c\left[\frac{1}{(R-\rho)^2}\int_{B_R(\bar{x})} |u(x)|^2\, dx + \int_{B_R(\bar{x})} |\mathbf{F}(x)|^2\, dx\right]. \tag{2.9}$$

2.3. Statement of the result

We now state precisely the main result of this paper.

Theorem 2.12. *Under Assumption (H), let the $a_{\alpha\beta}^{ij}$'s belong to $VMO_{loc}(\Omega)$ and let $\Omega' \Subset \Omega$, $2 < p < \infty$. Then there is a positive constant c depending on $\mathbb{G}$, μ, p, Ω, Ω' and the local VMO moduli of the $a_{\alpha\beta}^{ij}$'s in Ω' such that if $\mathbf{F} = \left(f_\alpha^i\right) \in L^p\left(\Omega; \mathbb{M}^{N\times q}\right)$ and $u \in HW^{1,2}\left(\Omega; \mathbb{R}^N\right)$ is a weak solution to (1.1) in Ω, then $u \in HW^{1,p}\left(\Omega'; \mathbb{R}^N\right)$ and*

$$\|u\|_{HW^{1,p}(\Omega';\mathbb{R}^N)} \le c\left(\|\mathbf{F}\|_{L^p(\Omega;\mathbb{M}^{N\times q})} + \|u\|_{L^2(\Omega;\mathbb{R}^N)}\right). \tag{2.10}$$

In order to prove Theorem 2.12, we will prove the following local result.

Theorem 2.13. *Under Assumption (H), for any $\bar{x} \in \Omega$, $R_0 > 0$ such that $B_{11R_0}(\bar{x}) \subset \Omega$ there exists $\delta = \delta(p, \mathbb{G}, R_0, \mu) > 0$ such that for any $R \le R_0$, if the coefficients $a_{\alpha\beta}^{ij}$ are $(\delta, 8R)$-vanishing in $B_R(\bar{x})$ and $p \in (2, \infty)$, then there is a positive $c = c(R, R_0, p, \mathbb{G})$ such that if $\mathbf{F} = \left(f_\alpha^i\right) \in L^p\left(B_{11R}(\bar{x}); \mathbb{M}^{N\times q}\right)$ and $u \in HW^{1,2}\left(B_{11R}(\bar{x}); \mathbb{R}^N\right)$ is a weak solution of (1.1) in $B_{11R}(\bar{x})$, then $u \in HW^{1,p}\left(B_R(\bar{x}); \mathbb{R}^N\right)$ and*

$$\|Xu\|_{L^p(B_R(\bar{x});\mathbb{R}^N)} \le c\left(\|\mathbf{F}\|_{L^p(B_{11R}(\bar{x});\mathbb{M}^{N\times q})} + \|Xu\|_{L^2(B_{11R}(\bar{x});\mathbb{R}^N)}\right). \tag{2.11}$$

Proof of Theorem 2.12 from Theorem 2.13. For fixed domains $\Omega' \Subset \Omega'' \Subset \Omega$, pick R_0 such that $B_{12R_0}(\bar{x}) \subset \Omega''$ for any $\bar{x} \in \Omega'$. For this R_0 and a fixed $p \in (2, \infty)$, let δ be as in Theorem 2.13. Since the $a_{\alpha\beta}^{ij}$'s belong to $VMO_{loc}(\Omega)$, there exists $R \le R_0$, R depending on Ω, Ω', R_0, δ, such that the $a_{\alpha\beta}^{ij}$'s are $(\delta, 8R)$-vanishing in $B_R(\bar{x})$. Therefore by Theorem 2.13, (2.11) holds for any such $\bar{x}$ and R. Next, we apply Caccioppoli's inequality (2.9), getting

$$\|Xu\|_{L^2(B_{11R}(\bar{x});\mathbb{R}^N)} \le c\left\{\frac{1}{R}\|u\|_{L^2(B_{12R}(\bar{x});\mathbb{R}^N)} + \|\mathbf{F}\|_{L^2(B_{12R}(\bar{x});\mathbb{M}^{N\times q})}\right\},$$

which inserted in (2.11) gives

$$\|Xu\|_{L^p\left(B_R(\overline{x});\mathbb{R}^N\right)} \leq c\,(R)\left\{\|\mathbf{F}\|_{L^p(B_{12R}(\overline{x});\mathbb{M}^{N\times q})} + \|u\|_{L^2(B_{12R}(\overline{x});\mathbb{R}^N)}\right\}. \tag{2.12}$$

On the other hand, by Poincaré's inequality (2.2) we have

$$\begin{aligned}\|u^\alpha\|_{L^p(B_R(\overline{x}))} &\leq \left\|u^\alpha - u^\alpha_{B_R(\overline{x})}\right\|_{L^p(B_R(\overline{x}))} + \left|u^\alpha_{B_R(\overline{x})}\right| |B_R(\overline{x})|^{1/p}\\ &\leq cR\,\|Xu^\alpha\|_{L^p(B_R(\overline{x}))} + \|u^\alpha\|_{L^2(B_R(\overline{x}))} |B_R(\overline{x})|^{1/p-1/2};\end{aligned}$$

hence

$$\|u\|_{L^p\left(B_R(\overline{x});\mathbb{R}^N\right)} \leq c\,(R,p)\left\{\|Xu\|_{L^p\left(B_R(\overline{x});\mathbb{R}^N\right)} + \|u\|_{L^2\left(B_R(\overline{x});\mathbb{R}^N\right)}\right\},$$

which together with (2.12) gives

$$\|u\|_{HW^{1,p}\left(B_R(\overline{x});\mathbb{R}^N\right)} \leq c\left\{\|\mathbf{F}\|_{L^p(B_{12R}(\overline{x});\mathbb{M}^{N\times q})} + \|u\|_{L^2(B_{12R}(\overline{x});\mathbb{R}^N)}\right\}.$$

A covering argument then gives (2.10). □

It is worthwhile to point out that, as we will see from the proof of Theorem 2.13 in Section 5, the following bound, stronger than (2.11), is actually established:

$$\left\|\mathcal{M}_{B_{11R}(\overline{x})}\left(|Xu|^2\right)\right\|^{1/2}_{L^{p/2}\left(B_R(\overline{x});\mathbb{R}^N\right)} \leq c\left\{\|\mathbf{F}\|_{L^p(B_{12R}(\overline{x});\mathbb{M}^{N\times q})} + \|u\|_{L^2(B_{12R}(\overline{x});\mathbb{R}^N)}\right\}, \tag{2.13}$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal function (see Section 4).

Remark 2.14. Note that what allows to exploit the VMO assumption on the coefficients is the fact that the number δ in Theorem 2.13 depends on R_0 but not on $R \leq R_0$, which allows shrinking R without changing δ, to get the (δ, R)-vanishing condition satisfied. Under this regard, our result is very different from those proved for instance in [9,8] where the parameter δ possibly depends on R, which makes the (δ, R)-vanishing assumption hard to check.

Dependence of constants. Throughout this paper, the letter c denotes a constant which may vary from line to line. The parameters which the constants depend on are declared in the statements or in the proofs of the theorems. When we write that c is an "absolute constant" we mean that it may depend on $\mathbb{G}$ and N.

3. Approximation by solutions of systems with constant coefficients

Notation 3.1. *In order to simplify notation, henceforth we will systematically write the norms and spaces of vector valued functions as*

$$\begin{aligned}&HW^{1,p}(B),\ \|u\|_{HW^{1,p}(B)},\ \|\mathbf{F}\|_{L^p(B)} \quad \textit{instead of}\\ &HW^{1,p}\left(B;\mathbb{R}^N\right),\ \|u\|_{HW^{1,p}\left(B;\mathbb{R}^N\right)},\ \|\mathbf{F}\|_{L^p\left(B;\mathbb{M}^{N\times q}\right)},\end{aligned}$$

and so on.

In this section we will prove a couple of theorems asserting that a solution to a system (1.1) with small datum **F** and coefficients with small oscillation, can be suitably approximated by a solution to a system with constant coefficients and zero datum. This approximation is one of the tools which will be used in the proof of Theorem 2.13.

Theorem 3.2. *Under Assumption (H) (see Section 2.2), for any $\varepsilon > 0$, $R_0 > 0$ there is a small $\delta = \delta(\varepsilon, R_0, \mu) > 0$ such that for any $R \leq R_0$, if u is a weak solution to the system (1.1) in $B_{4R} \Subset \Omega$ with*

$$\frac{1}{|B_{4R}|}\int_{B_{4R}} |Xu|^2\,dx \leq 1, \qquad \frac{1}{|B_{4R}|}\int_{B_{4R}}\left(|\mathbf{F}|^2 + \left|a^{ij}_{\alpha\beta} - \left(a^{ij}_{\alpha\beta}\right)_{B_{4R}}\right|^2\right)dx \leq \delta^2, \tag{3.1}$$

then there exists a weak solution v to the following homogeneous system with constant coefficients:

$$X_i\left(\left(a^{ij}_{\alpha\beta}\right)_{B_{4R}} X_j v^\beta(x)\right) = 0 \quad \textit{in } B_{4R} \tag{3.2}$$

such that

$$\frac{1}{R^2}\frac{1}{|B_{4R}|}\int_{B_{4R}} |u - v|^2\,dx \leq \varepsilon^2.$$

Proof. Let us first prove the result for a fixed R (and δ possibly depending on R), then we will show how to remove the dependence on R.

By contradiction, if the result does not hold, then there exist a constant $\varepsilon_0 > 0$, and sequences $\left\{a_{\alpha\beta}^{ijk}\right\}_{k=1}^{\infty}$ satisfying (2.7), $\{u_k\}_{k=1}^{\infty}$, $\{\mathbf{F}_k\}_{k=1}^{\infty}$ such that u_k is a weak solution to the system

$$X_i\left(a_{\alpha\beta}^{ijk}X_ju_k^{\beta}(x)\right) = X_if_{\alpha}^{ik}(x) \tag{3.3}$$

in B_{4R} with

$$\frac{1}{|B_{4R}|}\int_{B_{4R}}|Xu_k|^2\,dx \le 1, \qquad \frac{1}{|B_4|}\int_{B_{4R}}\left(|\mathbf{F}_k|^2 + \left|a_{\alpha\beta}^{ijk} - \left(a_{\alpha\beta}^{ijk}\right)_{B_{4R}}\right|^2\right)dx \le \frac{1}{k^2}, \tag{3.4}$$

but

$$\frac{1}{R^2}\frac{1}{|B_{4R}|}\int_{B_{4R}}|u_k - v_k|^2\,dx > \varepsilon_0^2 \tag{3.5}$$

for any weak solution v_k of

$$X_i\left(\left(a_{\alpha\beta}^{ijk}\right)_{B_{4R}}X_jv_k^{\beta}(x)\right) = 0 \quad \text{in } B_{4R}. \tag{3.6}$$

From (3.4) and Poincaré's inequality (2.2), we know that $\left\{u_k - (u_k)_{B_{4R}}\right\}_{k=1}^{\infty}$ is bounded in $HW^{1,2}(B_{4R})$, then Rellich's lemma allows us to find a subsequence of $\left\{u_k - (u_k)_{B_{4R}}\right\}$, still denoted by $\left\{u_k - (u_k)_{B_{4R}}\right\}$, such that

$$\frac{1}{R^2}\frac{1}{|B_{4R}|}\int_{B_{4R}}\left|u_k - (u_k)_{B_{4R}} - u_0\right|^2dx \to 0, \tag{3.7}$$

$$Xu_k \to Xu_0 \quad \text{weakly in } L^2, \tag{3.8}$$

as $k \to \infty$, for some $u_0 \in HW^{1,2}(B_{4R})$. Since $\left\{\left(a_{\alpha\beta}^{ijk}\right)_{B_{4R}}\right\}_{k=1}^{\infty}$ is bounded in $\mathbb{R}$, it allows a subsequence, still denoted by $\left\{\left(a_{\alpha\beta}^{ijk}\right)_{B_{4R}}\right\}_{k=1}^{\infty}$, such that

$$\left|\left(a_{\alpha\beta}^{ijk}\right)_{B_{4R}} - \bar{a}_{\alpha\beta}^{ij}\right| \to 0, \quad \text{as } k \to \infty, \tag{3.9}$$

for some constants $\bar{a}_{\alpha\beta}^{ij}$. By (3.4), it follows

$$a_{\alpha\beta}^{ijk} \to \bar{a}_{\alpha\beta}^{ij} \text{ in } L^2(B_{4R}), \quad \text{as } k \to \infty.$$

Next, we show that u_0 is a weak solution of

$$X_i\left(\bar{a}_{\alpha\beta}^{ij}X_ju^{\beta}(x)\right) = 0 \quad \text{in } B_{4R}. \tag{3.10}$$

We start from

$$\int_{B_{4R}}a_{\alpha\beta}^{ijk}(x)X_ju_k^{\beta}X_i\varphi^{\alpha}\,dx = \int_{B_{4R}}f_{\alpha}^{ik}X_i\varphi^{\alpha}\,dx \tag{3.11}$$

with $\varphi^{\alpha} \in C_0^{\infty}(\Omega)$, and take the limit for $k \to \infty$. By (3.4),

$$\int_{B_{4R}}f_{\alpha}^{ik}X_i\varphi^{\alpha}\,dx \to 0.$$

Moreover,

$$\int_{B_{4R}}a_{\alpha\beta}^{ijk}(x)X_ju_k^{\beta}X_i\varphi^{\alpha}\,dx = \int_{B_{4R}}\left[a_{\alpha\beta}^{ijk}(x) - \bar{a}_{\alpha\beta}^{ij}\right]X_ju_k^{\beta}X_i\varphi^{\alpha}\,dx + \int_{B_{4R}}\bar{a}_{\alpha\beta}^{ij}X_ju_k^{\beta}X_i\varphi^{\alpha}\,dx \equiv A_k + B_k.$$

Now,

$$|A_k| \le c\left\|a_{\alpha\beta}^{ijk}(x) - \bar{a}_{\alpha\beta}^{ij}\right\|_{L^2(B_{4R})}\left\|X_ju_k^{\beta}\right\|_{L^2(B_{4R})} \to 0,$$

because $a^{ijk}_{\alpha\beta}(x) \to \bar{a}^{ij}_{\alpha\beta}$ in L^2 and $\left\{X_j u^\beta_k\right\}$ is bounded in L^2. Finally, since $Xu_k \to Xu_0$ weakly in L^2,

$$B_k \to \int_{B_{4R}} \bar{a}^{ij}_{\alpha\beta} X_j u^\beta_0 X_i \varphi^\alpha dx;$$

hence

$$\int_{B_{4R}} \bar{a}^{ij}_{\alpha\beta} X_j u^\beta_0 X_i \varphi^\alpha dx = 0 \quad \text{for any } \varphi^\alpha \in C^\infty_0 (B_{4R}).$$

By density, this holds for any $\varphi^\alpha \in HW^{1,2}_0(B_{4R})$, so u_0 is a weak solution to (3.10).

Now, let v_k be the unique solution to the Dirichlet problem

$$\begin{cases} X_i\left(\left(a^{ijk}_{\alpha\beta}\right)_{B_{4R}} X_j v_k\right) = 0 & \text{in } B_{4R} \\ v_k - u_0 \in HW^{1,2}_0(B_{4R}) \end{cases} \tag{3.12}$$

(see Remark 2.9). By (2.7) and using $v_k - u_0$ as a test function in the definition of the solution to (3.12) we have

$$\begin{aligned} \mu \int_{B_{4R}} |Xv_k - Xu_0|^2 dx &\le \int_{B_{4R}} \left(a^{ijk}_{\alpha\beta}\right)_{B_{4R}} \left(X_j v^\beta_k - X_j u^\beta_0\right)\left(X_i v^\alpha_k - X_i u^\alpha_0\right) dx \\ &= -\int_{B_{4R}} \left(a^{ijk}_{\alpha\beta}\right)_{B_{4R}} X_j u^\beta_0 \left(X_i v^\alpha_k - X_i u^\alpha_0\right) dx, \end{aligned}$$

since u_0 is a weak solution to (3.10)

$$\begin{aligned} &= \int_{B_{4R}} \left(\bar{a}^{ij}_{\alpha\beta} - \left(a^{ijk}_{\alpha\beta}\right)_{B_{4R}}\right) X_j u^\beta_0 \left(X_i v^\alpha_k - X_i u^\alpha_0\right) dx \\ &\le \left|\left(\bar{a}^{ij}_{\alpha\beta} - \left(a^{ijk}_{\alpha\beta}\right)_{B_{4R}}\right)\right| \int_{B_{4R}} \left|X_j u^\beta_0\right| \left|X_i v^\alpha_k - X_i u^\alpha_0\right| dx \\ &\le c(N) \max_{i,j,\alpha,\beta} \left|\left(\bar{a}^{ij}_{\alpha\beta} - \left(a^{ijk}_{\alpha\beta}\right)_{B_{4R}}\right)\right| \left(\int_{B_{4R}} |Xu_0|^2 dx\right)^{1/2} \cdot \left(\int_{B_{4R}} |Xv_k - Xu_0|^2 dx\right)^{1/2}, \end{aligned}$$

which implies

$$\mu \left(\int_{B_{4R}} |Xv_k - Xu_0|^2 dx\right)^{1/2} \le c \max_{i,j,\alpha,\beta} \left|\left(\bar{a}^{ij}_{\alpha\beta} - \left(a^{ijk}_{\alpha\beta}\right)_{B_{4R}}\right)\right| \left(\int_{B_{4R}} |Xu_0|^2 dx\right)^{1/2}. \tag{3.13}$$

Inequalities (3.9) and (3.13) imply

$$\|Xv_k - Xu_0\|_{L^2(B_{4R})} \to 0 \quad \text{as } k \to 0.$$

This convergence, the fact that $v_k - u_0 \in HW^{1,2}_0(B_{4R})$ and (2.3) imply

$$\|v_k - u_0\|_{L^2(B_{4R})} \to 0 \quad \text{as } k \to 0. \tag{3.14}$$

By (3.7) and (3.14) we can write

$$\left\|v_k - \left(u_k - (u_k)_{B_{4R}}\right)\right\|_{L^2(B_{4R})} \le \left\|u_0 - \left(u_k - (u_k)_{B_{4R}}\right)\right\|_{L^2(B_{4R})} + \|v_k - u_0\|_{L^2(B_{4R})} \to 0. \tag{3.15}$$

On the other hand, $v_k + (u_k)_{B_{4R}}$ is still a weak solution to (3.6); hence (3.5) implies

$$\frac{1}{R^2}\frac{1}{|B_{4R}|}\int_{B_{4R}} \left|v_k - \left(u_k - (u_k)_{B_{4R}}\right)\right|^2 dx > \varepsilon_0^2,$$

which contradicts (3.15). So we have proved the assertion, for some δ possibly depending on ε, R, μ.

Let us now fix a particular R_0, and let R be any number $\le R_0$. Assume u is a weak solution to system (1.1) in $B_{4R} \Subset \Omega$ satisfying (3.1). Just to simplify notations, assume that the center of B_{4R} is the origin, and define

$$\widetilde{u}(x) = \frac{R_0}{R} u\left(D\left(\frac{R}{R_0}\right)x\right);$$

$$\widetilde{a}^{\alpha\beta}_{ij}(x) = a^{\alpha\beta}_{ij}\left(D\left(\frac{R}{R_0}\right)x\right);$$

$$\widetilde{f}^i_\alpha(x) = f^i_\alpha\left(D\left(\frac{R}{R_0}\right)x\right).$$

Then, one can check that the function $\widetilde{u}$ solves the system

$$X_i\left(\widetilde{a}^{ij}_{\alpha\beta}(x)X_j\widetilde{u}^{\beta}\right)=X_i\widetilde{f}^i_{\alpha}\quad \text{in } B_{4R_0}.$$

To see this, for any $\phi\in C_0^{\infty}(B_{4R})$, let $\widetilde{\phi}(x)=\frac{R_0}{R}\phi\left(D\left(\frac{R}{R_0}\right)x\right)$; then $\widetilde{\phi}\in C_0^{\infty}\left(B_{4R_0}\right)$ and

$$\begin{aligned}\int_{B_{4R_0}}\widetilde{a}^{\alpha\beta}_{ij}(x)\,X_j\widetilde{u}^{\beta}(x)\,X_i\widetilde{\phi}^{\alpha}(x)\,dx &= \int_{B_{4R_0}}a^{\alpha\beta}_{ij}\left(D\left(\frac{R}{R_0}\right)x\right)\left(X_ju^{\beta}\right)\left(D\left(\frac{R}{R_0}\right)x\right)(X_i\phi^{\alpha})\left(D\left(\frac{R}{R_0}\right)x\right)dx\\ &=\left(\frac{R_0}{R}\right)^{Q}\int_{B_{4R}}a^{\alpha\beta}_{ij}(y)\left(X_ju^{\beta}\right)(y)\,(X_i\phi^{\alpha})(y)\,dy\\ &=\left(\frac{R_0}{R}\right)^{Q}\int_{B_{4R}}f^i_{\alpha}(y)\,(X_i\phi^{\alpha})(y)\,dy\\ &=\int_{B_{4R_0}}f^i_{\alpha}\left(D\left(\frac{R}{R_0}\right)x\right)(X_i\phi^{\alpha})\left(D\left(\frac{R}{R_0}\right)x\right)dx\\ &=\int_{B_{4R_0}}\widetilde{f}^{\alpha}_i(x)\,X_i\widetilde{\phi}^{\alpha}(x)\,dx.\end{aligned}$$

Also, note that the $\widetilde{a}^{ij}_{\alpha\beta}$'s satisfy condition (2.7) with the same μ. Let $\delta=\delta(\varepsilon,R_0,\mu)$ be the number found in the first part of the proof, and assume that $u,\mathbf{F},a^{\alpha\beta}_{ij}$ satisfy (3.1) on B_{4R} for this δ; then $\widetilde{u},\widetilde{\mathbf{F}},\widetilde{a}^{\alpha\beta}_{ij}$ satisfy (3.1) on B_{4R_0} for the same δ:

$$\begin{aligned}\frac{1}{\left|B_{4R_0}\right|}\int_{B_{4R_0}}|X\widetilde{u}(x)|^2\,dx &= \frac{1}{\left|B_{4R_0}\right|}\int_{B_{4R_0}}\left|(Xu)\left(D\left(\frac{R}{R_0}\right)x\right)\right|^2 dx\\ &=\frac{1}{\left|B_{4R_0}\right|}\left(\frac{R_0}{R}\right)^{Q}\int_{B_{4R}}|(Xu)(y)|^2\,dy=\frac{1}{|B_{4R}|}\int_{B_{4R}}|(Xu)(y)|^2\,dy\le 1;\end{aligned}$$

$$\frac{1}{\left|B_{4R_0}\right|}\int_{B_{4R_0}}\left(\left|\widetilde{\mathbf{F}}\right|^2+\left|\widetilde{a}^{ij}_{\alpha\beta}-\left(\widetilde{a}^{ij}_{\alpha\beta}\right)_{B_{4R_0}}\right|^2\right)dx=\frac{1}{|B_{4R}|}\int_{B_{4R}}\left(|\mathbf{F}|^2+\left|a^{ij}_{\alpha\beta}-\left(a^{ij}_{\alpha\beta}\right)_{B_{4R}}\right|^2\right)dx\le\delta.$$

Hence, by the first part of the proof, there exists a weak solution $\widetilde{v}$ to the following homogeneous system with constant coefficients:

$$X_i\left(\left(\widetilde{a}^{ij}_{\alpha\beta}\right)_{B_{4R_0}}X_j\widetilde{v}^{\beta}(x)\right)=0\quad\text{in } B_{4R_0}$$

such that

$$\frac{1}{R_0^2}\frac{1}{\left|B_{4R_0}\right|}\int_{B_{4R_0}}|\widetilde{u}-\widetilde{v}|^2\,dx\le\varepsilon^2.$$

Then, the function

$$v(x)=\frac{R}{R_0}\widetilde{v}\left(D\left(\frac{R_0}{R}\right)x\right)$$

satisfies

$$X_i\left(\left(a^{ij}_{\alpha\beta}\right)_{B_{4R}}X_jv^{\beta}(x)\right)=0\quad\text{in } B_{4R}$$

and

$$\begin{aligned}\frac{1}{R^2}\frac{1}{|B_{4R}|}\int_{B_{4R}}|u(x)-v(x)|^2\,dx &= \frac{1}{R^2}\frac{1}{|B_{4R}|}\int_{B_{4R}}\left|\frac{R}{R_0}\widetilde{u}\left(D\left(\frac{R_0}{R}\right)x\right)-\frac{R}{R_0}\widetilde{v}\left(D\left(\frac{R_0}{R}\right)x\right)\right|^2dx\\ &=\frac{1}{R_0^2}\frac{1}{|B_{4R}|}\int_{B_{4R}}\left|\widetilde{u}\left(D\left(\frac{R_0}{R}\right)x\right)-\widetilde{v}\left(D\left(\frac{R_0}{R}\right)x\right)\right|^2dx\\ &=\frac{1}{R_0^2}\frac{1}{\left|B_{4R_0}\right|}\int_{B_{4R_0}}|\widetilde{u}-\widetilde{v}|^2\,dx\le\varepsilon^2.\end{aligned}$$

We have therefore proved that the assertion holds with δ depending on R_0 but independent of $R\le R_0$. $\square$

The following technical lemma is adapted from [11, Lemma 4.1, p.27].

Lemma 3.3. *Let $\psi(t)$ be a bounded nonnegative function defined on the interval $[T_0, T_1]$, where $T_1 > T_0 \geq 0$. Suppose that for any $T_0 \leq t \leq s \leq T_1$, ψ satisfies*

$$\psi(t) \leq \vartheta\psi(s) + \frac{A}{(s-t)^\beta} + B,$$

where ϑ, A, B, β are nonnegative constants, and $\vartheta < \frac{1}{3}$. Then

$$\psi(\rho) \leq c_\beta\left[\frac{A}{(R-\rho)^\beta} + B\right], \quad \forall\rho, T_0 \leq \rho < R \leq T_1,$$

where c_β only depends on β.

We are going to enforce the previous theorem with the following.

Theorem 3.4. *For any $\varepsilon > 0R_0 > 0$, there is a small $\delta = \delta(\varepsilon, R_0, \mu) > 0$ such that for any $R \leq R_0$, if u is a weak solution of system (1.1) in $B_{4R} \Subset \Omega$ and (3.1) holds, then there exists a weak solution v to (3.2) such that*

$$\frac{1}{|B_{2R}|}\int_{B_{2R}} |Xu - Xv|^2\,dx \leq \varepsilon^2.$$

Proof. By Theorem 3.2, we know that for any $\eta > 0$, there exist a small $\delta = \delta(\eta, R_0, \mu) > 0$ and a weak solution v of (3.2) in B_{4R}, such that

$$\frac{1}{R^2}\frac{1}{|B_{4R}|}\int_{B_{4R}} |u-v|^2\,dx \leq \eta^2, \tag{3.16}$$

provided (3.1) holds.

Let us note that $u - v$ is a weak solution to the system

$$X_i\left(a^{ij}_{\alpha\beta}(x)X_j\left(u^\beta - v^\beta\right)(x)\right) = X_i\left(f^i_\alpha(x) - \left(a^{ij}_{\alpha\beta}(x) - \left(a^{ij}_{\alpha\beta}\right)_{B_{4R}}\right)X_jv^\beta\right) \tag{3.17}$$

in B_{4R}. For any $2R \leq s < t \leq 3R$, we choose a cutoff function $\varphi(x)$ which satisfies

$$0 < \varphi(x) \leq 1 \quad \text{in } B_{3R}, \qquad \varphi(x) \equiv 1 \quad \text{in } B_s, \qquad \varphi(x) \equiv 0 \quad \text{in } B_{3R}\setminus B_t$$

and

$$|X\varphi(x)| \leq \frac{c}{t-s} \quad \text{in } B_{4R}.$$

Taking $(u-v)\varphi$ as a test function, it follows by (3.17) that

$$\begin{aligned}
\mu\int_{B_s} |X(u-v)|^2\,dx &\leq \int_{B_t} \varphi(x)\,a^{ij}_{\alpha\beta}(x)X_j\left(u^\beta - v^\beta\right)X_i\left(u^\alpha - v^\alpha\right)dx \\
&= \int_{B_t}\left(f^i_\alpha(x) - \left(a^{ij}_{\alpha\beta}(x) - \left(a^{ij}_{\alpha\beta}\right)_{B_{4R}}\right)X_jv^\beta\right)X_i\left(\left(u^\alpha - v^\alpha\right)\varphi\right)dx \\
&\quad - \int_{B_t} a^{ij}_{\alpha\beta}(x)\left(u^\alpha - v^\alpha\right)X_j\left(u^\beta - v^\beta\right)X_i\varphi dx.
\end{aligned}$$

By the properties of φ, Young's inequality and (2.7),

$$\begin{aligned}
\int_{B_s} |Xu - Xv|^2\,dx &\leq c\int_{B_t}\left(|\mathbf{F}| + \max_{i,j,\alpha,\beta}\left|a^{ij}_{\alpha\beta}(x) - \left(a^{ij}_{\alpha\beta}(x)\right)_{B_{4R}}\right||Xv|\right)^2 dx \\
&\quad + \frac{1}{4}\int_{B_t}|Xu - Xv|^2\,dx + \frac{c}{(t-s)^2}\int_{B_t}|u-v|^2\,dx \\
&\leq c\int_{4R}|\mathbf{F}|^2\,dx + \sup_{B_{3R}}|Xv|^2\cdot\max_{i,j,\alpha,\beta}\int_{B_{4R}}\left|a^{ij}_{\alpha\beta}(x) - \left(a^{ij}_{\alpha\beta}(x)\right)_{B_{4R}}\right|^2 dx \\
&\quad + \frac{c}{(t-s)^2}\int_{B_{4R}}|u-v|^2\,dx + \frac{1}{4}\int_{B_t}|Xu - Xv|^2\,dx.
\end{aligned}$$

Setting

$$\psi(s) = \int_{B_s} |Xu - Xv|^2\,dx,$$

$$B = c\int_{B_{4R}} |\mathbf{F}|^2\,dx + \sup_{B_{3R}} |Xv|^2 \cdot \max_{i,j,\alpha,\beta} \int_{B_{4R}} \left| a^{ij}_{\alpha\beta}(x) - \left(a^{ij}_{\alpha\beta}(x)\right)_{B_{4R}} \right|^2 dx,$$

$$A = \int_{B_{4R}} |u - v|^2\,dx, \quad \beta = 2,$$

by Lemma 3.3 we deduce

$$\int_{B_{2R}} |Xu - Xv|^2\,dx \le \frac{c}{R^2}\int_{B_{4R}} |u-v|^2\,dx + c\int_{B_{4R}} |\mathbf{F}|^2\,dx + c\sup_{B_{3R}} |Xv|^2 \cdot \max_{i,j,\alpha,\beta}\int_{B_{4R}} \left| \left(a^{ij}_{\alpha\beta}(x) - \left(a^{ij}_{\alpha\beta}(x)\right)_{B_{4R}} \right) \right|^2 dx. \tag{3.18}$$

By Theorem 2.10, since $v - u_{B_{4R}}$ is still a solution to the system (3.2) in B_{4R} we can write

$$\begin{aligned}\sup_{B_{3R}} |Xv| &\le \frac{c}{R}|B_R|^{-1/2}\left\|v - u_{B_{4R}}\right\|_{L^2(B_{4R})} \\ &\le \frac{c}{R}|B_R|^{-1/2}\left(\|u - v\|_{L^2(B_{4R})} + \left\|u - u_{B_{4R}}\right\|_{L^2(B_{4R})}\right)\end{aligned}$$

by (3.16), (2.2) and assumption (3.1) on u

$$\le c\eta + c|B_R|^{-1/2}\|Xu\|_{L^2(B_{4R})} \le c(\eta+1) \le N_0, \tag{3.19}$$

for some absolute constant N_0 when η is, say, any number ≤ 1.

By (3.18) and (3.19) we have

$$\frac{1}{|B_{2R}|}\int_{B_{2R}} |Xu - Xv|^2\,dx \le \frac{c}{|B_{4R}|}\int_{B_{4R}} |\mathbf{F}|^2\,dx + \frac{cN_0}{|B_{4R}|}\max_{i,j,\alpha,\beta}\int_{B_{4R}}\left| a^{ij}_{\alpha\beta}(x) - \left(a^{ij}_{\alpha\beta}\right)_{B_{4R}}\right|^2 dx + \frac{c}{R^2}\frac{1}{|B_{4R}|}\int_{B_{4R}} |u-v|^2\,dx,$$

by (3.16) and (3.1)

$$\begin{aligned}&\le \frac{c}{|B_{4R}|}\int_{B_{4R}}\left(|\mathbf{F}|^2 + \max_{i,j,\alpha,\beta}\left| a^{ij}_{\alpha\beta}(x) - \left(a^{ij}_{\alpha\beta}(x)\right)_{B_{4R}}\right|^2\right)dx + c\eta^2 \\ &\le c\left(\delta^2 + \eta^2\right) < \varepsilon^2,\end{aligned}$$

for a suitable choice of η, and after possibly diminishing δ. This ends the proof. □

4. Estimates on the maximal function of $|Xu|^2$

Definition 4.1. Let $B_R \Subset \Omega$. For every $f \in L^1(B_R)$, define the Hardy–Littlewood maximal function of f by

$$\mathcal{M}_{B_R}(f)(x) = \sup_{r>0}\frac{1}{|B_r(x)\cap B_R|}\int_{B_r(x)\cap B_R} |f(y)|\,dy.$$

Since (B_R, d_X, dx) is a space of homogeneous type (see Remark 2.7), by [16, Theorem 2.1 p. 71] the following holds.

Lemma 4.2. *Let $f \in L^1(B_R)$, then*

(i) *$\mathcal{M}_{B_R}(f)(x)$ is finite almost everywhere in B_R;*
(ii) *for every $\alpha > 0$,*

$$\left|\left\{x \in B_R : \mathcal{M}_{B_R}(f)(x) > \alpha\right\}\right| \le \frac{c_1}{\alpha}\int_{B_R} |f(y)|\,dy;$$

(iii) *if* $f \in L^p(B_R)$ *with* $1 < p < \infty$, *then* $\mathcal{M}_{B_R}(f) \in L^p(B_R)$ *and*

$$\left\| \mathcal{M}_{B_R}(f) \right\|_{L^p(B_R)} \le c_p \|f\|_{L^p(B_R)},$$

where the constants c_p *only depend on* p *and* $\mathbb{G}$ *(but are independent of* B_R*).*

The last statement about the dependence of the constants requires some explanation. In any space of homogeneous type these constants depend on the two constants of the space, namely the one appearing in the "quasitriangle inequality" (2.4) and the doubling constant appearing in (2.5). In our case the first constant is 1 (since d_X is a distance) and the second is independent of R, by Remark 2.7. Hence c_p is independent of R.

Theorem 4.3. *There exists an absolute constant* N_1 *such that for any* $\varepsilon > 0$, $R_0 > 0$, *there is a small* $\delta = \delta(\varepsilon, R_0, \mu) > 0$ *such that for any* $R \le R_0/2$, $z \in B_R(\overline{x}) \subset B_{11R}(\overline{x}) \Subset \Omega$ *and* $0 < r \le 2R$, *if* u *is a weak solution of* (1.1) *in* $B_{11R}(\overline{x})$ *with*

$$\begin{aligned} &B_r(z) \cap \left\{x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})}\left(|Xu|^2\right)(x) \le 1\right\} \\ &\quad \cap \left\{x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})}\left(|\mathbf{F}|^2\right)(x) \le \delta^2\right\} \ne \emptyset \end{aligned} \tag{4.1}$$

and the coefficients $a_{\alpha\beta}^{ij}(x)$ *are* $(\delta, 4r)$*-vanishing in* $B_R(\overline{x})$, *then*

$$\left|B_r(z) \cap \left\{x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})}\left(|Xu|^2\right)(x) > N_1^2\right\}\right| < \varepsilon |B_r(z)|. \tag{4.2}$$

Proof. Fix $\varepsilon, R_0 > 0$; the number δ will be chosen later. By (4.1), there exists a point $x_0 \in B_r(z)$, such that for any $\rho > 0$,

$$\frac{1}{\left|B_\rho(x_0) \cap B_{11R}(\overline{x})\right|} \int_{B_\rho(x_0) \cap B_{11R}(\overline{x})} |Xu|^2\, dx \le 1, \tag{4.3}$$

$$\frac{1}{\left|B_\rho(x_0) \cap B_{11R}(\overline{x})\right|} \int_{B_\rho(x_0) \cap B_{11R}(\overline{x})} |\mathbf{F}|^2\, dx \le \delta^2. \tag{4.4}$$

Since $z, x_0 \in B_R(\overline{x})$ and $r \le 2R$, we have the inclusions: $B_{4r}(z) \subset B_{5r}(x_0) \subset B_{11R}(\overline{x})$ and $B_{5r}(x_0) \subset B_{6r}(z)$. Then by (4.4) with $\rho = 5r$ we have that

$$\frac{1}{|B_{4r}(z)|} \int_{B_{4r}(z)} |\mathbf{F}|^2\, dx \le \frac{|B_{6r}(z)|}{|B_{4r}(z)|} \frac{1}{|B_{5r}(x_0)|} \int_{B_{5r}(x_0)} |\mathbf{F}|^2\, dx \le \left(\frac{6}{4}\right)^Q \delta^2. \tag{4.5}$$

Similarly, by (4.3) we find

$$\frac{1}{|B_{4r}(z)|} \int_{B_{4r}(z)} |Xu|^2\, dx \le \left(\frac{6}{4}\right)^Q. \tag{4.6}$$

By (4.5), (4.6) and the assumption on $a_{\alpha\beta}^{ij}(x)$, we can apply Theorem 3.4 (with u replaced by $\left(\frac{4}{6}\right)^Q u$ and $\mathbf{F}$ replaced by $\left(\frac{4}{6}\right)^Q \mathbf{F}$) on the ball $B_{4r}(z)$ (recall that $r \le R_0$) and obtain that for any $\eta > 0$, there exists a small $\delta = \delta(\eta, R_0, \mu)$ and a weak solution v to

$$X_i\left(\left(a_{\alpha\beta}^{ij}\right)_{B_{4r}(z)} X_j v\right) = 0 \quad \text{in } B_{4r}(z)$$

such that

$$\frac{1}{|B_{2r}(z)|} \int_{B_{2r}(z)} |X(u - v)|^2\, dx \le \eta^2. \tag{4.7}$$

Also, recall the interior $HW^{1,\infty}$ regularity of v (3.19):

$$\|Xv\|^2_{L^\infty(B_{3r}(z))} \le N_0^2. \tag{4.8}$$

Now, pick

$$N_1^2 = \max\left\{\frac{5^Q}{c_d}, 4N_0^2\right\}. \tag{4.9}$$

Then we claim that

$$\begin{aligned} &\left\{x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})}\left(|Xu|^2\right)(x) > N_1^2\right\} \cap B_r(z) \\ &\quad \subset \left\{x \in B_R(\overline{x}) : \mathcal{M}_{B_{2r}(z)}\left(|X(u - v)|^2\right)(x) > N_0^2\right\} \cap B_r(z). \end{aligned} \tag{4.10}$$

To see this, suppose

$$x_1 \in \left\{x \in B_R(\bar{x}) \cap B_r(z) : \mathcal{M}_{B_{2r}(z)}\left(|X(u-v)|^2\right)(x) \le N_0^2\right\}. \tag{4.11}$$

When $\rho \le r$, it follows that $B_\rho(x_1) \subset B_{2r}(z) \subset B_{5R}(\bar{x})$, then (4.11) and (4.8) imply

$$\begin{aligned}\frac{1}{\left|B_\rho(x_1) \cap B_{11R}(\bar{x})\right|}\int_{B_\rho(x_1)\cap B_{11R}(\bar{x})} |Xu|^2\,dx &= \frac{1}{\left|B_\rho(x_1)\right|}\int_{B_\rho(x_1)} |Xu|^2\,dx \\ &\le \frac{2}{\left|B_\rho(x_1)\right|}\int_{B_\rho(x_1)} \left(|X(u-v)|^2 + |Xv|^2\right)dx \le 4N_0^2 \le N_1^2. \end{aligned} \tag{4.12}$$

When $\rho > r$, since $x_1, x_0 \in B_r(z)$ we have $d(x_1, x_0) < 2r < 2\rho$; it follows that $B_\rho(x_1) \subset B_{3\rho}(x_0) \subset B_{5\rho}(x_1)$. Then by Remark 2.7 and (4.3) we have

$$\begin{aligned}\frac{1}{\left|B_\rho(x_1) \cap B_{11R}(\bar{x})\right|}\int_{B_\rho(x_1)\cap B_{11R}(\bar{x})} |Xu|^2\,dx &\le \frac{1}{c_d\left|B_\rho(x_1)\right|}\int_{B_{3\rho}(x_0)\cap B_{11R}(\bar{x})} |Xu|^2\,dx \\ &= \frac{5^Q}{c_d\left|B_{5\rho}(x_1)\right|}\int_{B_{3\rho}(x_0)\cap B_{11R}(\bar{x})} |Xu|^2\,dx \\ &\le \frac{5^Q}{c_d\left|B_{3\rho}(x_0)\cap B_{11R}(\bar{x})\right|}\int_{B_{3\rho}(x_0)\cap B_{11R}(\bar{x})} |Xu|^2\,dx \\ &\le \frac{5^Q}{c_d} \le N_1^2. \end{aligned} \tag{4.13}$$

By (4.12) and (4.13), we have

$$x_1 \in \left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right) \le N_1^2\right\} \cap B_r(z). \tag{4.14}$$

Thus, inclusion (4.10) follows from the fact that (4.11) implies (4.14).

By (4.10), Lemma 4.2 (ii) and (4.7), we have

$$\begin{aligned}&\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > N_1^2\right\} \cap B_r(z)\right| \\ &\quad\le \left|\left\{x \in B_{2r}(z) : \mathcal{M}_{B_{2r}(z)}\left(|X(u-v)|^2\right)(x) > N_0^2\right\}\right| \\ &\quad\le \frac{c}{N_0^2}\int_{B_{2r}(z)} |X(u-v)|^2\,dx \\ &\quad\le c\eta^2\left|B_{2r}(z)\right| = c2^Q\eta^2\left|B_r(z)\right| \\ &\quad= \varepsilon^2\left|B_r(z)\right|.\end{aligned}$$

For a fixed ε, we have finally chosen η so that $c2^Q\eta^2 = \varepsilon^2$ and picked the corresponding δ depending on R_0, μ and η, that is on R_0, μ, ε. This finishes our proof. □

Corollary 4.4. *For any $\varepsilon > 0$, $R_0 > 0$, there is a small $\delta = \delta(\varepsilon, R_0, \mu) > 0$ such that for any $R \le R_0/2$, $z \in B_R(\bar{x})$, $0 < r \le 2R$, if u is a weak solution of (1.1) in $B_{11R}(\bar{x}) \Subset \Omega$, the coefficients $a_{\alpha\beta}^{ij}(x)$ are $(\delta, 4r)$-vanishing in $B_R(\bar{x})$ and*

$$\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > N_1^2\right\} \cap B_r(z)\right| \ge \varepsilon\left|B_r(z)\right|,$$

then

$$B_r(z) \cap B_R(\bar{x}) \subset \left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > 1\right\} \cup \left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}|^2\right)(x) > \delta^2\right\}.$$

5. L^p estimate on $|Xu|$

In this section we exploit the local estimates on the maximal function of $|Xu|^2$ proved in the previous section in order to prove the desired L^p bound. The starting point is the following useful lemma about the estimate of the L^p norm of a function by means of its distribution function.

Lemma 5.1 (See [10, p. 62]). *Let $\theta > 0$, $m > 1$ be constants, $p \in (1, \infty)$. Then there exists $c > 0$ such that for any nonnegative and measurable function f in Ω,*

$$f \in L^p(\Omega) \quad \text{if and only if} \quad S = \sum_{l \ge 1} m^{lp}\left|\left\{x \in \Omega : f(x) > \theta m^l\right\}\right| < \infty$$

and

$$\frac{1}{c} S \le \|f\|^p_{L^p(\Omega)} \le c\left(|\Omega| + S\right).$$

Lemma 5.2 (*Vitali*)**.** *Let $\mathcal{F}$ be a family of d_X-balls in $\mathbb{R}^n$ with bounded radii. There exists a finite or countable sequence $\{B_i\} \subset \mathcal{F}$ of mutually disjoint balls such that*

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_i 5B_i$$

where $5B$ *is the ball with the same center as* B *and radius five times big.*

The proof is identical to that of the Euclidean case, with the Euclidean distance replaced by d_X here.

Lemma 5.3. *Let $0 < \varepsilon < 1$, C and D be two measurable sets satisfying $C \subset D \subset B_R(\bar{x}) \subset \Omega$, $|C| < \varepsilon\,|B_R(\bar{x})|$ and the following property:*

$$\forall x \in B_R(\bar{x}), \quad \forall r \le 2R, \qquad |C \cap B_r(x)| \ge \varepsilon\,|B_r(x)| \Longrightarrow B_r(x) \cap B_R(\bar{x}) \subset D. \tag{5.1}$$

Then

$$|C| \le \varepsilon \frac{5^Q}{c_d}\,|D|,$$

where c_d is the constants in (2.6).

Proof. For any $x \in C$, $C \subset B_R(\bar{x}) \subset B_{2R}(x)$; hence

$$|C \cap B_{2R}(x)| = |C| < \varepsilon\,|B_R(\bar{x})| < \varepsilon\,|B_{2R}(x)|.$$

On the other hand, by Lebesgue differentiation theorem, for a.e. $x \in C$,

$$\lim_{r\to 0} \frac{|C \cap B_r(x)|}{|B_r(x)|} = 1;$$

hence for a.e. $x \in C$ there is an $r_x \le 2R$ such that for all $r \in (r_x, 2R)$ it holds

$$\left|C \cap B_{r_x}(x)\right| \ge \varepsilon\left|B_{r_x}(x)\right| \quad \text{and} \quad |C \cap B_r(x)| < \varepsilon\,|B_r(x)|. \tag{5.2}$$

By Lemma 5.2, there are $x_1, x_2, \ldots \in C$, such that $B_{r_{x_1}}(x_1), B_{r_{x_2}}(x_2), \ldots$ are mutually disjoint and satisfy

$$\bigcup_k B_{5r_{x_k}}(x_k) \cap B_R(\bar{x}) \supset C.$$

By (5.2) and (2.1), we know

$$\left|C \cap B_{5r_{x_k}}(x_k)\right| < \varepsilon\left|B_{5r_{x_k}}(x_k)\right| = \varepsilon 5^Q \left|B_{r_{x_k}}(x_k)\right|.$$

Also,

$$\begin{aligned} |C| &= \left|\bigcup_k B_{5r_{x_k}}(x_k) \cap C\right| \le \sum_k \left|B_{5r_{x_k}}(x_k) \cap C\right| \\ &\le \varepsilon 5^Q \sum_k \left|B_{r_{x_k}}(x_k)\right| \\ &\le \varepsilon \frac{5^Q}{c_d} \sum_k \left|B_{r_{x_k}}(x_k) \cap B_R(\bar{x})\right|, \end{aligned}$$

where the last inequality follows since $B_R(\bar{x})$ is d_X-regular (see Remark 2.7). Moreover since the $B_{r_{x_k}}(x_k)$ are mutually disjoint the last quantity equals

$$= \varepsilon \frac{5^Q}{c_d} \left|\bigcup_k \left(B_{r_{x_k}}(x_k) \cap B_R(\bar{x})\right)\right| \le \varepsilon \frac{5^Q}{c_d}\,|D|,$$

since, by assumption (5.1), $B_{r_{x_k}}(x_k) \cap B_R(\bar{x}) \subset D$. This completes the proof. □

Theorem 5.4. *For any $\varepsilon > 0, R_0 > 0$ there is a small $\delta = \delta(\varepsilon, R_0, \mu) > 0$ such that for any $R \le R_0/2$, if u is a weak solution of (1.1) in $B_{11R}(\bar{x}) \Subset \Omega$, the coefficients $a^{ij}_{\alpha\beta}(x)$ are $(\delta, 8R)$-vanishing in $B_R(\bar{x})$ and*

$$\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > N_1^2\right\}\right| < \varepsilon\,|B_R(\bar{x})| \tag{5.3}$$

(where N_1 is like in Theorem 4.3), then for any positive integer m,

$$\begin{aligned}\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > N_1^{2m}\right\}\right| \le{}& \sum_{i=1}^{m}\varepsilon_1^i\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}|^2\right)(x) > \delta^2 N_1^{2(m-i)}\right\}\right| \\ &+ \varepsilon_1^m\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > 1\right\}\right|\end{aligned}$$

where $\varepsilon_1 = \varepsilon 5^Q/c_d$.

Proof. Fix $\varepsilon, R_0 > 0$ and pick $\delta = \delta(\varepsilon, R_0, \mu)$ as in Corollary 4.4. We will prove this assertion by induction on m. For $m = 1$, we want to apply Lemma 5.3 to

$$\begin{aligned}C &:= \left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > N_1^2\right\},\\ D &:= \left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}|^2\right)(x) > \delta^2\right\} \cup \left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > 1\right\}.\end{aligned}$$

Since $N_1 \ge 1$, $C \subset D \subset B_R(\bar{x})$. Also, by assumption $|C| < \varepsilon\,|B_R(\bar{x})|$. Let $x \in B_R(\bar{x})$ such that

$$|C \cap B_r(x)| \ge \varepsilon\,|B_r(x)|.$$

Then by Corollary 4.4

$$B_r(x) \cap B_R(\bar{x}) \subset D;$$

hence by Lemma 5.3

$$|C| \le \varepsilon\frac{5^Q}{c_d}\,|D|$$

which is our assertion for $m = 1$.

Now assume that the assertion is valid for some m. Let u be a weak solution to (1.1) in $B_{11R}(\bar{x})$ satisfying (5.3). Set $u_1 = u/N_1$ and $\mathbf{F}_1 = \mathbf{F}/N_1$, then u_1 is a weak solution of

$$X_i\left(a^{ij}_{\alpha\beta}(x)\,X_j u_1\right) = X_i\mathbf{F}_1$$

in $B_{11R}(\bar{x}) \Subset \Omega$, and satisfies

$$\begin{aligned}\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu_1|^2\right)(x) > N_1^2\right\}\right| &= \left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > N_1^4\right\}\right|\\ &< \left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > N_1^2\right\}\right| < \varepsilon\,|B_R(\bar{x})|.\end{aligned}$$

By the induction assumption on m, we have

$$\begin{aligned}&\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > N_1^{2(m+1)}\right\}\right| = \left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu_1|^2\right)(x) > N_1^{2m}\right\}\right|\\ &\quad\le \sum_{i=1}^{m}\varepsilon_1^i\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}_1|^2\right)(x) > \delta^2 N_1^{2(m-i)}\right\}\right| + \varepsilon_1^m\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu_1|^2\right)(x) > 1\right\}\right|\\ &\quad= \sum_{i=1}^{m}\varepsilon_1^i\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}|^2\right)(x) > \delta^2 N_1^{2(m+1-i)}\right\}\right| + \varepsilon_1^m\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > N_1^2\right\}\right|.\end{aligned} \tag{5.4}$$

On the other hand, by the assertion valid for $m = 1$,

$$\begin{aligned}\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > N_1^2\right\}\right| \le{}& \varepsilon_1\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}|^2\right)(x) > \delta^2\right\}\right|\\ &+ \varepsilon_1\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > 1\right\}\right|.\end{aligned} \tag{5.5}$$

Putting (5.5) into (5.4) we get

$$\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > N_1^{2(m+1)}\right\}\right| \le \sum_{i=1}^{m} \varepsilon_1^i \left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}|^2\right)(x) > \delta^2 N_1^{2(m+1-i)}\right\}\right|$$
$$+ \varepsilon_1^{m+1}\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > 1\right\}\right| + \varepsilon_1^{m+1}\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}|^2\right)(x) > \delta^2\right\}\right|$$
$$= \sum_{i=1}^{m+1} \varepsilon_1^i \left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}|^2\right)(x) > \delta^2 N_1^{2(m+1-i)}\right\}\right| + \varepsilon_1^{m+1}\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > 1\right\}\right|$$

which is the desired assertion for $m + 1$. This completes the proof. □

We can finally come to the following.

Proof of Theorem 2.13. Fix R_0, let $\varepsilon > 0$ to be chosen later, and pick $\delta = \delta(\varepsilon, R_0, \mu)$ as in Theorem 5.4. For $\lambda > 0$, let $u_\lambda = \frac{u}{\lambda}$, $\mathbf{F}_\lambda = \frac{\mathbf{F}}{\lambda}$. We claim that we can take λ large enough (depending on ε, u and $\mathbf{F}$) so that

$$\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu_\lambda|^2\right)(x) > N_1^2\right\}\right| < \varepsilon\,|B_R(\bar{x})| \tag{5.6}$$

and

$$\sum_{k=1}^{\infty} N_1^{kp}\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}_\lambda|^2\right)(x) > \delta N_1^{2k}\right\}\right| \le 1. \tag{5.7}$$

Actually, since $\mathbf{F} \in L^p\left(B_{11R}(\bar{x}); \mathbb{M}^{N\times q}\right)$ with $p > 2$, we have $\mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}_\lambda|^2\right)(x) \in L^{\frac{p}{2}}(B_{11R}(\bar{x}))$ by Lemma 4.2. Applying Lemma 5.1 with $f = \mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}_\lambda|^2\right)$, $\theta = \delta$, $m = N_1^2$, $\Omega = B_R(\bar{x})$ and p replaced by $p/2$, there is a positive constant c depending only on δ, p and N_1, such that

$$\sum_{k=1}^{\infty} N_1^{kp}\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}_\lambda|^2\right)(x) > \delta N_1^{2k}\right\}\right| \le c\left\|\mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}_\lambda|^2\right)\right\|^{p/2}_{L^{p/2}(B_{11R}(\bar{x}))} \le c\,\|\mathbf{F}_\lambda\|^p_{L^p(B_{11R}(\bar{x}))}.$$

Also, by Lemma 4.2 we have

$$\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu_\lambda|^2\right)(x) > N_1^2\right\}\right| = \left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right)(x) > \lambda^2 N_1^2\right\}\right| \le \frac{c}{\lambda^2 N_1^2}\|Xu\|^2_{L^2(B_{11R}(\bar{x}))}$$

Hence we can take

$$\lambda = c\left(\frac{\|Xu\|_{L^2(B_{11R}(\bar{x});\mathbb{R}^N)}}{\varepsilon^{1/2}\,|B_R(\bar{x})|^{1/2}} + \|\mathbf{F}\|_{L^p(B_{11R}(\bar{x}))}\right) \tag{5.8}$$

for some constant c depending on δ, p, N_1; hence $c = (\varepsilon, R_0, p, \mathbb{G})$, and get (5.6) and (5.7) satisfied.

Next, by (5.6) we can apply Theorem 5.4 to u_λ for this large λ, writing

$$\sum_{k=1}^{\infty} N_1^{kp}\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu_\lambda|^2\right)(x) > N_1^{2k}\right\}\right|$$
$$\le \sum_{k=1}^{\infty} N_1^{kp}\left(\sum_{i=1}^{k}\varepsilon_1^i\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}_\lambda|^2\right)(x) > \delta^2 N_1^{2(k-i)}\right\}\right|\right.$$
$$\left. + \varepsilon_1^k\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu_\lambda|^2\right)(x) > 1\right\}\right|\right)$$
$$= \sum_{i=1}^{\infty}\left(N_1^p\varepsilon_1\right)^i \sum_{k=i}^{\infty} N_1^{p(k-i)}\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|\mathbf{F}_\lambda|^2\right)(x) > \delta^2 N_1^{2(k-i)}\right\}\right|$$
$$+ \sum_{i=1}^{\infty}\left(N_1^p\varepsilon_1\right)^i\left|\left\{x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu_\lambda|^2\right)(x) > 1\right\}\right|$$

by (5.7)

$$= \sum_{i=1}^{\infty}\left(N_1^p\varepsilon_1\right)^i\left(1 + |B_R(\bar{x})|\right) < 1 + |B_R(\bar{x})|$$

taking ε so that $N_1^p \varepsilon_1 = 1/2$. We have finally chosen ε small enough, depending on p and $\mathbb{G}$, and a corresponding $\delta = \delta(\varepsilon, R_0, \mu) = \delta(p, \mathbb{G}, R_0, \mu)$.

Therefore we can apply Lemma 5.1 to $f = \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu_\lambda|^2\right)(x)$ and $m = N_1^2$ getting

$$\left\| \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu_\lambda|^2\right) \right\|_{L^{p/2}(B_R(\bar{x}))}^{p/2} \le c\left(1 + R^Q\right)$$

with $c = c(p, \mathbb{G})$, which by (5.8) implies

$$\left\| \mathcal{M}_{B_{11R}(\bar{x})}\left(|Xu|^2\right) \right\|_{L^{p/2}(B_R(\bar{x}))}^{1/2} \le c\left\{ \|Xu\|_{L^2(B_{11R}(\bar{x}))} + \|\mathbf{F}\|_{L^p(B_{11R}(\bar{x}))} \right\}$$

with $c = c(R, R_0, p, \mathbb{G})$, and recalling that $|f(x)| \le \mathcal{M}_{B_{11R}(\bar{x})}(f)(x)$ for a.e. x, we get

$$\|Xu\|_{L^p(B_R(\bar{x}))} \le c\left\{ \|Xu\|_{L^2(B_{11R}(\bar{x}))} + \|\mathbf{F}\|_{L^p(B_{11R}(\bar{x}))} \right\}.$$

This completes the proof of Theorem 2.13. □

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