Basic properties of nonsmooth Hörmander's vector fields and Poincaré's inequality*

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April 8, 2009

Abstract

We consider a family of vector fields

$$X_i = \sum_{j=1}^{p} b_{ij}(x) \partial_{x_j}$$

$(i = 1, 2, ..., n;\ n < p)$ defined in some bounded domain $\Omega \subset \mathbb{R}^p$ and assume that the X_i's satisfy Hörmander's rank condition of some step r in Ω, and $b_{ij} \in C^{r-1}(\overline{\Omega})$. We extend to this nonsmooth context some results which are well-known for smooth Hörmander's vector fields, namely: some basic properties of the distance induced by the vector fields, the doubling condition, Chow's connectivity theorem, and, under the stronger assumption $b_{ij} \in C^{r-1,1}(\Omega)$, Poincaré's inequality. By known results, these facts also imply a Sobolev embedding. All these tools allow to draw some consequences about second order differential operators modeled on these nonsmooth Hörmander's vector fields:

$$\sum_{i,j=1}^{n} X_i^* (a_{ij}(x) X_j)$$

where $\{a_{ij}\}$ is a uniformly elliptic matrix of $L^\infty(\Omega)$ functions.

Contents

*2000 AMS Classification: Primary 53C17; Secondary 46E35, 26D10. **Keywords**: nonsmooth Hörmander's vector fields, Poincaré inequality

1 Introduction

1.1 The problem

Let us consider a family of real valued vector fields

$$X_i = \sum_{j=1}^{p} b_{ij}\left(x\right) \partial_{x_j}$$

$(i = 0, 1, 2, ..., n;\ n < p)$ defined in some domain $\Omega \subset \mathbb{R}^p$. For the moment, we do not specify the regularity of the b_{ij}'s, but just assume that these coefficients have all the derivatives involved in the formulae which we will write. Let us define the *commutator* of two vector fields:

$$[X, Y] = XY - YX.$$

We also call *commutator of length* r an iterated commutator of the kind:

$$\left[X_{i_1}, \left[X_{i_2}, ... \left[X_{i_{r-1}}, X_{i_r}\right] ...\right]\right].$$

One says that the system of vector fields $X_0, \ldots, X_n$ satisfies *Hörmander's condition of step* r in Ω if the vector space spanned by the vector fields X_i's and their commutators of length up to r is the whole $\mathbb{R}^p$ at each point of Ω. A famous theorem by Hörmander, 1967, [26], states that if the X_i's are real valued, have C^∞ coefficients, and satisfy Hörmander's condition of some step r in Ω, then the linear second order differential operator:

$$L = \sum_{i=1}^{n} X_i^2 + X_0. \tag{1.1}$$

is hypoelliptic in Ω. This means, by definition, that whenever the equation $Lu = f$ is satisfied in Ω in distributional sense, then for any open subset $A \subset \Omega$,

$$f \in C^{\infty}(A) \Longrightarrow u \in C^{\infty}(A).$$

Another consequence of Hörmander's condition, which is known since the 1930's, is the *connectivity property*: any two points of Ω can be joined by a sequence of arcs of integral lines of the vector fields ("Chow's theorem", 1939 [2]; see also Rashevski, 1938 [46]). This fact suggests that one can define a distance induced by the vector fields, as the infimum of the lengths of the "admissible lines" (tangent at every point to some linear combination of the X_i's) connecting two points.

Starting from Hörmander's theorem, many other important properties have been proved, in the last 40 years, both regarding systems of Hörmander's vector fields and the metric they induce, and regarding second order differential operators structured on Hörmander's vector fields, like (1.1). In the first group of results, we recall:

- the doubling property of the Lebesgue measure with respect to the metric balls (Nagel-Stein-Wainger [43]);
- Poincaré's inequality with respect to the vector fields (Jerison [28]).

In the second group of results, we recall:

- the "subelliptic estimates" of $H^{\varepsilon,2}$ norm of u in terms of L^2 norms of Lu and u (Kohn [31]);
- the "$W^{2,p}$ estimates", involving second order derivatives with respect to the vector fields X_i, in terms of L^p norms of Lu and u (Folland [12], Rothschild-Stein [47]);
- estimates on the fundamental solution of L or $\partial_t - L$ (again [43], Sanchez-Calle [50], Jerison-Sanchez-Calle [29], Fefferman-Sanchez-Calle [11]).

Now, it is fairly natural to ask whether part of the previous theory still holds for a family of vector fields having only a partial regularity. Here are just a few facts which suggest this question:

(i) to check Hörmander's condition of step r one has to compute derivatives of order up to $r-1$ of the coefficients of vector fields;

(ii) the definition of distance induced by a system of vector fields makes sense as soon as the vector fields are, say, locally Lipschitz continuous (in this general case, however, the distance of two points could be infinite, and proving connectivity, studying the volume of metric balls, proving the doubling condition and so on are open problems);

(iii) apart from Hörmander's theorem about hypoellipticity, which is meaningful in the context of operators with C^{∞} coefficients, several important results about second order differential operators built on Hörmander's vector fields are stated in a form which makes sense also for vector fields with a limited regularity (e.g., Poincaré inequality, a priori estimates on X_iX_ju in L^p or Hölder spaces, *etc.*).

1.2 Previous results

Several authors have studied the subject of nonsmooth Hörmander's vector fields, approaching the problem under different points of view. We give a brief account of the main lines of research, without any attempt to quote neither all the papers nor all the authors who have given contributions in these directions.

1. *Nonsmooth diagonal vector fields*

$$X_i = a_i(x)\, \partial_{x_i}.$$

Here the typical assumptions are the following:

- the number of vector fields equals the dimension of the space;
- the i-th vector field involves only the derivative in the i-th direction;
- the coefficients a_i can vanish, so the operator $\sum X_i^2$ can be degenerate;
- the coefficients can be nonsmooth (typically, they are Lipschitz continuous, and satisfy some other structural assumptions).

These operators have been first studied in several papers of the 1980's by Franchi and Lanconelli, see [16], [17], [18], [19], [20], and also the more recent paper [13]. A recent work by Sawyer-Wheeden [52] deals extensively with these operators. Clearly, the particular structure of these vector fields allows to use ad-hoc techniques which cannot be employed in the general (non-diagonal) case.

2. *"Axiomatic theories" of general Lipschitz vector fields*, and the metrics induced by them. This means that, for instance, one assumes axiomatically the validity of a connectivity theorem, a doubling property for the metric balls, a Poincaré's inequality for the "gradient" defined by the system of vector fields, and proves as a consequence other interesting properties of the metric or of second order PDE's structured on the vector fields. A good deal of papers have been written in this spirit; we just quote some of the Authors and some of the papers on this subject, which are a good starting point for further bibliographic references: Capogna, Danielli, Franchi, Gallot, Garofalo, Gutierrez, Lanconelli, Morbidelli, Nhieu, Serapioni, Serra Cassano, Wheeden; see [1], [9], [14], [15], [22], [23], [24], [33]; see also the already quoted paper [52] and the one by Hajlasz-Koskela [25].

3. *Nonsmooth vector fields of step two.* The two papers by Montanari-Morbidelli [39], [40] consider vector fields with Lipschitz continuous coefficients, satisfying Hörmander's condition of step two, plus some other structural condition. The goal of these papers is to prove Poincaré's and Sobolev' type inequalities for these vector fields. Rampazzo-Sussman [45] adopt a different point of view; here the assumptions are very weak, considering Lipschitz vector fields satisfying (at step 2) a "set valued Lie bracket condition" previously introduced by the same Authors in the context of control theory; the Authors then establish some basic properties of this weak "commutator".

4. *"Nonlinear vector fields".* In the context of Levi-type equations, several Authors have considered vector fields with $C^{1,\alpha}$ coefficients, having a particular

structure, and satisfying Hörmander's condition of step 2; the final goal is to get a regularity theory for certain classes of nonlinear equations, which can be written as sum of squares of "nonlinear vector fields", i.e. vector fields whose coefficients depend on the first order derivatives of the solution. Assuming that the solution is $C^{2,\alpha}$, these vector fields become $C^{1,\alpha}$, and a good regularity theory for the corresponding linear equation then implies, by a bootstrap argument, the smoothness of the solution. Some results of this kind also involve higher steps. We refer to the papers by Citti [4], Citti-Montanari [6], [7], [8], Montanari [35], [36], Citti-Lanconelli-Montanari [5], Montanari-Lanconelli [37], Montanari-Lascialfari [38], and references therein.

We also quote some papers by Vodopyanov-Karmanova (see [30], [53] and references therein), where the Authors study the geometry of nonsmooth vector fields, and in particular establish a connectivity theorem assuming that the highest order commutators have $C^{1,\alpha}$ coefficients.

1.3 Aim of the present research and main results

Summarizing the discussion of the last paragraph, most of the previous results about nonsmooth vector fields *either* hold only for the step 2 case, *or* for vector fields with a particular structure, *or* assume axiomatically some important properties of the metric induced by the vector fields themselves.

Our aim is to develop a theory for any system of vector fields satisfying "Hörmander's condition", at any step, requiring that the coefficients of the vector fields possess the minimal number of derivatives necessary to check Hörmander's condition.

More precisely, our assumptions consist in asking that, for some integer $r \geq 2$, the vector fields $X_1, ..., X_n$ possess $C^{r-1,1}(\Omega)$ coefficients, and satisfy Hörmander's condition at step r. Under these assumptions, we prove some basic properties of the distance induced by the X_i's (see Propositions 2.3 and 5.8, Theorem 5.10), the doubling condition (Theorem 3.5 and Theorem 5.10), Chow's connectivity theorem (Theorem 5.5), and Poincaré's inequality (Theorem 7.2). Actually, most of our results (with the relevant exception of Poincaré's inequality) hold under the weaker assumption that $X_i \in C^{r-1}(\Omega)$. We will make precise our assumptions later.

These results constitute a first set of tools regarding "nonsmooth Hörmander's vector fields" which is enough to draw some interesting consequences. For instance, by known results, these facts also imply a Sobolev embedding (Theorem 8.1). All these tools then allow to prove some properties of solutions to second order differential equations of the kind:

$$\sum_{i,j=1}^{n} X_i^* \left(a_{ij}(x) X_j u\right) = 0$$

where $\{a_{ij}\}$ is a uniformly elliptic matrix of $L^\infty(\Omega)$ functions, and X_i are nonsmooth Hörmander's vector fields (see Theorem 8.2). Many other problems in this direction remain open, which we hope to address in a future.

Another feature of our work which we would like to point out here, is that we take into account explicitly the possibility of *weighted* vector fields. To explain this point, we recall that Hörmander's theorem refers to an operator of the kind

$$\sum_{i=1}^{n} X_i^2 + X_0$$

and that, both in dealing with the metric induced by the X_i's and in dealing on a priori estimates for second order operators, the field X_0 has "weight two", compared with $X_1, X_2, ..., X_n$ which have "weight one": in some sense, X_0 plays the role of a second order derivative, in a similar way as the time derivative enters the heat equation. In §2, making precise our assumptions and notation, we will explain how we take into account this fact.

1.4 Logical structure of the paper

The paper is sequenced into three parts. In the first part, consists in §§ 2-3, some results about nonsmooth vector fields are deduced from analogous results which are known to hold for smooth vector fields. Namely, in §2, after introducing notation and making precise our assumptions, we prove a first basic inequality relating the subelliptic metric d induced by nonsmooth vector fields and the Euclidean one. Then, in §3 we introduce, in a standard way, a family of smooth vector fields which approximate the nonsmooth ones in the neighborhood of a point (by Taylor's expansion of their coefficients), and prove that the metric balls of the distances induced by smooth and nonsmooth vector fields are comparable. In view of the doubling condition which holds in the smooth case, this implies the doubling condition also for the metric d. This approximation technique for nonsmooth vector fields has been already used by several authors, see for instance [5], [8].

In the second part, consisting in §§4-5, we study extensively exponential and "quasiexponential" maps built with our vector fields. In contrast with the style of the first part, here we do not use any approximation argument, but have to work directly with nonsmooth vector fields. The key result in §4 is Theorem 4.2, which says that the quasiexponential maps built composing in a suitable way the exponentials of our basic vector fields are approximated by the exponentials of commutators. As a consequence of this result, in §5 we can prove Theorem 5.1, which states that the set of points which are reachable moving along integral curves of the vector fields from a fixed point, is diffeomorphic to a neighborhood of the origin. The two theorems we have just quoted are perhaps the technical core of the paper, and the possibility of proving them under our mild smoothness assumptions relies on a careful study of regularity matters related to exponential and quasiexponential maps. These two theorems have several interesting consequences. The first is a version of Chow's connectivity theorem (Theorem 5.5). A second one is the proof of the local equivalence of the "control distance" d_1 attached to our system of vector fields (and defined without reference to the commutators) with d, as in the smooth case (Theorem

5.10). In the "axiomatic theories" of vector fields with Lipschitz continuous coefficients, d_1 is the natural distance that can be defined, while d (which involves commutators) is generally meaningless. Therefore, the equivalence of d and d_1 is a crucial point, because it allows to link the abstract results of axiomatic theories with our more concrete setting (this fact will be actually useful in §6). A third consequence is the possibility of controlling the increment of a function by means of its gradient with respect to the vector fields X_i (Theorem 5.11).

The third part of the paper consisting in §§6-8. Here we prove Poincaré's inequality and draw some consequences from the whole theory developed so far. In this part we set $X_0 \equiv 0$ (as is natural in the context of Poincaré-type inequalities) and strengthen our assumptions on the X_i's, asking them to belong to $C^{r-1,1}$, instead of C^{r-1} (recall that r is the maximum length of commutators required to check Hörmander's condition). Part 3 is in some sense a mix of the techniques employed in Part 1 and Part 2: namely, we make use of the approximation by smooth vector fields and apply some known results which hold in the smooth case (as in Part 1) but also have to make explicit computation with nonsmooth vector fields (as in Part 2). Our strategy to prove Poincaré's inequality is to exploit the general approach developed by Lanconelli-Morbidelli in [33], as well as Jerison's method of proving Poincaré's inequality first for the lifted vector fields and then in the general case. We will say more about this in §§6-7; here we just want to stress that all the results proved in this paper before Poincaré's inequality are needed, in order to apply the results in [33] and derive this result in our context.

Finally, in §8, we show some of the facts which immediately follow from our results, thanks to the existing "axiomatic theories": a Sobolev embedding, p-Poincaré's inequality, and Moser's iteration for variational second order operators structured on nonsmooth vector fields

The paper ends with an Appendix where we collect some miscellaneous known results about ordinary differential equations, which are used throughout the paper, together with the justification of a Claim made in §3.

Acknowledgements. We wish to thank Ermanno Lanconelli and Giovanna Citti for some useful conversation on the subject of this research.

While we were completing this paper, Annamaria Montanari and Daniele Morbidelli told us that they were working on similar problems, and have proved some results similar to ours in [41]. We thank these authors for sharing with us this information, and Daniele Morbidelli for having made important remarks on a preprint of this paper.

2 The subelliptic metric

Notation. Let $X_0, X_1, ..., X_n$ be a system of real vector fields, defined in a domain of $\mathbb{R}^p$. Let us assign to each X_i a *weight* p_i, saying that

$$p_0 = 2 \text{ and } p_i = 1 \text{ for } i = 1, 2, ...n.$$

The following standard notation, will be used throughout the paper. For any multiindex

$$I = (i_1, i_2, ..., i_k)$$

we define the *weight* of I as

$$|I| = \sum_{j=1}^{k} p_{i_j}.$$

Sometimes, we will also use the (usual) *length* of I,

$$\ell(I) = k.$$

For any multiindex $I = (i_1, i_2, ..., i_k)$ we set:

$$X_I = X_{i_1} X_{i_2} ... X_{i_k}$$

and

$$X_{[I]} = \left[X_{i_1}, \left[X_{i_2}, ... \left[X_{i_{k-1}}, X_{i_k}\right] ...\right]\right].$$

If $I = (i_1)$, then

$$X_{[I]} = X_{i_1} = X_I.$$

As usual, $X_{[I]}$ can be seen either as a differential operator or as a vector field. We will write

$$X_{[I]} f$$

to denote the differential operator $X_{[I]}$ acting on a function f, and

$$\left(X_{[I]}\right)_x$$

to denote the vector field $X_{[I]}$ evaluated at the point x.

Assumptions (A). We assume that for some integer $r \geq 2$ and some bounded domain (i.e., connected open subset) $\Omega \subset \mathbb{R}^p$ the following hold:

(A1) The coefficients of the vector fields $X_1, X_2, ..., X_n$ belong to $C^{r-1}\left(\overline{\Omega}\right)$, while the coefficients of X_0 belong to $C^{r-2}\left(\overline{\Omega}\right)$. Here and in the following, C^k stands for the classical space of functions with continuous derivatives up to order k.

(A2) The vectors $\left\{\left(X_{[I]}\right)_x\right\}_{|I| \leq r}$ span $\mathbb{R}^p$ at every point $x \in \Omega$.

Assumptions (A) will be in force throughout this section and the following. These assumptions are consistent in view of the following

Remark 2.1 *Under the assumption (A1) above, for any $1 \leq k \leq r$, the differential operators*

$$\{X_I\}_{|I| \leq k}$$

are well defined, and have C^{r-k} coefficients. The same is true for the vector fields $\left\{X_{[I]}\right\}_{|I| \leq k}$.

Dependence of the constants. We will often write that some constant depends on the vector fields X_i's and some fixed domain $\Omega' \Subset \Omega$. (Actually, the dependence on the X_i's will be usually left understood). Explicitly, this will mean that the constant depends on:

(i) Ω';

(ii) the norms $C^{r-1}\left(\overline{\Omega}\right)$ of the coefficients of X_i $(i = 1, 2, ..., n)$ and the norms $C^{r-2}\left(\overline{\Omega}\right)$ of the coefficients of X_0;

(iii) the moduli of continuity on $\overline{\Omega}$ of the highest order derivatives of the coefficients of the X_i's $(i = 0, 1, 2, ..., n)$.

(iv) a positive constant c_0 such that the following bound holds:

$$\inf_{x\in\Omega'} \max_{|I_1|,|I_2|,...,|I_p|\leq r} \left|\det\left(\left(X_{[I_1]}\right)_x, \left(X_{[I_2]}\right)_x, ..., \left(X_{[I_p]}\right)_x\right)\right| \geq c_0$$

(where "det" denotes the determinant of the $p \times p$ matrix having the vectors $\left(X_{[I_i]}\right)_x$ as rows).

Note that (iv) is a quantitative way of assuring the validity of Hörmander's condition, uniformly in Ω'.

The subelliptic metric introduced by Nagel-Stein-Wainger [43], in this situation is defined as follows:

Definition 2.2 *For any $\delta > 0$, let $C(\delta)$ be the class of absolutely continuous mappings $\varphi : [0, 1] \longrightarrow \Omega$ which satisfy*

$$\varphi'(t) = \sum_{|I|\leq r} a_I(t) \left(X_{[I]}\right)_{\varphi(t)} \quad a.e.$$

with $a_I : [0, 1] \to \mathbb{R}$ measurable functions,

$$|a_I(t)| \leq \delta^{|I|}.$$

Then define

$$d(x, y) = \inf \{\delta > 0 : \exists\varphi \in C(\delta) \text{ with } \varphi(0) = x, \varphi(1) = y\}.$$

The following property can be proved exactly like in the smooth case (see for instance Proposition 1.1 in [43]). We present a proof for the sake of completeness.

Proposition 2.3 (Relation with the Euclidean distance) *Assume (A1)-(A2). Then the function $d : \Omega \times \Omega \to \mathbb{R}$ is a (finite) distance. Moreover, there exist a positive constant c_1 depending on Ω and the X_i's and, for every $\Omega' \Subset \Omega$, a positive constant c_2 depending on Ω' and the X_i's, such that*

$$c_1 |x - y| \leq d(x, y) \leq c_2 |x - y|^{1/r} \text{ for any } x, y \in \Omega'. \tag{2.1}$$

Hence, in particular, the distance d induces Euclidean topology.

Proof. It is clear by definition that d is a distance. Namely, this follows from the fact that the union of two consecutive admissible curves can be reparametrized to give an admissible curve. To prove (2.1), let $\varphi \in C(\rho)$, for some ρ, be any curve joining x to y, contained in Ω, then:

$$\varphi'(t) = \sum_{|I| \le r} a_I(t) \left(X_{[I]}\right)_{\varphi(t)}$$
$$\varphi(0) = x, \varphi(1) = y, |a_I(t)| \le \rho^{|I|}.$$

Hence

$$|y - x| = \left| \int_0^1 \varphi'(t)\, dt \right| \le \int_0^1 \sum_{|I| \le r} \left| a_I(t) \left(X_{[I]}\right)_{\varphi(t)} \right| dt \le$$
$$\le \sup_{|I| \le r, z \in \Omega} \left| \left(X_{[I]}\right)_z \right| \cdot \sum_{|I| \le r} \rho^{|I|} \le c\rho.$$

By the definition of d, taking the infimum over ρ we get the first inequality in (2.1).

To prove the second inequality, fix $x_0 \in \Omega'$, and select a subset η of multi-indices I, $|I| \le r$, such that $\left\{X_{[I]}\right\}_{I \in \eta}$ is a basis of $\mathbb{R}^p$ at x_0, and therefore in a small neighborhood $U(x_0) \Subset \Omega$; by continuity of the vector fields $\left\{X_{[I]}\right\}_{I \in \eta}$, we can take $U(x_0)$ small enough so that the $p \times p$ matrix

$$\{\alpha_{IJ}(x)\}_{I,J \in \eta}, \text{ with } \alpha_{IJ}(x) = \left(X_{[I]}\right)_x \cdot \left(X_{[J]}\right)_x$$

be uniformly positive in $U(x_0)$:

$$\sum_{I,J \in \eta} \alpha_{IJ}(x) \xi_I \xi_J \ge c_0 |\xi|^2 \text{ for any } x \in U(x_0), \xi \in \mathbb{R}^p. \tag{2.2}$$

Now, for any $x, y \in U(x_0)$, let γ be any C^1 curve contained in $U(x_0)$, such that $\gamma(0) = x, \gamma(1) = y$, and such that length$(\gamma) \le c|x - y|$; moreover, we take γ of constant speed, hence

$$|\gamma'(t)| = \text{length}(\gamma). \tag{2.3}$$

Since the vector fields $\left\{X_{[I]}\right\}_{I \in \eta}$ are a basis of $\mathbb{R}^p$ at any point of $U(x_0)$, we can write

$$\gamma'(t) = \sum_{I \in \eta} a_I(t) \left(X_{[I]}\right)_{\gamma(t)}$$

for suitable functions a_I, where, by (2.2)

$$|\gamma'(t)|^2 \ge c_0 \sum_{J \in \eta} |a_J(t)|^2.$$

Hence, by (2.3),

$$|a_I(t)| \le c|\gamma'(t)| \le c|x - y| \le c|x - y|^{|I|/r}. \tag{2.4}$$

Therefore $\gamma \in C\left(c|x - y|^{1/r}\right)$ and the second inequality in (2.1) follows, for any $x, y \in U(x_0)$. A compactness argument gives the general case. ■

3 Approximating vector fields and the doubling condition

A key tool in the study of the nonsmooth vector fields X_i is to approximate them, locally, with smooth vector fields, as we shall explain in this section.

Let us start with the following general remark, which follows from the standard Taylor formula.

For any $f \in C^k(\Omega)$ and $\Omega' \Subset \Omega$, let us define the following moduli of continuity:

$$\omega_\alpha(\delta) = \sup\{|D^\alpha f(x) - D^\alpha f(y)| : x, y \in \Omega', |x-y| \le \delta\} \text{ for any } |\alpha| = k;$$
$$\omega_k(\delta) = \max_{|\alpha|=k} \omega_\alpha(\delta).$$

Then the following holds:

$$f(x) = \sum_{|\alpha| \le k} \frac{D^\alpha f(x_0)}{\alpha!}(x-x_0)^\alpha + O\left(|x-x_0|^k \omega_k(|x-x_0|)\right) \text{ for any } x, x_0 \in \Omega'.$$

The error term $O\left(|x-x_0|^k \omega_k(|x-x_0|)\right)$ can be rewritten as $o\left(|x-x_0|^k\right)$, where this symbol means that

$$\frac{o\left(|x-x_0|^k\right)}{|x-x_0|^k} \to 0 \text{ for } x \to x_0,$$

uniformly for x_0 ranging in Ω'. We stress that, although elementary, this remark is crucial in allowing us to prove the doubling condition assuming the coefficients of X_i just in C^{r-1} (and not, for instance, in $C^{r-1,1}$, as we shall do later).

Now, fix a point $x_0 \in \Omega$; for any $i = 0, 1, 2, ..., n$, let us consider the vector field

$$X_i = \sum_{j=1}^p b_{ij}(x)\,\partial_{x_j};$$

let $p_{ij}^r(x)$ be the Taylor polynomial of $b_{ij}(x)$ of center x_0 and order $r - p_i$; note that, under assumption (A1), and by the above remark,

$$b_{ij}(x) = p_{ij}^r(x) + o\left(|x-x_0|^{r-p_i}\right) \tag{3.1}$$

with the above meaning of the symbol $o(\cdot)$.

Set

$$S_i^{x_0} = \sum_{j=1}^p p_{ij}^r(x)\,\partial_{x_j}.$$

We will often write S_i in place of $S_i^{x_0}$, leaving the dependence on the point x_0 implicitly understood.

From (3.1) immediately follows:

Proposition 3.1 *Assume (A1) (see §2). Then the* $S_i^{x_0}$*'s* $(i = 0, 1, 2, ..., n)$ *are smooth vector fields defined in the whole space, satisfying:*

$$(S_I)_{x_0} = (X_I)_{x_0} \text{ and } \left(S_{[I]}\right)_{x_0} = \left(X_{[I]}\right)_{x_0} \text{ for any } I \text{ with } |I| \leq r.$$

Moreover,

$$X_{[I]} - S_{[I]} = \sum_{j=1}^{p} c_I^j(x)\, \partial_{x_j} \text{ with } c_I^j(x) = o\left(|x - x_0|^{r-|I|}\right). \tag{3.2}$$

We also need to check that the $S_i^{x_0}$'s satisfy Hörmander's condition in a neighborhood of x_0, with some uniform control on the diameter of this neighborhood:

Lemma 3.2 *For every domain* $\Omega' \Subset \Omega$ *there exists a constant* $\delta > 0$ *depending on* Ω' *and the* X_i*'s, such that for any* $x_0 \in \Omega'$ *the smooth vector fields* $S_1^{x_0}, S_2^{x_0}, ..., S_n^{x_0}$ *satisfy Hörmander's condition in*

$$U_\delta(x_0) = \{x \in \Omega : |x - x_0| < \delta\}.$$

Proof. Let $f(x, x_0) = \max_\eta \left| \det \left\{ \left(S_{[I]}^{x_0}\right)_x \right\}_{I \in \eta} \right|$ where the maximum is taken over all the possible choices of family η of p multiindices I with $|I| \leq r$. Writing the explicit form of the $S_i^{x_0}$'s:

$$S_i^{x_0} = \sum_{j=1}^{n} \left(\sum_{|\alpha| \leq r-1} \frac{D^\alpha b_{ij}(x_0)}{\alpha!} (x - x_0)^\alpha \right) \partial_{x_j}, \text{ where}$$

$$X_i = \sum_{j=1}^{n} b_{ij}(x)\, \partial_{x_j},$$

we see that the function f is continuous in $\Omega \times \Omega$. Also observe now that, since $\left(S_{[I]}^{x_0}\right)_{x_0} = \left(X_{[I]}\right)_{x_0}$ we have

$$f(x_0, x_0) = \max_\eta \left| \det \left\{ \left(X_{[I]}\right)_{x_0} \right\}_{I \in \eta} \right| \geq c_0 > 0 \;\forall x_0 \in \Omega'.$$

The uniform continuity of f (in a suitable domain that contains $\Omega' \times \Omega'$) allows to find $\delta > 0$ such that $f(x, x_0) \geqslant \frac{1}{2} c_0$ is $|x - x_0| \leqslant \delta$. This proves that in $U_\delta(x_0)$ the $S_i^{x_0}$ satisfy Hörmander's condition. ■

The family $\{S_i^{x_0}\}_{i=1}^n$ will be a key tool for us. Namely, throughout the paper we will apply to the $S_i^{x_0}$'s four important results proved by Nagel-Stein-Wainger [43] for smooth Hörmander's vector fields, namely: the estimate on the volume of metric balls; the doubling condition (both contained in [43, Theorem 1]); the equivalence between two different distances induced by the vector fields ([43, Theorem 4]), and a more technical result which we will recall later as

Theorem 7.5. Since, on the other hand, for every different point $x_0 \in \Omega'$ we are considering a *different system* of smooth vector fields, we are obliged to check that the constants appearing in Nagel-Stein-Wainger's estimates depend on the smooth vector fields in a way that allows to keep them under uniform control, for x_0 ranging in $\Omega' \Subset \Omega$. This is possible in view of the following:

Claim 3.3 *Let $S_1, S_2, ..., S_n$ be a system of smooth Hörmander's vector fields of step r in some neighbourhood Ω of a bounded domain $\Omega' \subset \mathbb{R}^p$. Then all the constants appearing in the estimates proved in [43] depend on the S_i's only through the following quantities:*

1. *an upper bound on the $C^k\left(\overline{\Omega'}\right)$ norms of the coefficients of the S_i's, for some "large" k only depending on the numbers p, n, r;*
2. *a positive lower bound on*

$$\inf_{x\in\Omega'} \max_{|I_1|,|I_2|,...,|I_p|\le r} \left|\det\left(\left(S_{[I_1]}\right)_x, \left(S_{[I_2]}\right)_x, ..., \left(S_{[I_p]}\right)_x\right)\right|.$$

A justification of this Claim will be sketched in the Appendix. Here we just recall that a similar claim (specifically referring to the doubling condition proved in [43]) was first made by Jerison in [28].

Now, let us fix a domain $\Omega' \Subset \Omega$; for any $x_0 \in \Omega'$ we can see by the explicit form of the $S_i^{x_0}$ that

1. the $C^k\left(\overline{\Omega'}\right)$ norms of the coefficients of the $S_i^{x_0}$'s are bounded, for all k, by a constant only depending on the $C^{r-p_i}\left(\overline{\Omega'}\right)$ norms of the coefficients of the vector fields X_i, the numbers r, p and the diameter of Ω'.

Moreover, from the proof of Lemma 3.2 we read that:

2. there exists a constant $c_0 > 0$ such that for any $x_0 \in \Omega'$, if $U_\delta(x_0)$ is the neighborhood appearing in Lemma 3.2, where the $S_i^{x_0}$ satisfy Hörmander's condition, then

$$\inf_{x\in U_\delta(x_0)} \max_{|I_j|\le r} \left|\det\left(\left(S_{[I_1]}^{x_0}\right)_x, \left(S_{[I_2]}^{x_0}\right)_x, ..., \left(S_{[I_p]}^{x_0}\right)_x\right)\right| \ge c_0.$$

The constant c_0 depends on vector fields X_i's only through the $C^{r-p_i}\left(\overline{\Omega'}\right)$ norms of the coefficients, the moduli of continuity on Ω' of the highest order derivatives of the coefficients, and the positive quantity

$$\inf_{x\in\Omega'} \max_{|I_j|\le r} \left|\det\left(\left(X_{[I_1]}\right)_x, \left(X_{[I_2]}\right)_x, ..., \left(X_{[I_p]}\right)_x\right)\right|.$$

The above discussion allows us to assure that every time we will apply to the system of approximating vector fields $S_i^{x_0}$ some results proved in [43] for smooth Hörmander's vector fields, the constants appearing in these estimates will be bounded, uniformly for x_0 ranging in Ω', in terms of quantities related to our original nonsmooth vector fields X_i.

In order to prove the doubling condition for the balls defined by the distance induced by nonsmooth vector fields, the simplest way is to compare this distance to the one induced by the smooth approximating vector fields S_i. We will show that these two distances are locally equivalent, in a suitable pointwise sense, which will be enough to deduce the doubling condition:

Theorem 3.4 (Approximating balls) *Assume (A1)-(A2). For any fixed $x_0 \in \Omega' \Subset \Omega$, let $S_i^{x_0}$ be the smooth vector fields defined as above. Let us denote by d_X and d_S the distances induced by the X_i's and the S_i's, respectively, and by B_X and B_S the corresponding metric balls. There exist positive constants c_1, c_2, r_0 depending on Ω, Ω' and the X_i's, but not on x_0, such that*

$$B_{S^{x_0}}(x_0, c_1\rho) \subset B_X(x_0,\rho) \subset B_{S^{x_0}}(x_0, c_2\rho)$$

for any $\rho < r_0$.

Proof. Let $x \in B_{S^{x_0}}(x_0,\rho)$. This means there exists $\phi(t)$ such that

$$\begin{cases} \phi'(t) = \sum_{|I|\leq r} a_I(t) \left(S_{[I]}^{x_0}\right)_{\phi(t)} \\ \phi(0) = x_0, \phi(1) = x \end{cases}$$

with $|a_I(t)| \leq \rho^{|I|}$. Let $\gamma(t)$ be *a solution* to the system

$$\begin{cases} \gamma'(t) = \sum_{|I|\leq r} a_I(t) \left(X_{[I]}\right)_{\gamma(t)} \\ \gamma(0) = x_0, \end{cases}$$

and set $x' = \gamma(1)$. (Note that in general we don't have *uniqueness*, because the $X_{[I]}$'s are just continuous if $|I| = r$ and the functions $a_I(\cdot)$ can be merely measurable; however, existence is granted by Carathéodory's theorem, see the Appendix).

By definition, $x' \in B_X(x_0,\rho)$. We have

$$\begin{aligned} &|\gamma(t) - \phi(t)| = \left|\int_0^t (\gamma'(s) - \phi'(s))\, ds\right| \leq \\ &\leq \sum_{|I|\leq r} \int_0^t |a_I(s)| \left|\left(X_{[I]}\right)_{\gamma(s)} - \left(S_{[I]}^{x_0}\right)_{\phi(s)}\right| ds \leq \\ &\leq \sum_{|I|\leq r} \int_0^t |a_I(s)| \left\{\left|\left(X_{[I]}\right)_{\gamma(s)} - \left(X_{[I]}\right)_{\phi(s)}\right| + \left|\left(X_{[I]}\right)_{\phi(s)} - \left(S_{[I]}^{x_0}\right)_{\phi(s)}\right|\right\} ds \\ &\equiv A + B. \end{aligned}$$

By (3.2) we have, for $\rho \leq r_0$, r_0 small enough

$$B \leq c \sum_{|I|\leq r} \int_0^t \rho^{|I|} \left|\phi(s) - x_0\right|^{r-|I|} ds.$$

Since, by (2.1), $|\phi(s) - x_0| \leq c d_{S^{x_0}}(\phi(s), x_0) \leq c\rho$,we obtain

$$B \leq \sum_{|I| \leq r} c\rho^{|I|}\rho^{r-|I|} = c\rho^r.$$

Moreover,

$$\begin{aligned} A &\leq \sum_{|I| \leq r} \int_0^t \rho^{|I|} \left| \left(X_{[I]}\right)_{\gamma(s)} - \left(X_{[I]}\right)_{\phi(s)} \right| ds \leq \\ &\leq c \sum_{|I|<r} \int_0^t \rho^{|I|} |\gamma(s) - \phi(s)| \, ds + \sum_{|I|=r} \rho^r \cdot 2 \sup_{z \in \overline{\Omega}} \left| \left(X_{[I]}\right)_z \right| \\ &\leq c\rho \int_0^t |\gamma(s) - \phi(s)| \, ds + c\rho^r \end{aligned}$$

where we used the fact that the $X_{[I]} \in C^1\left(\overline{\Omega}\right)$ for $|I| < r$ and $X_{[I]} \in C^0\left(\overline{\Omega}\right)$ for $|I| = r$. Therefore we have, for any $t \in (0,1)$

$$|\gamma(t) - \phi(t)| \leq c\rho \int_0^t |\gamma(s) - \phi(s)| \, ds + c\rho^r.$$

By Gronwall's Lemma (see the Appendix) this implies

$$|\gamma(t) - \phi(t)| \leq c\rho^r$$

for any $t \in (0,1)$, and so

$$|x - x'| \leq c\rho^r$$

which, again by (2.1), implies

$$d_X(x, x') \leq c|x - x'|^{1/r} \leq c\rho.$$

Since we already know that $x' \in B_X(x_0, \rho)$, we infer $x \in B_X(x_0, c_1\rho)$. In other words,

$$B_{S^{x_0}}(x_0, \rho) \subset B_X(x_0, c_1\rho).$$

We can now repeat the same argument exchanging the roles of $d_X, d_{S^{x_0}}$, and get

$$B_X(x_0, \rho) \subset B_{S^{x_0}}(x_0, c_2\rho).$$

Actually, in this case, some arguments simplify, due to the smoothness of the vector fields $S_i^{x_0}$. ■

By the doubling condition which holds for the balls induced by smooth vector fields, proved by Nagel-Stein-Wainger (see [43, Thm.1]), the above Theorem immediately implies the following

Theorem 3.5 (Doubling condition) *Assume (A1)-(A2). For any domain $\Omega' \Subset \Omega$, there exist positive constants c, r_0, depending on Ω, Ω' and the X_i's, such that*

$$|B_X(x_0, 2\rho)| \leq c|B_X(x_0, \rho)|$$

for any $x_0 \in \Omega'$, $\rho < r_0$.

Nagel-Stein-Wainger in [43] deduce the doubling condition from a sharp result about the volume of metric balls, which we recall here. Also this result follows, in the case of nonsmooth vector fields, by the above theorem about approximating balls. Even though, in our approach, the volume estimate of nonsmooth balls is not necessary to prove the nonsmooth doubling condition, it can be of independent interest.

Theorem 3.6 (Volume of metric balls) *Let η be any family of p multiindices $I_1, I_2, ..., I_p$ with $|I_j| \leq r$; let $|\eta| = \sum_{j=1}^{p} |I_j|$. Let $\lambda_\eta(x)$ be the determinant of the $p \times p$ matrix of rows $\left\{ \left(X_{[I_j]} \right)_x \right\}_{I_j \in \eta}$. For any $\Omega' \Subset \Omega$ there exist positive constants c_1, c_2, r_0 depending on Ω, Ω' and the X_i's, such that*

$$c_1 \sum_\eta |\lambda_\eta(x)| \rho^{|\eta|} \leq |B_X(x, \rho)| \leq c_2 \sum_\eta |\lambda_\eta(x)| \rho^{|\eta|}$$

for any $\rho < r_0$, $x \in \Omega'$, where the sum is taken over any family η with the above properties.

Proof. By Nagel-Stein-Wainger's theorem (see [43, Thm.1]),

$$c_1 \sum_\eta |\lambda_\eta(x)| \rho^{|\eta|} \leq |B_S(x, \rho)| \leq c_2 \sum_\eta |\lambda_\eta(x)| \rho^{|\eta|}$$

where B_S is the ball induced by the smooth vector fields S_i^x. Note that $\left(X_{[I]} \right)_x = \left(S_{[I]}^x \right)_x$ for $|I| \leq r$, therefore the quantity $\lambda_\eta(x)$ computed for the system X_i is the same of that computed for the system S_i^x. Moreover, the constants c_1, c_2, r_0 do not depend on the point x, but only on Ω, Ω' and the X_i's. Then the result follows by Theorem 3.4. ∎

4 Exponential and quasiexponential maps

In this section we slightly strengthen our assumptions as follows:

Assumptions (B). We keep assumptions (A) but, in the case $r = 2$, we also require X_0 to have Lipschitz continuous (instead of merely continuous) coefficients.

Accordingly, the constants in our estimates will depend on the X_i through the quantities stated in §2 (see "Dependence of the constants") and the Lipschitz norms of the coefficients of X_0.

Let us recall the standard definition of *exponential of a vector field*. We set:

$$\exp(tX)(x_0) = \varphi(t)$$

where φ is the solution to the Cauchy problem

$$\begin{cases} \varphi'(\tau) = X_{\varphi(\tau)} \\ \varphi(0) = x_0 \end{cases} \tag{4.1}$$

The point $\exp(tX)(x_0)$ is uniquely defined for $t \in \mathbb{R}$ small enough, as soon as X has Lipschitz continuous coefficients, by the classical Cauchy's theorem about existence and uniqueness for solutions to Cauchy problems. For a fixed $\Omega' \Subset \Omega$, a t-neighborhood of zero where $\exp(tX)(x_0)$ is defined can be found uniformly for x_0 ranging in Ω' (see the Appendix).

Equivalently, we can write

$$\exp(tX)(x_0) = \phi(1)$$

where ϕ is the solution to the Cauchy problem

$$\begin{cases} \phi'(\tau) = tX_{\phi(\tau)} \\ \phi(0) = x_0. \end{cases}$$

By definition of subelliptic distance, this implies in particular that

$$d\left(\exp\left(t^{p_i} X_i\right)(x_0), x_0\right) \leq ct \tag{4.2}$$

(recall that p_i is the weight of X_i, defined in §2).

Let us also define the following *quasiexponential maps* (for $i_1, i_2, ... \in \{0, 1, ..., n\}$):

$$\begin{aligned} C_1(t, X_{i_1}) &= \exp\left(t^{p_{i_1}} X_{i_1}\right); \\ C_2(t, X_{i_1} X_{i_2}) &= \exp\left(-t^{p_{i_2}} X_{i_2}\right) \exp\left(-t^{p_{i_1}} X_{i_1}\right) \exp\left(t^{p_{i_2}} X_{i_2}\right) \exp\left(t^{p_{i_1}} X_{i_1}\right); \\ &... \\ C_l(t, X_{i_1} X_{i_2} ... X_{i_l}) &= \\ &= C_{l-1}(t, X_{i_2} ... X_{i_l})^{-1} \exp\left(-t^{p_{i_1}} X_{i_1}\right) C_{l-1}(t, X_{i_2} ... X_{i_l}) \exp\left(t^{p_{i_1}} X_{i_1}\right) \end{aligned}$$

Remark 4.1 *Note that, in this definition, the exponential is taken only on the vector fields X_i ($i = 1, 2, ..., n$), which have at least C^1 coefficients, and X_0, which has at least Lipschitz continuous coefficients, hence C_l is well defined, for t small enough. Also, note that $C_{\ell(I)}(t, X_I)$ is the product of a fixed number (depending on $\ell(I)$) of factors of the kind $\exp(\pm t^{p_i} X_i)$ with $i = 0, 1, 2, ..., n$ (and $p_i = 2, 1, 1, ..., 1$, respectively). In particular, this implies that each map*

$$x \longmapsto C_{\ell(I)}(t, X_I)(x)$$

is invertible, for t small enough.

Moreover, if

$$x = C_l(t, X_{i_1} X_{i_2} ... X_{i_l})(x_0),$$

this means that the points x_0, x can be joined by a curve composed by a finite number of integral curves of the X_i's, and that $d(x, x_0) \leq ct$ (see (4.2)). We are going to show that every point x in a small neighborhood of x_0 can be obtained in this way: this will imply Chow's connectivity theorem.

The key result about the maps C_l defined above is the following:

Theorem 4.2 (Approximation of quasiexponential maps with commutators) *For any fixed* $x_0 \in \Omega$, *let* η *be a set of* p *multiindices* I *with* $|I| \leq r$ *such that* $\left\{ \left(X_{[I]} \right)_{x_0} \right\}_{I \in \eta}$ *is a basis of* $\mathbb{R}^p$. *Then there exists a neighborhood* U *of* x_0 *such that for any* $x \in U$ *and any* $I \in \eta$

$$C_{\ell(I)} \left(t, X_I \right) (x) = x + t^{|I|} \left(X_{[I]} \right)_x + o \left(t^{|I|} \right) \text{ as } t \to 0$$
$$C_{\ell(I)} \left(t, X_I \right)^{-1} (x) = x - t^{|I|} \left(X_{[I]} \right)_x + o \left(t^{|I|} \right) \text{ as } t \to 0$$

where the remainder $o \left(t^{|I|} \right)$ *is a map* $x \mapsto f(t)(x)$ *such that*

$$\sup_{x \in \overline{U}} \frac{|f(t)(x)|}{t^{|I|}} \to 0 \text{ as } t \to 0.$$

The above theorem says that moving in a suitable way along a chain of integral lines of the X_i's can give, as a net result, a displacement approximately in the direction of any commutator of the vector fields. Since the commutators span, this will imply that we can reach any point in this way.

The proof of the above theorem is organized in several steps. First of all, we have to get some sharp information about the degree of regularity of exponential maps. We start with the following classical result:

Theorem 4.3 *Let* $F(t,x) = \exp(tX)(x)$, *i.e.*

$$\begin{cases} \frac{\partial F}{\partial t} = X_{F(t,x)} \\ F(0,x) = x \end{cases} \tag{4.3}$$

where X *is* C^k *in a neighborhood of* x_0. *Then, the function* $(t,x) \longmapsto F(t,x)$ *is* C^k *in a neighborhood of* $(0, x_0)$.

Proof. In [44, §21 chap.3 and §29 in chap.4], it is proved that the derivatives

$$\frac{\partial^\alpha F}{\partial x^\alpha}(t,x)$$

are continuous in a neighborhood of $(0, x_0)$, for $|\alpha| \leq k$. To complete the proof, we have to check the continuity of mixed derivatives

$$\frac{\partial^{\alpha+\beta} F}{\partial x^\alpha \partial t^\beta}(t,x)$$

for $|\alpha| + |\beta| \leq k, |\beta| \geq 1$. If $\alpha = 0$, the required regularity is read from the equation. The general case requires an inductive reasoning. To fix ideas, let us consider the case $k = 2$. Then we have:

$$\frac{\partial^2 F(t,x)}{\partial x_i \partial t} = \frac{\partial}{\partial x_i} \left(X \left(F(t,x) \right) \right) = J_X \left(F(t,x) \right) \cdot \frac{\partial F(t,x)}{\partial x_i}$$

where J_X denotes the Jacobian matrix of the map $x \mapsto X_x$. Since we already know that $\frac{\partial F(t,x)}{\partial x_i}$ and J_X are continuous, continuity of $\frac{\partial^2 F(t,x)}{\partial x_i \partial t}$ follows. The general case can be treated analogously. ■

Corollary 4.4 *If X is C^k in a neighborhood of x_0 and F is as in (4.3), then*

$$\frac{\partial^{m+\alpha}F}{\partial t^m \partial x^\alpha} \in C\left(U\left(0,x_0\right)\right)$$

for some neighborhood $U\left(0,x_0\right)$, if $m \geq 1$ and $m+|\alpha| \leq k+1$.

Proof. If $m+|\alpha| \leq k$ this is contained in the previous theorem, so let $m+|\alpha| = k+1$, $m \geq 1$. Since F is C^k by the previous theorem, X is C^k by assumption, and

$$\frac{\partial F\left(t,x\right)}{\partial t} = X_{F(t,x)},$$

then $\frac{\partial F}{\partial t} \in C^k\left(U\left(0,x_0\right)\right)$. Hence

$$\frac{\partial^{m+\alpha}F}{\partial t^m \partial x^\alpha} = \frac{\partial^{m-1+\alpha}}{\partial t^{m-1}\partial x^\alpha}\frac{\partial F}{\partial t} \in C\left(U\left(0,x_0\right)\right).$$

■

Corollary 4.5 *If X is C^k in a neighborhood of x_0 and F is as in (4.3), then F is $k+1$ times differentiable at $(0,x)$, for any x in that neighborhood of x_0.*

Proof. By Theorem 4.3, we are left to prove that

$$\frac{\partial^{m+\alpha}F}{\partial t^m \partial x^\alpha} \text{ is differentiable at } (0,x) \text{ for } m+|\alpha| = k.$$

If $m \geq 1$, this fact is contained in Corollary 4.4, hence we have to prove that

$$\frac{\partial^\alpha F}{\partial x^\alpha} \text{ is differentiable at } (0,x) \text{ for } |\alpha| = k.$$

Let us write

$$\begin{aligned} &\frac{\partial^\alpha F}{\partial x^\alpha}\left(t,x+h\right) - \frac{\partial^\alpha F}{\partial x^\alpha}\left(0,x\right) = \\ &= \left[\frac{\partial^\alpha F}{\partial x^\alpha}\left(t,x+h\right) - \frac{\partial^\alpha F}{\partial x^\alpha}\left(0,x+h\right)\right] + \left[\frac{\partial^\alpha F}{\partial x^\alpha}\left(0,x+h\right) - \frac{\partial^\alpha F}{\partial x^\alpha}\left(0,x\right)\right] \\ &\equiv A+B. \end{aligned}$$

By Corollary 4.4, $\frac{\partial}{\partial t}\left(\frac{\partial^\alpha F}{\partial x^\alpha}\right)$ is continuous, hence for some $\tau \in (0,t)$ we have

$$A = t\frac{\partial^{\alpha+1}F}{\partial t \partial x^\alpha}\left(\tau,x+h\right) = t\left[\frac{\partial^{\alpha+1}F}{\partial t\partial x^\alpha}\left(0,x\right) + o\left(1\right)\right] \text{ for } (t,h) \to 0.$$

On the other hand, $F\left(0,x\right) = x$ for every x, hence

$$B = \frac{\partial^\alpha}{\partial x^\alpha}\left(x+h-x\right) = 0$$

so

$$\frac{\partial^{\alpha} F}{\partial x^{\alpha}}(t, x+h)-\frac{\partial^{\alpha} F}{\partial x^{\alpha}}(0, x)=t \frac{\partial^{\alpha+1} F}{\partial t \partial x^{\alpha}}(0, x)+o\left(\sqrt{t^{2}+h^{2}}\right),$$

and $\frac{\partial^{\alpha} F}{\partial x^{\alpha}}$ is differentiable at $(0, x)$. ■

In order to apply the previous results to quasiexponential maps, it is convenient to express the following step in an abstract way:

Proposition 4.6 *For some positive integer l, let us consider a family of functions*

$$F(t_1, t_2, ..., t_l)(\cdot)$$

defined in a neighborhood of x_0, with values in $\mathbb{R}^p$, depending on l scalar parameters $t_1, t_2, ..., t_l$, ranging in a neighborhood of 0, in such a way that:

$$F(t_1, t_2, ..., t_l)(x) = x$$

as soon as at least one of the t_j is equal to 0, and

$$(t_1, t_2, ..., t_l, x) \mapsto F(t_1, t_2, ..., t_l)(x)$$

is C^{l-1} in a neighborhood of $(0, 0, ..., 0, x)$ and differentiable l times at $(0, 0, ..., 0, x)$. Then the following expansion holds:

$$F(t_1, t_2, ..., t_l)(x) = x + t_1 t_2 ... t_l \frac{\partial^l F}{\partial t_1 \partial t_2 ... \partial t_l}(0, 0, ..., 0)(x) + o(t_1 t_2 ... t_l)$$

as $(t_1, t_2, ..., t_l) \to 0$.

Proof. Since $F(t_1, t_2, ..., t_l)(x) = x$ if $t_i = 0$ for some i, then

$$\frac{\partial F}{\partial t_j}(t_1, t_2, ..., t_l)(x) = 0 \text{ if } t_i = 0 \text{ for some } i \neq j.$$

Then we can write, since F is C^{l-1},

$$\begin{aligned}
F(t_1, t_2, ..., t_l)(x) &= x + \int_0^{t_1} \frac{\partial F}{\partial t_1}(u_1, t_2, ..., t_l)\, du_1 \\
&= x + \int_0^{t_1} \left[\frac{\partial F}{\partial t_1}(u_1, t_2, ..., t_l) - \frac{\partial F}{\partial t_1}(u_1, 0, t_3, ..., t_l)\right] du_1 \\
&= x + \int_0^{t_1} \int_0^{t_2} \frac{\partial^2 F}{\partial t_1 \partial t_2}(u_1, u_2, t_3 ..., t_l)\, du_2 du_1 \\
&= ... \\
&= x + \int_0^{t_1} ... \int_0^{t_{l-1}} \frac{\partial^{l-1} F}{\partial t_1 \partial t_2 ... \partial t_{l-1}}(u_1, u_2, ..., u_{l-1}, t_l)\, du_{l-1} ... du_1.
\end{aligned} \tag{4.4}$$

By assumption, $\frac{\partial^{l-1}F}{\partial t_1 \partial t_2 ... \partial t_{l-1}}$ is differentiable at $(0,0,...,0)(x)$, and

$$\frac{\partial}{\partial t_j}\frac{\partial^{l-1}F}{\partial t_1 \partial t_2 ... \partial t_{l-1}}(0,0,...,0)(x) = 0 \text{ for any } j \neq l,$$

hence the last expression in (4.4) equals

$$x + \int_0^{t_1} ... \int_0^{t_{l-1}} \left[t_l \frac{\partial^{l-1}F}{\partial t_1 \partial t_2 ... \partial t_{l-1}}(0,0,...,0) + o\left(\sqrt{u_1^2 + ...u_{l-1}^2 + t_l^2}\right)\right] du_{l-1}...du_1.$$

However, performing if necessary the integration with respect to the variables u_i in a different order, we can always assume that $t_l \geq \max(t_1, t_2, ..., t_{l-1})$, so that $o\left(\sqrt{u_1^2 + ...u_{l-1}^2 + t_l^2}\right) = o(t_l)$ and we get

$$\begin{aligned} & F(t_1, t_2, ..., t_l)(x) \\ &= x + \int_0^{t_1} ... \int_0^{t_{l-1}} \left[t_l \frac{\partial^{l-1}F}{\partial t_1 \partial t_2 ... \partial t_{l-1}}(0,0,...,0) + o(t_l)\right] du_{l-1}...du_1 \\ &= x + t_1 t_2 ... t_l \frac{\partial^l F}{\partial t_1 \partial t_2 ... \partial t_l}(0,0,...,0)(x) + o(t_1 t_2 ... t_l). \end{aligned}$$

■

We now come back to our vector fields. For fixed $\ell \leq r$ and $X_{i_1}, X_{i_2}, ..., X_{i_\ell}$, with $\{i_1, i_2, ..., i_\ell\} \subset \{0, 1, 2, ..., n\}$, let us define recursively the following maps:

$$\begin{aligned} \mathcal{C}_1(t_1)(x) &= \exp(t_1 X_{i_1})(x) \\ \mathcal{C}_2(t_1, t_2)(x) &= \exp(-t_2 X_{i_2}) \exp(-t_1 X_{i_1}) \exp(t_2 X_{i_2}) \exp(t_1 X_{i_1})(x) \\ &\vdots \\ \mathcal{C}_l(t_1, \dots t_l)(x) &= \mathcal{C}_{l-1}(t_2, \dots, t_l)^{-1} \exp(-t_1 X_1) \mathcal{C}_{\ell-1}(t_2, \dots, t_l) \exp(t_1 X_1)(x) \end{aligned}$$

Note that $\mathcal{C}_\ell(t^{p_{i_1}}, t^{p_{i_2}}, \dots, t^{p_{i_l}})$ coincides with the map $C_l(t, X_{i_1} X_{i_2} ... X_{i_l})$ previously defined. The previous Proposition implies the following:

Theorem 4.7 *For any multiindex I with $|I| \leq r$, $l = \ell(I)$ we have*

$$\mathcal{C}_l(t_1, \dots t_l)(x) = x + t_1 t_2 \dots t_l \frac{\partial^l \mathcal{C}_l(0,0,...,0)}{\partial t_1 \partial t_2 \cdots \partial t_l}(x) + o(t_1 t_2 \dots t_l) \tag{4.5}$$

as $(t_1, t_2, \dots, t_l) \to 0$. In particular:

$$C_l(t, X_{i_1} X_{i_2} ... X_{i_l})(x) = x + t^{|I|} \frac{\partial^l \mathcal{C}_l(0,0,...,0)}{\partial t_1 \partial t_2 \cdots \partial t_l}(x) + o\left(t^{|I|}\right) \tag{4.6}$$

as $t \to 0$, where the symbol $o(\cdot)$ has the meaning explained in Theorem 4.2.

Proof. We start noting that $\mathcal{C}_l(t_1, \dots t_l)$ reduces to the identity if at least one component of $(t_1, \dots t_l)$ vanishes. This can be proved inductively, as follows.

For $l=1$, this is just the identity $\exp(0)(x)=x$. Assume this holds up to $l-1$. Then, if $t_1=0$ we have

$$\begin{aligned}\mathcal{C}_l(0,t_2,\dots t_l)(x) &= \mathcal{C}_{l-1}(t_2,\dots,t_l)^{-1}\exp(0)\,\mathcal{C}_{l-1}(t_2,\dots,t_l)\exp(0)(x)\\ &= \mathcal{C}_{l-1}(t_2,\dots,t_l)^{-1}\,\mathcal{C}_{l-1}(t_2,\dots,t_l)(x)=x\end{aligned}$$

On the other side, if $(t_2,\dots,t_l)$ has some component that vanishes then

$$\begin{aligned}&\mathcal{C}_l(t_1,\dots t_l)(x)=\\ &=\mathcal{C}_{l-1}(t_2,\dots,t_l)^{-1}\exp(-t_1X_1)\,\mathcal{C}_{l-1}(t_2,\dots,t_l)\exp(t_1X_1)(x)\\ &=\mathbf{Id}\exp(-t_1X_1)\,\mathbf{Id}\exp(t_1X_1)(x)=\exp(-t_1X_1)\exp(t_1X_1)(x)=x.\end{aligned}$$

Now, assume first that the multiindex I does not contain any 0; then $\ell(I)=|I|\le r$; each vector field X_{i_j} is C^{r-1}, so that the function $\mathcal{C}_l(t_1,\dots t_l)(x)$ is C^{r-1} in a neighborhood of $(x,0)$ and, by Corollary 4.5 r times differentiable at $t=0$. Hence we can apply Proposition 4.6 and conclude (4.5).

If, instead, the multiindex I contains some 0, since X_0 is just C^{r-2}, by Corollary 4.5 we will conclude that the function $\mathcal{C}_l(t_1,\dots t_l)(x)$ is only $r-1$ times differentiable at $t=0$. On the other hand, in this case $\ell(I)\le|I|-1\le r-1$ (because X_0 has weight 2), so the function $\mathcal{C}_l(t_1,\dots t_l)(x)$ is still l times differentiable at $t=0$, and Proposition 4.6 still implies (4.5).

Finally, (4.5) implies (4.6) letting $t_i=t^{p_i}$ for $i=1,2,...,l$. ■

The identity (4.6) in Theorem 4.7 will imply Theorem 4.2 as soon as we will prove the following:

Theorem 4.8 *For any multiindex I with $|I|\le r$, $l=\ell(I)$ we have*

$$\frac{\partial^l\mathcal{C}_l(0,0,...,0)}{\partial t_1\cdots\partial t_l}(x)=\left(X_{[I]}\right)_x. \tag{4.7}$$

In order to prove Theorem 4.8, still another abstract Lemma is useful:

Lemma 4.9 *Let O be an open subset of $\mathbb{R}^p$ and $A,B:(-\varepsilon,\varepsilon)\times O\to\mathbb{R}^p$ be two C^1 funtions (for some $\varepsilon>0$); assume that for every $x\in O$ we have $A(0,x)=x$ and $B(0,x)=x$. Since $\frac{\partial A}{\partial x}(0,x)=\mathbf{Id}$ it follows that for every t sufficiently small $A(t,\cdot)$ is invertible. We denote with $A^{-1}(t,x)$ the inverse of this function. Similarly let $B^{-1}(t,x)$ the inverse of $B(t,\cdot)$. Let*

$$F(t,s,x)=A^{-1}\left(t,B^{-1}(s,A(t,B(s,x)))\right).$$

Assume that for every $x\in O$, A and B are two times differentiable at $(0,x)$. Then

$$\frac{\partial^2F}{\partial t\partial s}(0,0,x)=\frac{\partial^2A}{\partial x\partial t}(0,x)\frac{\partial B}{\partial s}(0,x)-\frac{\partial^2B}{\partial s\partial x}(0,x)\frac{\partial A}{\partial t}(0,x).$$

We have used a compact matrix notation, where for instance $\frac{\partial^2 A}{\partial x \partial t}(0,x)$ stands for the Jacobian of the map

$$x \mapsto \frac{\partial A}{\partial t}(0,x).$$

Proof. Since $\frac{\partial A}{\partial x}(0,x) = \mathbf{Id}$ we have $\frac{\partial^2 A}{\partial x_i \partial x_j}(0,x) = 0$ and the same holds with A replaced by A^{-1}, B and B^{-1}. Then

$$\frac{\partial F}{\partial t}(0,s,x) = \frac{\partial A^{-1}}{\partial t}\left(0, B^{-1}(s, A(0, B(s,x)))\right) \\ + \frac{\partial A^{-1}}{\partial x}\left(0, B^{-1}(s, A(t, B(s,x)))\right) \frac{\partial B^{-1}}{\partial x}(s, A(0, B(s,x))) \frac{\partial A}{\partial t}(0, B(s,x)).$$

Since $B^{-1}(s, A(0, B(s,x))) = B^{-1}(s, B(s,x)) = x$ and $\frac{\partial A^{-1}}{\partial x}(0,x) = \mathbf{Id}$ the above equation reduces to

$$\frac{\partial F}{\partial t}(0,s,x) = \frac{\partial A^{-1}}{\partial t}(0,x) + \frac{\partial B^{-1}}{\partial x}(s, B(s,x)) \frac{\partial A}{\partial t}(0, B(s,x)).$$

Let us compute

$$\frac{\partial F(0,0,x)}{\partial t \partial s} = \left[\frac{\partial B^{-1}}{\partial s \partial x}(0, B(0,x)) + \frac{\partial^2 B^{-1}}{\partial x^2}(0, B(0,x)) \frac{\partial B}{\partial s}(0,x)\right] \frac{\partial A}{\partial t}(0, B(0,x)) \\ + \frac{\partial B^{-1}}{\partial x}(0, B(0,x)) \frac{\partial^2 A}{\partial x \partial t}(0, B(0,x)) \frac{\partial B}{\partial s}(0,x)$$

Since $\frac{\partial B^{-1}}{\partial x}(0, B(0,x)) = \mathbf{Id}$ and $\frac{\partial^2 B^{-1}}{\partial x^2}(0, B(0,x)) = 0$ we have

$$\frac{\partial F(0,0,x)}{\partial t \partial s} = \frac{\partial^2 B^{-1}}{\partial s \partial x}(0,x) \frac{\partial A}{\partial t}(0,x) + \frac{\partial^2 A}{\partial x \partial t}(0,x) \frac{\partial B}{\partial s}(0,x).$$

Finally since $B^{-1}(s, B(s,x)) = x$ a simple computation shows that $\frac{\partial B^{-1}}{\partial s}(0,x) = -\frac{\partial B}{\partial s}(0,x)$ and therefore

$$\frac{\partial F(0,0,x)}{\partial t \partial s} = -\frac{\partial^2 B}{\partial s \partial x}(0,x) \frac{\partial A}{\partial t}(0,x) + \frac{\partial^2 A}{\partial x \partial t}(0,x) \frac{\partial B}{\partial s}(0,x).$$

■

Proof of Theorem 4.8. We prove the theorem by induction on l. For $l = 1$ the Theorem is trivial:

$$\frac{\partial \mathcal{C}_1(0)(x)}{\partial t} = \frac{\partial}{\partial t} \exp(tX_i)_{/t=0}(x) = (X_i)_x.$$

Assume the theorem holds for $l-1$ and let us prove it for $l \geq 2$. Let

$$A(t_2, \ldots t_l, x) = C_{\ell-1}(t_2, \ldots t_l)(x)$$

and

$$B(t_1,x) = \exp(t_1 X_{i_1})(x).$$

In order to apply Lemma 4.9, we have to check that A, B are C^1 and twice differentiable at $t=0$. Let us distinguish the following cases:

a) The multiindex I does not contain any 0. Then all the X_{i_j} are C^{r-1}, that is at least C^1, hence A, B are C^1 and, by Corollary 4.5, twice differentiable at $t=0$.

b) The multiindex I contains at least a 0. Then, since we are commuting at least two vector fields, at least one of which is X_0, we have that $r \geq 3$. Therefore all the X_i's are at least C^1, hence A, B are C^1 and, by Corollary 4.5, twice differentiable at $t=0$.

We can then apply Lemma 4.9 to A, B, with respect to the variables t_1, t_2 (regarding $t_3, .., t_l$ as parameters), obtaining:

$$\begin{aligned}
&\frac{\partial^2 C_l}{\partial t_1 \partial t_2}(0,0,t_3,\dots,t_l)(x) \\
&= -\frac{\partial^2 B}{\partial t_1 \partial x}(0,x)\frac{\partial A}{\partial t_2}(0,x) + \frac{\partial^2 A}{\partial x \partial t_2}(0,x)\frac{\partial B}{\partial t_1}(0,x) \\
&= -\frac{\partial X_{i_1}}{\partial x}(x)\frac{\partial C_{l-1}(0,t_3,\dots,t_\ell)}{\partial t_2}(x) + \frac{\partial}{\partial x}\frac{\partial C_{l-1}(0,t_3,\dots,t_\ell)}{\partial t_2}(x) X_{i_1}(x)
\end{aligned}$$

We can now compute the remaining $\ell-2$ derivatives in 0 (observe that by Theorem 4.7 we already know that we can compute r derivatives of C_{l-1} at $t=0$). This yields

$$\begin{aligned}
&\frac{\partial^l C_\ell}{\partial t_1 \cdots \partial t_l}(0,\dots,0) = \\
&= -\frac{\partial X_{i_1}}{\partial x}(x)\frac{\partial C_{l-1}(0,\dots,0)}{\partial t_2 \cdots \partial t_l}(x) + \frac{\partial}{\partial x}\frac{\partial C_{l-1}(0,\dots,0)}{\partial t_2 \cdots \partial t_l}(x) X_{i_1}(x).
\end{aligned}$$

Since by inductive assumption we have $\frac{\partial C_{l-1}(0,\dots,0)}{\partial t_2 \cdots \partial t_l}(x) = \left[X_{i_2}, \dots \left[X_{i_{\ell-1}}, X_{i_\ell}\right]\right]_x$ the Theorem follows. ■

As already noted, from Theorem 4.7 and Theorem 4.8, Theorem 4.2 follows.

5 Connectivity and equivalent distances

In this section we have to further strengthen our assumption on X_0:

Assumptions (C). We keep assumptions (A) but, in the case $r=2$, we also require X_0 to have C^1 coefficients (instead of merely continuous, as in §2 or Lipschitz continuous, as in §4) .

Let us define the maps:

$$E_I(t) = \begin{cases} C_{\ell(I)}\left(t^{1/|I|}, X_I\right) & \text{if } t \geq 0 \\ C_{\ell(I)}\left(|t|^{1/|I|}, X_I\right)^{-1} & \text{if } t < 0 \end{cases}.$$

for any $I \in \eta$ (where η is like in Theorem 4.2). By Theorem 4.2, the following expansion holds:

$$E_I(t)(x) = x + t\left(X_{[I]}\right)_x + o(t) \text{ as } t \to 0. \tag{5.1}$$

We are now in position to state the main result of this section:

Theorem 5.1 *Let $\Omega' \Subset \Omega$, $x_0 \in \Omega'$ and let $\left\{X_{[I_j]}\right\}_{I_j\in\eta}$ be any family of p commutators (with $|I_j| \leq r$) which span $\mathbb{R}^p$ at x_0, satisfying*

$$\left|\det\left\{\left(X_{[I_j]}\right)_{x_0}\right\}_{I_j\in\eta}\right| \geq (1-\varepsilon)\max_{\zeta}\left|\det\left\{\left(X_{[I_j]}\right)_{x_0}\right\}_{I_j\in\zeta}\right| \tag{5.2}$$

for some $\varepsilon \in (0,1)$. Then there exist constants $\delta_1, \delta_2 > 0$, depending on Ω', ε and the X_i's, such that the map

$$(h_1, h_2, ..., h_p) \mapsto E_{I_1}(h_1)\,E_{I_2}(h_2)\,...E_{I_p}(h_p)(x_0)$$

is a C^1 diffeomorphism of a neighborhood of the origin $\{h : |h| < \delta_1\}$ onto a neighborhood $U(x_0)$ of x_0 containing $\{x : |x - x_0| < \delta_2\}$. It is also a C^1 map in the joint variables $h_1, h_2, ..., h_p, x$ for $x \in \Omega'$ and $|h| < \delta_1$.

To stress the dependence of this diffeomorphism on the system of vector fields $\{X_i\}$, the choice of the basis η, and the point x, we will write

$$E_\eta^X(x,h) = E_{I_1}(h_1)\,E_{I_2}(h_2)\cdots E_{I_p}(h_p)(x).$$

Proof. First of all, let us check that the map

$$(x,h) \longmapsto E_{I_1}(h_1)\,E_{I_2}(h_2)\cdots E_{I_p}(h_p)(x) \tag{5.3}$$

is of class C^1 for $x \in \overline{\Omega'}$ and $|h| \leq \delta$, for some $\delta > 0$. This will follow, by composition, if we prove that for any multiindex I, the map

$$(t,x) \mapsto E_I(t)(x)$$

is C^1.

Assume first that the multiindex I does not contain any 0, so that each vector field which enters the definition of $E_I(t)(x)$ is of class C^{r-1}. Then, by Corollary 4.4, and Corollary 4.5 we know that each function

$$(t,x) \longmapsto C_I(t)(x)$$

is C^{r-1} in the joint variables, for t in a neighborhood of the origin and x in a fixed neighborhood of some x_0, and differentiable r times at $(0,x)$. By composition, the map $(t,x) \mapsto E_I(t)(x)$ is C^1 in x, and has continuous t derivative for $t \neq 0$. Moreover, the expansion (5.1) shows that there exists

$$\frac{\partial E_I(0)(x)}{\partial t} = \left(X_{[I]}\right)_x. \tag{5.4}$$

It remains to prove that

$$\frac{\partial E_I(t)(x)}{\partial t} \to \left(X_{[I]}\right)_x \text{ for } t \to 0. \tag{5.5}$$

Since $C_I(t)(x)$ is differentiable r times at $t = 0$ (and $r \geq |I|$), the expansion of $C_I(t)(x)$ given by Theorem 4.2 also says that

$$\frac{\partial C_I(t)(x)}{\partial t} = |I|\, t^{|I|-1} \left(X_{[I]}\right)_x + o\left(t^{|I|-1}\right)$$

Then we can compute:

$$\lim_{t\to 0} \frac{\partial E_I(t)(x)}{\partial t} = \lim_{t\to 0} \frac{1}{|I|\, t^{1-1/|I|}} \frac{\partial C_I\left(t^{1/|I|}\right)(x)}{\partial t} =$$

$$= \lim_{t\to 0} \frac{1}{|I|\, t^{|I|-1}} \frac{\partial C_I(t)(x)}{\partial t} = \lim_{t\to 0} \frac{1}{|I|\, t^{|I|-1}} \left(|I|\, t^{|I|-1} \left(X_{[I]}\right)_x + o\left(t^{|I|-1}\right)\right) = \left(X_{[I]}\right)_x$$

and this allows to conclude that the map (5.3) is C^1.

Let us now consider the case when the multiindex I also contains some 0, so that the vector field X_0, of class C^{r-2}, enters the definition of $E_I(t)(x)$. This case requires a more careful inspection. First of all, by our Assumptions (C) all the X_i's ($i = 0, 1, ..., n$) are at least C^1, and the function $E_I(t)(x)$ is C^1 in the joint variables for $t \neq 0$. Again, (5.4) is in force, and we are reduced to proving (5.5). Assume $|I| = r$ (the case $|I| < r$ is easier). Since X_0 has weight two, we have $l = \ell(I) < r$.

Let us consider the function $\mathcal{C}_l(t_1, \ldots, t_l)(x)$ introduced in the previous section; we have:

$$g(t) \equiv E_I(t)(x) = \mathcal{C}_l\left(t^{p_{k_1}/r}, \ldots, t^{p_{k_l}/r}\right)(x) \tag{5.6}$$

where each p_{k_i} is the weight of a vector field ($p_{k_i} = 1$ or 2), and $p_{k_1} + p_{k_2} + ... + p_{k_l} = r$. Our goal consists in proving that $g'(t) \to g'(0)$ as $t \to 0$.

Claim 5.2 *The function*

$$(t_1, \ldots, t_l, x) \mapsto \mathcal{C}_l(t_1, \ldots t_l)(x)$$

is C^{l-1}, and is l times differentiable at any point $(t_1, \ldots, t_l, x)$ such that $t_j = 0$ if $p_{k_j} = 2$.

Proof of the Claim. We know that

$$\mathcal{C}_l(t_1, \ldots t_l)(x) = \prod_{j=1}^{N} \exp\left(\pm t_{k_j} X_{k_j}\right)(x).$$

If $p_{k_j} = 1$, then $X_{k_j} \in C^{r-1} \subset C^l$ (since $l < r$), and the function

$$(t, x) \mapsto \exp\left(\pm t X_{k_j}\right)(x)$$

is l times differentiable at any point (t,x);

if $p_{k_j}=2$, then $X_{k_j}\in C^{r-2}\subset C^{l-1}$ (since $l<r$), and the function

$$(t,x)\mapsto \exp\left(\pm tX_{k_j}\right)(x)$$

is l times differentiable at any point $(0,x)$, by Corollary 14. By composition, the Claim follows. ■

Claim 5.3 *The function*

$$t_h\mapsto \frac{\partial^l\mathcal{C}_l}{\partial t_1...\partial t_l}(0,...,t_h,...,0)(x)$$

is continuous if $p_{i_h}=1$.

Proof of the Claim. By the previous claim, this derivative actually exists; we have to prove its continuity. Indeed, one can easily check by induction that the derivative

$$\frac{\partial^l\mathcal{C}_l}{\partial t_1...\partial t_l}(0,...,t_h,...,0)(x)$$

is a polynomial in variables of the form

$$\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}\left(\exp\left(\pm t_iX_i\right)\right)\left(\prod_j \exp\left(\pm t_{k_j}X_{k_j}\right)(x)\right) \tag{5.7}$$

with $|\alpha|\leqslant \ell$ and

$$\frac{\partial^{|\alpha|+1}}{\partial t_i\partial x^{\alpha}}\left(\exp\left(\pm t_iX_i\right)\right)\left(\prod_j \exp\left(\pm t_{k_j}X_{k_j}\right)(x)\right)=\pm\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}X_i\left(\prod_j \exp\left(\pm t_{k_j}X_{k_j}\right)(x)\right) \tag{5.8}$$

with $|\alpha|\leqslant \ell-1$. These derivatives should be evaluated for $t_i=0$ when $i\neq h$.

Let us consider the derivatives of the form (5.7) and assume first $i\neq h$ so that $t_i=0$. In this case the map $\exp\left(\pm t_iX_i\right)$ reduces to the identity and the derivative is obsiously continuous. When $i=h$ the continuity follows from the fact that $X_h\in C^{\ell}$.

The continuity of the derivatives of the form (5.8) follows from the fact that in this case $|\alpha|\leqslant \ell-1$ and $X_i\in C^{\ell-1}$. ■

By Proposition 4.6, we know that

$$\mathcal{C}_l\left(t_1,t_2,...,t_l\right)(x)=x+t_1t_2...t_l\frac{\partial^l\mathcal{C}_l}{\partial t_1\partial t_2...\partial t_l}(0,0,...,0)(x)+o\left(t_1t_2...t_l\right).$$

Assume for a moment that we can prove an analogous expansion for first order derivatives of $\mathcal{C}_l$, namely

$$\frac{\partial\mathcal{C}_l\left(t_1,\dots t_l\right)(x)}{\partial t_1}=t_2t_3...t_l\frac{\partial^l\mathcal{C}_l}{\partial t_1\partial t_2...\partial t_{l-1}\partial t_l}(0,0,...,0)(x)+o\left(t_2t_3...t_l\right). \tag{5.9}$$

Then we could easily conclude the proof as follows.

Let us compute

$$g'(t)=\sum_{j=1}^{l}\frac{p_{k_j}}{r}t^{p_{k_j}/r-1}\frac{\partial\mathcal{C}_l\left(t^{p_{k_1}/r},\dots,t^{p_{k_l}/r}\right)(x)}{\partial t_j}=$$

applying (5.9) to every j-derivative of $\mathcal{C}_l$

$$\begin{aligned}
&=\sum_{j=1}^{l}\frac{p_{k_j}}{r}t^{p_{k_j}/r-1}\left[\frac{\partial\mathcal{C}_l(0,0,...,0)(x)}{\partial t_j}t^{\sum_{i\neq j}p_{k_j}/r}+o\left(t^{\sum_{i\neq j}p_{k_j}/r}\right)\right]\\
&=\sum_{j=1}^{l}\frac{p_{k_j}}{r}t^{p_{k_j}/r-1}\left[\frac{\partial\mathcal{C}_l(0,0,...,0)(x)}{\partial t_j}t^{1-p_{k_j}/r}+o\left(t^{1-p_{k_j}/r}\right)\right]\\
&=\frac{\partial\mathcal{C}_l(0,0,...,0)(x)}{\partial t_j}\sum_{j=1}^{l}\frac{p_{k_j}}{r}\left[1+o(1)\right]=\frac{\partial\mathcal{C}_l(0,0,...,0)(x)}{\partial t_j}+o(1)\\
&\to\frac{\partial\mathcal{C}_l(0,0,...,0)(x)}{\partial t_j}=g'(0)\ \text{ as } t\to 0.
\end{aligned}$$

So we are left to prove (5.9). What we can actually prove is a slightly less general assertion, which is enough to perform the above computation:

Claim 5.4 *The expansion (5.9) holds if*

$$t_i=t^{p_{k_i}/r}\ \textit{for}\ i=1,2,...,l,\ \textit{and}\ t\to 0.$$

Proof of the Claim. As we have seen in the proof of Proposition 4.6, since $\mathcal{C}_l$ is C^{l-1} we can write

$$\mathcal{C}_l(t_1,\dots t_l)(x)=x+\int_0^{t_1}...\int_0^{t_{l-1}}\frac{\partial^{l-1}\mathcal{C}_l}{\partial t_1\partial t_2...\partial t_{l-1}}(u_1,u_2,...,u_{l-1},t_l)(x)\,du_{l-1}...du_1$$

and hence, differentiating with respect to t_1,

$$\frac{\partial\mathcal{C}_l(t_1,\dots t_l)(x)}{\partial t_1}=\int_0^{t_2}...\int_0^{t_{l-1}}\frac{\partial^{l-1}\mathcal{C}_l}{\partial t_1\partial t_2...\partial t_{l-1}}(t_1,u_2,...,u_{l-1},t_l)(x)\,du_{l-1}...du_1 \tag{5.10}$$

$$\begin{aligned}
&=\int_0^{t_2}...\int_0^{t_{l-1}}\left[t_l\frac{\partial^{l}\mathcal{C}_l}{\partial t_1\partial t_2...\partial t_{l-1}\partial t_l}(0,0,...,0)(x)+\right.\\
&\qquad\left.+o\left(\sqrt{t_1^2+u_2^2+...+u_{l-1}^2+t_l^2}\right)\right]du_{l-1}...du_1\\
&=t_2t_3...t_l\frac{\partial^{l}\mathcal{C}_l}{\partial t_1\partial t_2...\partial t_{l-1}\partial t_l}(0,0,...,0)(x)+t_2t_3...t_{l-1}\cdot o(|t|)\,.
\end{aligned}$$

Now, if

$$\max_{j=1,\dots,l}|t_j|=|t_i|\ \text{ with } i\neq 1, \tag{5.11}$$

without loss of generality we can suppose that this maximum is assumed for $i = l$. In this case we can write

$$\frac{\partial \mathcal{C}_l(t_1,\dots t_l)(x)}{\partial t_1} = t_2 t_3 ... t_l \frac{\partial^l \mathcal{C}_l}{\partial t_1 \partial t_2 ... \partial t_{l-1} \partial t_l}(0,0,...,0)(x) + o(t_2 t_3 ... t_l).$$

Note that, for $t_i = t^{p_{k_i}/r}$ and $t \to 0$, condition (5.11) just means $p_{k_1} = 2$.

Assume, instead, that

$$\max_{j=1,\dots,l} |t_j| = |t_1|, \text{ that is } p_{k_1} = 1.$$

In this case, we start again with (5.10) but now we exploit the fact that $\frac{\partial^{l-1}\mathcal{C}_l}{\partial t_1 \partial t_2 ... \partial t_{l-1}}$ is differentiable at $(t_1, 0, ..., 0)(x)$ (see the previous Claim). Hence

$$\frac{\partial \mathcal{C}_l(t_1,\dots t_l)(x)}{\partial t_1} = \int_0^{t_2} ... \int_0^{t_{l-1}} \frac{\partial^{l-1}\mathcal{C}_l}{\partial t_1 \partial t_2 ... \partial t_{l-1}}(t_1, u_2, ..., u_{l-1}, t_l)(x)\, du_{l-1} ... du_1$$

$$= \int_0^{t_2} ... \left[\int_0^{t_{l-1}} t_l \frac{\partial^l \mathcal{C}_l}{\partial t_1 \partial t_2 ... \partial t_{l-1} \partial t_l}(t_1, 0, ..., 0)(x) \right.$$

$$\left. + o\left(\sqrt{u_2^2 + ... + u_{l-1}^2 + t_l^2}\right)\right] du_{l-1} ... du_1$$

$$= t_2 t_3 ... t_l \frac{\partial^l \mathcal{C}_l}{\partial t_1 \partial t_2 ... \partial t_{l-1} \partial t_l}(t_1, 0, ..., 0)(x) + o(t_2 ... t_l).$$

Finally, by the first Claim we have proved, $t_1 \mapsto \frac{\partial^l \mathcal{C}_l}{\partial t_1 ... \partial t_l}(t_1, 0, ..., 0)(x)$ is continuous since $p_{i_1} = 1$. Hence

$$\frac{\partial \mathcal{C}_l(t_1,\dots t_l)(x)}{\partial t_1} = t_2 t_3 ... t_l \left[\frac{\partial^l \mathcal{C}_l}{\partial t_1 ... \partial t_l}(0,0,...,0)(x) + o(1) \right]$$

$$= t_2 t_3 ... t_l \frac{\partial^l \mathcal{C}_l}{\partial t_1 ... \partial t_l}(0,0,...,0)(x) + o(t_2 t_3 ... t_l)$$

which completes the proof of the Claim. ■

We have therefore proved that the map $(t, x) \mapsto E_I(t)(x)$ is C^1. To prove that it is a diffeomorphism from a neighborhood of the origin onto a neighborhood of x_0, then, it will be enough to show that the Jacobian determinant of $E_\eta^X(x_0, \cdot)$ at the origin is nonzero. Let us compute

$$\frac{\partial}{\partial h_i} E_{I_1}(h_1) E_{I_2}(h_2) \cdots E_{I_p}(h_p)(x_0)_{/h=0} =$$

$$= E_{I_1}(0) E_{I_2}(0) \cdots \frac{\partial E_{I_i}}{\partial h_i}(0) \cdots E_{I_p}(0)(x_0) = \left(\frac{\partial E_{I_i}(0)}{\partial h_i}\right)(x_0) = \left(X_{[I_i]}\right)_{x_0}$$

Hence the Jacobian of (5.3) at zero is the matrix having as rows the vectors $\left(X_{[I_i]}\right)_{x_0}$; since the $\left(X_{[I_i]}\right)_{x_0}$ are a basis for $\mathbb{R}^p$, the Jacobian is nonsingular.

Moreover, the same Jacobian is uniformly continuous for $x \in \overline{\Omega'}, |h| \le \delta$; therefore from the standard proof of the inverse mapping theorem (see e.g. [48, p.221]) one can see that our map is a diffeomorphism of a neighborhood of the origin $\{h : |h| < \delta_1\}$ onto a neighborhood $U(x_0)$ of x_0 containing $\{x : |x - x_0| < \delta_2\}$, with δ_1, δ_2 depending on the number ε and the X_i's. ■

The above theorem has important consequences, the first of which is the following:

Theorem 5.5 (Chow's theorem for nonsmooth vector fields)

1. (Local statement of connectivity). For any $x_0 \in \Omega$ there exist two neighborhoods of x_0, $U \subset V \subset \Omega$, such that any two points of U can be connected by a curve contained in V, which is composed by a finite number of arcs, integral curves of the vector fields X_i for $i = 0, 1, 2, ..., n$.
2. (Global statement of connectivity). If Ω is connected, for any couple of points $x, y \in \Omega$ there exists a curve joining x to y and contained in Ω, which is composed by a finite number of arcs, integral curves of the vector fields X_i for $i = 0, 1, 2, ..., n$.

Proof. 1. For any fixed $x_0 \in \Omega$, let $U(x_0)$ be a neighborhood of x_0 where, by Theorem 5.1, the diffeomorphism $E_\eta^X(x_0, \cdot)$ is well defined for a suitable choice of η. More precisely, we can choose a neighborhood of the kind

$$U_\delta(x_0) = \left\{E_\eta^X(x_0, h) : |h| < \delta\right\}$$

(with δ small enough so that $U_\delta(x_0) \subset \Omega$). By definition of the map $E_\eta^X(x_0, \cdot)$, this means that every point of $U(x_0)$ can be joined to x_0 with a curve which is composed by a finite number of arcs, integral curves of the vector fields X_i for $i = 0, 1, 2, ..., n$ with coefficients of the order of $\delta^{1/r}$. Then we can also say that any two points of $U(x_0)$ can be joined by a curve in a similar way. Moreover, for each point $\gamma(t)$ of such a curve we have

$$|\gamma(t) - x_0| \le c d(\gamma(t), x_0) < c\delta^{1/r}.$$

Let us choose h small enough so that

$$V_\delta(x_0) = \left\{x : |x - x_0| < c\delta^{1/r}\right\} \subset \Omega;$$

then we have the statement 1, choosing $U = U_\delta(x_0), V = U \cup V_\delta(x_0)$.

2. Now we can cover any compact connected subset Ω' of Ω with a finite number of neighborhoods $U(x_i) \subset V(x_i) \subset \Omega$ in such a way that any two points of Ω' can be joined by a curve as above, contained in the union of the $V(x_i)$'s, and therefore in Ω. ■

Theorem 5.5 shows that it is possible to join any two points of Ω using only integral lines of the vector fields X_i. This justifies the following:

Definition 5.6 *For any $\delta > 0$, let $C_1(\delta)$ be the class of absolutely continuous mappings $\varphi : [0, 1] \longrightarrow \Omega$ which satisfy*

$$\varphi'(t) = \sum_{i=0}^{n} a_i(t) (X_i)_{\varphi(t)} \quad a.e.$$

with

$$|a_0(t)| \le \delta^2, |a_i(t)| \le \delta \text{ for } i = 1, 2, ..., n.$$

We define

$$d_1(x, y) = \inf \{\delta > 0 : \exists \varphi \in C_1(\delta) \text{ with } \varphi(0) = x, \varphi(1) = y\}.$$

Remark 5.7 *In the smooth case, and for $X_0 \equiv 0$, the distance d_1 has been introduced in [43]. The Authors also prove the equivalence of d and d_1 (note that our d_1 is the distance called ρ_4 in [43]). One can easily check that this equivalence, in the smooth case, still holds in presence of a vector field X_0 of weight 2.*

By the last theorem, the quantity $d_1(x, y)$ is finite for every $x, y \in \Omega$. It is easy to see that d_1 is a distance (it is still true that the union of two consecutive admissible curves can be reparametrized to give an admissible curve) and, just by definition, one always has

$$d(x, y) \le d_1(x, y).$$

We also have the following:

Proposition 5.8 *For any $\Omega' \Subset \Omega$ there exist positive constants c_1, c_2 such that*

$$c_1 |x - y| \le d_1(x, y) \le c_2 |x - y|^{1/r} \text{ for any } x, y \in \Omega'.$$

Proof. The first inequality is obvious because we already know that d satisfies it, and $d \le d_1$. So let us prove the second one.

Fix $x_0 \in \Omega'$, and let us consider the map $E_\eta^X(x_0, h)$ defined in Theorem 5.1, for a suitable choice of η. Since

$$h \longmapsto E_\eta^X(x_0, h)$$

is a diffeomorphism, there exist positive constants k_1, k_2 such that, for $x = E_\eta^X(x_0, h)$ in a suitable neighborhood of x_0, we have:

$$k_1 |x - x_0| \le \max_{i=1,\dots,p} |h_i| \le k_2 |x - x_0|.$$

On the other hand, saying that $x = E_\eta^X(x_0, h)$, by definition means that there exists a curve γ joining x_0 to x, which is composed by a finite number N (this number being under control) of arcs of integral curves of vector fields of the kinds

$$\pm h_j^{p_i/|I_j|} X_i$$

for $i = 0, 1, 2, ..., n$, $j = 1, 2, ..., p$, where $\left\{ \left(X_{[I_j]}\right)_{x_0} \right\}$ is a basis of $\mathbb{R}^p$ (see Remark 4.1). This means that γ satisfies

$$\begin{cases} \gamma'(\tau) = \sum_{i=0}^n a_i(\tau) (X_i)_{\gamma(\tau)} \\ \gamma(0) = x_0, \gamma(1) = x \end{cases}$$

with

$$|a_i(\tau)| \le c \left(\max_{j=1,\dots,p} |h_j| \right)^{p_i/r} \le c\,|x - x_0|^{p_i/r}.$$

This implies that $\gamma \in C_1\left(c\,|x - x_0|^{1/r}\right)$, that is

$$d_1(x, x_0) \le c\,|x - x_0|^{1/r}. \qquad (5.12)$$

So far, we have proved that every point x_0 has a neighborhood U such that for any $x \in U$ one has (5.12), where c is locally uniformly bounded with respect to x_0. Then one can also say that every point x_0 has a neighborhood V such that for any $x, y \in U$ one has

$$d_1(x, y) \le c\,|x - y|^{1/r}.$$

A covering argument then implies the desired statement. ■

We now want to prove the local equivalence of d and d_1. To this aim, we fix a point $x_0 \in \Omega' \Subset \Omega$ and make use once again of the smooth approximating vector fields $S_i^{x_0}$. Let us now denote by

$$d_S, d_{S,1}, d_X, d_{X,1}$$

the distances d and d_1 induced by the systems $\{S_i^{x_0}\}$ and $\{X_i\}$, respectively. The above proposition allows us to repeat also for the distances $d_{S,1}, d_{X,1}$ the proof of Theorem 3.4, and get the following:

Theorem 5.9 *For any $\Omega' \Subset \Omega$, there exist positive constants c_1, c_2, r_0 such that*

$$B_S^1(x_0, c_1\rho) \subset B_X^1(x_0, \rho) \subset B_S^1(x_0, c_2\rho)$$

for any $x_0 \in \Omega', \rho < r_0$, where B_S^1, B_X^1 denote the metric balls with respect to $d_{S,1}, d_{X,1}$, respectively.

Since, by the smooth theory, we already know that $d_{S,1}$ is locally equivalent to d_S (see Remark 5.7) the last theorem immediately implies the following result, which strengthens in a quantitative way the connectivity result contained in Chow's theorem:

Theorem 5.10 *The distances $d_{X,1}$ and d_X are locally equivalent in Ω'. More precisely there exist positive constants ρ_0 and C such that for every $w \in \Omega'$ and $y, z \in B_X(w, \rho_0)$ we have*

$$d_{X,1}(y, z) \le C d_X(y, z).$$

(The reverse inequality $d(y, z) \le d_1(y, z)$ obviously holds by definition of d, d_1). As a consequence, the doubling condition of Theorem 3.5 still holds with respect to d_1.

Another quantitative consequence of the connectivity property is the possibility of a pointwise control of the increment of a function f by means of its "gradient" $Xf = (X_i f)_{i=0}^{n}$ (that is, $|Xf|$ is an upper gradient in the terminology of [25]).

Theorem 5.11 *Let $f \in C^1 (B_1 (x_0, \rho))$, let $x \in B_1 (x_0, \rho)$ and let $\gamma \in C_1 (\rho)$ be a curve that joins x_0 with x. Then*

$$|f(x) - f(x_0)| \leqslant \sqrt{n}\rho \int_0^1 |Xf(\gamma(t))|\, dt.$$

As a consequence we also have

$$|f(x) - f(x_0)| \leq \sqrt{n} d_1 (x, x_0) \cdot \sup_{B_1(x_0,\rho)} |Xf| \qquad \forall x \in B_1 (x_0, \rho),$$
$$|f(x) - f(y)| \leq \sqrt{n} d_1 (x, y) \cdot \sup_{B_1(x_0,\rho)} |Xf| \qquad \forall x, y \in B_1 \left(x_0, \frac{\rho}{3}\right).$$

Proof. Let $x \in B_1 (x_0, \rho)$, then there exists a curve $\gamma (t)$ such that

$$\gamma(0) = x_0; \gamma(1) = x$$
$$\gamma'(t) = \sum_{i=1}^{n} a_i (t) (X_i)_{\gamma(\tau)}$$

with $|a_i (t)| \leq \rho$, then

$$\begin{aligned} f(x) - f(x_0) &= \int_0^1 \frac{d}{dt} (f(\gamma(t)))\, dt = \\ &= \int_0^1 \sum_{i=1}^{n} a_i (t) (X_i)_{\gamma(t)} \cdot \nabla f(\gamma(t))\, dt \\ &= \int_0^1 \sum_{i=1}^{n} a_i (t) (X_i f)(\gamma(t))\, dt \end{aligned}$$

$$\begin{aligned} |f(x) - f(x_0)| &\leq \int_0^1 \sqrt{\sum_{i=1}^{n} a_i (t)^2} \cdot \sqrt{\sum_{i=1}^{n} (X_i f)(\gamma(t)) dt} \\ &\leq \sqrt{n}\rho \int_0^1 |Xf(\gamma(t))|\, dt \\ &\leq \sqrt{n}\rho \sup_{B_1(x_0,\rho)} |Xf|. \end{aligned}$$

For any $x, y \in B_1 (x_0, \rho/3)$, now, we have $y \in B_1 (x, 2\rho/3)$ and

$$|f(x) - f(y)| \leq \sqrt{n} d(x, y) \sup_{B_1(x,2\rho/3)} |Xf| \leq \sqrt{n} d(x, y) \sup_{B_1(x_0,\rho)} |Xf|.$$

■

6 Lifting of nonsmooth vector fields

In the proof of Poincaré's inequality we will use an idea of Jerison (see [28]) which consists in deriving such inequality first for free vector fields and then in the general case. This approach requires to develop in the context on nonsmooth vector fields both Rothschild-Stein's lifting technique [47] and an estimate for the volume of lifted balls which was originally proved by Sanchez-Calle [50] and Nagel-Stein-Wainger [43]. These tools can also be of independent interest.

In what follows we keep the previous assumptions on the vector fields $X_1, X_2, \ldots, X_n$ and we take $X_0 \equiv 0$. Recall that a system of n vector fields is said to be *free up to step* r if the vector fields and their commutators of length at most r do not satisfy any linear dependence relations except those which follow from anticommutativity and Jacobi identity.

Theorem 6.1 (Lifting theorem) *For every $x_0 \in \Omega$, there exist a neighborhood $U(x_0)$, an integer m and vector fields of the form*

$$\widetilde{X}_k = X_k + \sum_{j=1}^{m} u_{kj}(x,t) \frac{\partial}{\partial t_j} \quad (k = 1, 2, ..., n) \tag{6.1}$$

defined for $(x,t) \in U(x_0) \times I$ where I is a neighborhood of $0 \in \mathbb{R}^m$, which are free up to step r and such that $\left\{\widetilde{X}_{[I]}(x,t)\right\}_{|I| \leq r}$ span $\mathbb{R}^{p+m}$ for every $(x,t) \in U(x_0) \times I$. Moreover the $u_{kj}(x,t)$ can be taken as polynomials of degree at most $r-1$.

After the original paper [47], alternative proofs of this result (for smooth vector fields) have been given by several authors. For our purposes the most useful is the one given by Hörmander-Melin [27]. Indeed, a careful inspection of their proof shows that it actually requires only the C^{r-1} regularity of the coefficients and therefore it applies also to our nonsmooth context.

In the sequel we will use both the lifting procedure and our approximation procedure by Taylor expansion of degree $r-1$ (see §3). Since the coefficients $u_{kj}(x,t)$ are polynomials of degree $\leq r-1$ one can easily see that these two procedures commute. We will denote by $\widetilde{S}_i^x$ the "lifted approximating field".

The next theorem contains a comparison between the volume of balls with respect to the original vector fields X_i and their lifted $\widetilde{X}_i$; we will denote these balls with the symbol $B, \widetilde{B}$, respectively. Thanks to the results proved in §5, here the distance induced by each system of vector fields can be either d (definition 2.2) or d_1 (definition 5.6). To prove Poincaré's inequality we will apply the result for d_1.

Theorem 6.2 *Let x_0 and $U(x_0), I$ be as in the above theorem. There exist positive constants c_1, c_2, r_0, and $\delta \in (0,1)$ such that for any $(x,h) \in U(x_0) \times I$, any $y \in B(x, \delta\rho)$, $0 < \rho < r_0$, we have, denoting by $|\cdot|$ the volume of a ball in* *the appropriate dimension,*

$$c_1 \frac{\left|\widetilde{B}\left((x,h),\rho\right)\right|}{\left|B\left(x,\rho\right)\right|} \leqslant \int_{\mathbb{R}^m} \chi_{\widetilde{B}((x,h),\rho)}\left(y,s\right) ds \leqslant c_2 \frac{\left|\widetilde{B}\left((x,h),\rho\right)\right|}{\left|B\left(x,\rho\right)\right|}. \tag{6.2}$$

Actually the second inequality holds for every $y \in \mathbb{R}^p$. Also, the projection of $\widetilde{B}\left((x,h),\rho\right)$ on $\mathbb{R}^p$ is exactly $B\left(x,\rho\right)$.

Proof. As already noted, for smooth vector fields (6.2) has been proved in [50] and [43]. See also [28] where the result is stated exactly in this form. Let us denote by $B_{S^x}, \widetilde{B}_{S^x}$ the balls defined with respect to the vector fields S_i^x and $\widetilde{S}_i^x$ respectively. Since, by Theorem 5.9,

$$B_{S^x}\left(x,k_1\rho\right) \subset B\left(x,\rho\right) \subset B_{S^x}\left(x,k_2\rho\right)$$

and

$$\widetilde{B}_{S^x}\left((x,h),k_1\rho\right) \subset \widetilde{B}\left((x,h),\rho\right) \subset \widetilde{B}_{S^x}\left((x,h),k_2\rho\right),$$

the result follows from (6.2) applied to the smooth vector fields S_i^x and $\widetilde{S}_i^x$ and the doubling property of Theorem 5.10. Also, since the lifted vector field $\widetilde{X}_i$ projects onto X_i, by definition of distance the projection of $\widetilde{B}\left((x,h),\rho\right)$ on $\mathbb{R}^p$ is exactly $B\left(x,\rho\right)$. ■

7 Poincaré's inequality

For smooth Hörmander's vector fields, Poincaré's inequality has been proved by Jerison in [28]. Lanconelli-Morbidelli in [33] have developed a general approach to Poincaré's inequality for (possibly nonsmooth) vector fields: they first prove an abstract result, which deduces Poincaré's inequality from a property which they call "representability of balls by means of controllable almost exponential maps", and then show how to apply this general result in several different situations. One of these situations is the classical case of smooth Hörmander's vector fields.

We will prove Poincaré's inequality in our context applying the aforementioned abstract result. To check the assumption of this theorem, we will exploit all our previous theory, plus some results and arguments used in [33] to handle the smooth case. Also, for technical reasons which will be explained later, we need to apply this abstract result to free vector fields, and then derive Poincaré's inequality in the general case from that proved in the free case, as already done by Jerison [28] in the smooth setting. By the way, we remark that it seems hard to apply directly Jerison's argument to the nonsmooth vector fields (without relying on Lanconelli-Morbidelli's theory), since this would require also Rothschild-Stein's approximation technique, which in our nonsmooth setting is not presently available.

Henceforth we will further strengthen our assumption as follows:

Assumptions (D). We assume that for some integer $r \geq 2$ and some bounded domain $\Omega \subset \mathbb{R}^p$ the following hold:

(D1) The coefficients of the vector fields $X_1, X_2, ..., X_n$ belong to $C^{r-1,1}(\Omega)$, while $X_0 \equiv 0$. Here and in the following, $C^{k,1}$ stands for the classical space of functions with Lipschitz continuous derivatives up to order k.

(D2) The vectors $\left\{\left(X_{[I]}\right)_x\right\}_{|I|\leq r}$ span $\mathbb{R}^p$ at every point $x \in \Omega$.

Assumptions (D) will be in force throughout this section and the following. See, however, our Remark 7.13 at the end of this section, for an inequality that holds under weaker assumptions.

Remark 7.1 *The lacking of X_0 is a natural assumption dealing with Poincaré-type inequalities. Note that, since $X_0 \equiv 0$, length and weight of a multiindex now coincide. Also note that under the assumption (D1) above, for any $1 \leq k \leq r$, the differential operators*

$$\{X_I\}_{|I|\leq k}$$

are well defined, and have $C^{r-k,1}$ coefficients. The same is true for the vector fields $\left\{X_{[I]}\right\}_{|I|\leq k}$.

Dependence of the constants. Whenever we will write that some constant depends on the vector fields X_i's and some fixed domain $\Omega' \Subset \Omega$, this will mean that the constant depends on:
(i) $\operatorname{diam}(\Omega')$;
(ii) the norms $C^{r-1,1}(\Omega)$ of the coefficients of X_i $(i = 1, 2, ..., n)$;
(iii) a positive constant c_0 such that the following bound holds:

$$\inf_{x\in\Omega'} \max_{|I_1|,|I_2|,...,|I_p|\leq r} \left|\det\left(\left(X_{[I_1]}\right)_x, \left(X_{[I_2]}\right)_x, ..., \left(X_{[I_p]}\right)_x\right)\right| \geq c_0.$$

In the following we will sometimes work with the lifted vector fields, defined in §6, so that the constants appearing in our results will also depend on the constants in (ii)-(iii) associated to the lifted vector fields. However, as observed by Jerison [28, Lemma (4.2) p.511], these in turn only depend on the constants in (ii)-(iii) corresponding to the original vector fields X_i's.

Let us state our main result:

Theorem 7.2 (Poincaré's inequality) *For any $\Omega' \Subset \Omega$ there exist constants $c, r_0 > 0, \lambda \geq 1$ such that for any $d_{X,1}$-ball $B = B(x,\rho)$, with $\rho \leq r_0$, $x \in \Omega'$, any $u \in C^1\left(\overline{\lambda B}\right)$, with $\lambda B = B(x, \lambda\rho)$, we have*

$$\int_{B\times B} |u(y) - u(x)|\, dydx \leq c\rho\, |B| \int_{\lambda B} |Xu(y)|\, dy \tag{7.1}$$

where:

$$|Xu(y)| = \sqrt{\sum_{j=1}^{n} |X_j u(y)|^2}.$$

Note that (7.1) is equivalent to the following (perhaps more familiar) form of Poincaré's inequality:

$$\int_B |u(y) - u_B|\, dy \leq c\rho \int_{\lambda B} |Xu(y)|\, dy \tag{7.2}$$

where, as usual, u_B denotes the average of u over B.

We start showing that Poincaré's inequality for free vector fields implies the same result in the general case.

So, fix $x_0 \in \Omega$ and a neighborhood $U(x_0) \subset \Omega$ where the lifting theorem 6.1 can be applied, and let $\widetilde{X}_i$ be the lifted free vector fields in $U(x_0) \times I$. Then for any $U'(x_0) \Subset U(x_0)$, $x \in U'(x_0)$, if $\widetilde{B} = \widetilde{B}((x,h),\rho)$ is a $d_{\widetilde{X},1}$-ball with $\rho \leq r_0$ and $\lambda\widetilde{B} = \widetilde{B}((x,h),\lambda\rho)$, we have

$$\int_{\widetilde{B}\times\widetilde{B}} |u(y,t) - u(z,s)|\, dydzdsdt \leq c\rho \left|\widetilde{B}\right| \int_{\lambda\widetilde{B}} \left|\widetilde{X}u(y,t)\right| dydt \tag{7.3}$$

for any $u \in C^1\left(\overline{\lambda\widetilde{B}}\right)$. If we apply (7.3) to a function $u(y)$ independent of t, $u \in C^1\left(\overline{\lambda B}\right)$, then by (6.1) we get

$$\int_{\widetilde{B}\times\widetilde{B}} |u(y) - u(z)|\, dydzdsdt \leq c\rho \left|\widetilde{B}\right| \int_{\lambda\widetilde{B}} |Xu(y)|\, dydt$$

that is (since $\widetilde{B}$ projects onto B)

$$\int_{B\times B} |u(y) - u(z)|\, dydz \int_{\mathbb{R}^m} \chi_{\widetilde{B}((x,h),\rho)}(y,t)\, dt \int_{\mathbb{R}^m} \chi_{\widetilde{B}((x,h),\rho)}(z,s)\, ds \leq$$
$$\leq c\rho \left|\widetilde{B}\right| \int_{\lambda B} |Xu(y)|\, dy \int_{\mathbb{R}^m} \chi_{\widetilde{B}((x,h),\rho)}(y,t)\, dt$$

which, by Theorem 6.2, implies (7.1).

A standard compactess argument then gives theorem 7.2.

We now proceed to prove theorem 7.2 under the additional assumption that the vector fields X_i are free up to step r.

Let us start fixing some notation. Throughout this section we will assume fixed a bounded domain $\Omega' \Subset \Omega$. Let $\left\{X_{[I_j]}\right\}_{I_j\in\eta}$ be any particular family of p commutators of our vector fields $\{X_i\}_{i=1}^n$, with $|I_j| \leq r$; let

$$|\eta| = \sum_{j=1}^p |I_j|;$$
$$\|h\|_\eta = \max_{j=1,\dots,p} |h_j|^{1/|I_j|} \text{ for any } h \in \mathbb{R}^p;$$
$$Q_\eta(\rho) = \left\{h \in \mathbb{R}^p : \|h\|_\eta \leq \rho\right\} \text{ for any } \rho > 0.$$

Let us recall that (see Theorem 5.1) $E_\eta^X(x,h)$ is C^1 in the joint variables (x,h) for $x \in \Omega'$ and $|h| < \delta_1$. Moreover, if $\left\{X_{[I_j]}\right\}_{I_j \in \eta}$ is a family of commutators which spans $\mathbb{R}^p$ at $x_0 \in \Omega'$ (and therefore in the whole Ω', since the X_i's are free) and satisfies (5.2), then $h \longmapsto E_\eta^X(x,h)$ is a diffeomorphism of a neighborhood of the origin $\{h : |h| < \delta_1\}$ onto a neighborhood $U(x_0)$ of x_0 containing $\{x : |x - x_0| < \delta_2\}$.

We also denote by

$$D_{E_\eta^X(x,h)}$$

the modulus of the Jacobian determinant of the mapping $h \longmapsto E_\eta^X(x,h)$. The function $D_{E_\eta^X(x,h)}$ is continuous for $x \in \Omega'$ and $|h| < \delta_1$; moreover,

$$D_{E_\eta^X(x,0)} = \left|\det\left\{\left(X_{[I_j]}\right)_x\right\}_{j=1}^p\right| > 0 \text{ for any } x \in \Omega'.$$

For any fixed $x_0 \in \Omega'$, let now $\{S_i^{x_0}\}$ be the system of smooth approximating vector fields, introduced in §3. We know that (see Lemma 3.2) the $S_i^{x_0}$'s satisfy Hörmander's condition in

$$U_\delta(x_0) = \{x \in \Omega : |x - x_0| < \delta\}.$$

For any $x_0 \in \Omega'$ we can therefore consider the system of smooth Hörmander's vector fields $\{S_i^{x_0}\}$ in $U_\delta(x_0)$, and perform for this system, as in §5, the construction of the maps

$$C_{\ell(I)}(t, S_I^{x_0})(x)$$

and that of the corresponding diffeomorphism

$$E_\eta^{S^{x_0}}(x,h).$$

It is now time to recall what is the abstract result proved in [33] regarding Poincaré inequality, how the Authors use it to deduce Jerison's Poincaré inequality (in the smooth case), and how we can adapt their arguments to our context. As we will see, a key feature of our approach is a suitable mix of the two systems of vector fields $\{X_i\}, \{S_i^{x_0}\}$, and of the corresponding maps $E_\eta^X, E_\eta^{S^{x_0}}$.

Let us start recalling a definition from [33].

Definition 7.3 *Let O be a bounded open set in $\mathbb{R}^p$, and Q a neighborhood of the origin. We say that a function*

$$E : O \times Q \to \mathbb{R}^p$$

is an almost exponential map *if*

(i) $E(x,0) = x \; \forall x \in O$

(ii) the map $h \longmapsto E(x,h)$ *is* C^1 *and* 1-1 *on* Q

(iii) the following condition holds:

$$\frac{1}{a} D_{E(x,0)} \leq D_{E(x,h)} \leq a D_{E(x,0)} \forall (x,h) \in O \times Q, \text{ some constant } a > 1$$

(where $D_{E(x,h)}$ stands for the modulus of the Jacobian determinant of the mapping $h \longmapsto E(x,h)$).

The abstract theorem proved by Lanconelli-Morbidelli reads as follows:

Theorem 7.4 (Theorem 2.1 in [33]) *Let $B = B_1^X(x_0, \rho)$ be a fixed ball. Assume there exists an open set $O \subset B$, an almost exponential map $E : O \times Q \to \mathbb{R}^p$ and two positive constants α, β satisfying the following conditions:*

(i) $|B| \leq \alpha |O|$ and $B \subset E(x, Q)$ for every $x \in O$;

(ii) E is X-controllable with a hitting time $T \leq \alpha\rho$;

(iii) $|(\alpha + 1) B| \leq \beta |O|$.

Then there exists $c > 0$ such that

$$\int_{B \times B} |u(y) - u(x)| \, dy dx \leq c\rho |B| \int_{(1+\alpha)B} |Xu(y)| \, dy$$

for any $u \in C^1\left(\overline{(1+\alpha) B}\right)$, where $(1+\alpha) B = B(x_0, (1+\alpha)\rho)$. The constant c only depends on the numbers α, β, the constant a appearing in Definition 7.3 and the constant b appearing in Definition 7.7.

We will recall and comment later the definition of "X-controllable map".

Our strategy consists in showing that the map

$$(x,h) \longmapsto E_\eta^{S^x}(x,h)$$

satisfies the assumption of the previous theorem, on suitable domains O, Q, for a suitable choice of η. Note that this map, built upon the system $\{S_i^{x_0}\}$, will be shown to satisfy the assumptions of the theorem *with respect to the system* $\{X_i\}$. Also, note that in the definition of this map E the point x_0 where the system $\{S_i^{x_0}\}$ approximates $\{X_i\}$ is taken equal to x, that is "unfrozen". Therefore our map E will be only Lipschitz continuous with respect to x. These facts will require some care.

First of all, we can apply to the system of smooth Hörmander's vector fields $S_i^{x_0}$, in the domain $U_\delta(x_0)$, Theorem 4.1 in [33]:

Theorem 7.5 *For any $x_0 \in \Omega' \Subset \Omega$ there exist positive constants r_0, c_1, c_2, with $c_2 < c_1 < 1$, such that for any family η of p commutators, $x \in U_\delta(x_0), \rho \leq r_0$ satisfying the inequality*

$$D_{E_\eta^{S^{x_0}}(x,0)} \rho^{|\eta|} \geq \frac{1}{2} \max_\zeta D_{E_\zeta^{S^{x_0}}(x,0)} \rho^{|\zeta|}$$

the following assertions hold:

(a) If $h \in Q_\eta(c_1\rho)$ then

$$\frac{1}{4} D_{E_\eta^{S^{x_0}}(x,0)} \leq D_{E_\eta^{S^{x_0}}(x,h)} \leq 4 D_{E_\eta^{S^{x_0}}(x,0)}$$

(b) $B^1_{S^{x_0}}(x, c_2\rho) \subset E_\eta^{S^{x_0}}(x, Q_\eta(c_1\rho))$.

(c) The function $E_\eta^{S^{x_0}}(x, \cdot)$ is one-to-one on the set $Q_\eta(c_1\rho)$.

Here $B^1_{S^{x_0}}$ stands for the metric ball with respect to the distance $d_{S^{x_0},1}$.

As the Authors write in [33], the above theorem, in the case of smooth Hörmander's vector fields, has a proof similar to that of Theorem 7 in [43], which is written in detail in [42]. Note that the constants r_0, c_1, c_2 in the above Theorem only depend on the X_i's and Ω', and not on x_0 (see the discussion in § 3). We can then set $x = x_0$ in the above theorem, obtaining the following:

Theorem 7.6 *For any $\Omega' \Subset \Omega$ there exist positive constants r_0, c_1, c_2, with $c_2 < c_1 < 1$, such that for any family η of p commutators, $x \in \Omega', \rho \leq r_0$ satisfying the inequality*

$$D_{E_\eta^{S^x}(x,0)} \rho^{|\eta|} \geq \frac{1}{2} \max_\zeta D_{E_\zeta^{S^x}(x,0)} \rho^{|\zeta|} \tag{7.4}$$

the following assertions hold:

(a') If $h \in Q_\eta(c_1\rho)$ then

$$\frac{1}{4} D_{E_\eta^{S^x}(x,0)} \leq D_{E_\eta^{S^x}(x,h)} \leq 4 D_{E_\eta^{S^x}(x,0)} \tag{7.5}$$

(b') $B^1_X(x, c_2\rho) \subset E_\eta^{S^x}(x, Q_\eta(c_1\rho))$.

(c') The function $E_\eta^{S^x}(x, \cdot)$ is one-to-one on the set $Q_\eta(c_1\rho)$.

Here B^1_X stands for the metric ball with respect to the distance $d_{X,1}$.

Note that (b') also exploits Theorem 5.9 (with possibly a smaller value of c_2).

In the following we will need to shrink the constant r_0 appearing in this theorem; however this is not restrictive.

In order to find the sets O, Q to which we will apply Theorem 7.4 we now proceed like in [33, p.336]: let B be a d_1^X-ball centered at some $x_0 \in \Omega'$ of radius $\rho < c_2 r_0/2$. For any family η of p commutators of lenght $\leq r$, we define

$$\Omega_\eta = \left\{ x \in B : D_{E_\eta(x,0)} \left(\frac{2\rho}{c_2}\right)^{|\eta|} > \frac{1}{2} \max_\zeta D_{E_\zeta(x,0)} \left(\frac{2\rho}{c_2}\right)^{|\zeta|} \right\}. \tag{7.6}$$

Here $D_{E_\eta(x,0)}$ stands for both $D_{E_\eta^{S^x}(x,0)}$ and $D_{E_\eta^X(x,0)}$ (since the two quantities coincide). At least one of the sets Ω_η satisfies

$$|\Omega_\eta| \geq \frac{1}{N} |B| \tag{7.7}$$

where N is the total number of p-tuples available. Let us choose one of such η's and denote by Q the box

$$Q = \left\{ h \in \mathbb{R}^p : \|h\|_\eta < \frac{2c_1}{c_2}\rho \right\}.$$

From now on, the basis η is chosen once and for all. By (a') and (c') of Theorem 7.6, the function

$$\begin{aligned} E : \Omega_\eta \times Q &\to \mathbb{R}^p \\ (x,h) &\mapsto E_\eta^{S^x}(x,h) \end{aligned} \tag{7.8}$$

is an *almost exponential map.* The fact that $h \mapsto E_\eta^{S^x}(x,h)$ is C^1 follows from Theorem 5.1 applied to the smooth vector fields S_i^x (for any frozen x).

We will show that the almost exponential map E we have just built satisfies assumptions (i),(ii),(iii) in Theorem 7.4. This will imply our Poincaré's inequality.

By (7.7), $|B| \le N|\Omega_\eta|$, while by (b') of Theorem 7.6, $B \subset E(x,Q)$ for every $x \in \Omega_\eta$. Thus assumption (i) in Theorem 7.4 is satisfied, while assumption (iii) follows from the doubling condition for d_1^X balls, which we have proved in Theorem 5.10, plus inequality $|B| \le N|\Omega_\eta|$.

It remains to prove that the map E is "X-controllable with a hitting time $T \le \alpha\rho$", that is, condition (ii). Let us first recall the definition of this concept, as appears in [33, p.330]:

Definition 7.7 *We say that an almost exponential map $E : O \times Q \to \mathbb{R}^p$ is X-controllable with a hitting time T if there exists a function $\gamma : O \times Q \times [0,T] \to \mathbb{R}^p$ such that*

(C1) For any $(x,h) \in O \times Q$, $t \mapsto \gamma(x,h,t)$ is an X-subunit path connecting x and $E(x,h)$, that is

$$\begin{cases} \frac{d}{dt}\gamma(x,h,t) = \sum a_j(t)(X_j)_{\gamma(x,h,t)} \text{ for suitable } a_j \text{ with } \sum |a_j(t)|^2 \le 1 \\ \gamma(x,h,0) = x; \gamma(x,h,T(x,h)) = E(x,h) \end{cases}$$

for a suitable $T(x,h) \le T$.

(C2) For any $(h,t) \in Q \times [0,T]$, $x \mapsto \gamma(x,h,t)$ is a one-to-one C^1 map having jacobian determinant bounded away from zero, i.e.

$$b \equiv \inf_{O \times Q \times [0,T]} \left| \frac{\partial \gamma}{\partial x} \right| > 0.$$

We start noting that condition (C2) is used in [33, p.330] only once, in the following change of variable:

$$\int_O |Xu(\gamma(x,h,t))| \, dx \le \frac{1}{b} \int_{B(x_0,(\alpha+1)\rho)} |Xu(z)| \, dz.$$

It is then apparent that (C2) can be replaced by the weaker assumption:

(C2') For any $(h,t) \in Q \times [0,T]$, $x \mapsto \gamma(x,h,t)$ is a one-to-one bilipschitz map having jacobian determinant bounded away from zero, i.e.

$$b \equiv \inf_{O \times Q \times [0,T]} \left| \frac{\partial \gamma}{\partial x} \right| > 0.$$

In view of Theorem 7.4, our proof of Theorem 7.2 will be completed as soon as we will prove the following:

Proposition 7.8 *The almost exponential map E defined in (7.8) is X-controllable with a hitting time $T \leq \alpha\rho$, in the sense of the above definition with (C1), (C2'), where α and ρ are as above.*

Remark 7.9 *Before going on, we have to make an important observation, in order to explain the role played by the fact that our vector fields X_i are free. In the following, we need to choose a basis $\left\{X_{[I]}\right\}_{I\in\eta}$ satisfying* simultaneously *the condition expressed in (7.6), which is fundamental to apply the theory of Lanconelli-Morbidelli, and the condition (5.2) in Theorem 5.1, which allows us to have a quantitative control on the diffeomorphism induced by the basis. This could impossible for a general family of Hörmander's vector fields, but it is easy as soon as they are free.*

Namely, since the vector fields X_i are free up to step r, they satisfy the same commutation relations (up to step r) at any point of Ω. Therefore all the possible families of p vector fields chosen among the commutators of length $\leq r$ of the X_i's can be grouped in two classes:

$$\mathcal{E} = \left\{\eta : D_{E_\eta(x,0)} \neq 0 \text{ for all } x \in \Omega\right\};$$
$$\mathcal{E}^c = \left\{\eta : D_{E_\eta(x,0)} = 0 \text{ for all } x \in \Omega\right\}.$$

If we choose a basis η satisfying the relation

$$D_{E_\eta(x,0)} \left(\frac{2\rho}{c_2}\right)^{|\eta|} > \frac{1}{2} \max_\zeta D_{E_\zeta(x,0)} \left(\frac{2\rho}{c_2}\right)^{|\zeta|},$$

this means that $\eta \in \mathcal{E}$, hence $D_{E_\eta(x,0)} \neq 0$ in the whole Ω. In order to apply Theorem 5.1 with a control on the constants which are involved, we need to know that, for some fixed ε,

$$D_{E_\eta(x,0)} > (1-\varepsilon) \max_\zeta D_{E_\zeta(x,0)} \tag{7.9}$$

Now,

$$D_{E_\eta(x,0)} \geq \min_{\zeta\in\mathcal{E}} D_{E_\zeta(x,0)} > (1-\varepsilon) \max_\zeta D_{E_\zeta(x,0)}$$

because: since the vector fields are free, all the determinants relative to different bases control each other by universal constants. This means that (7.9) holds with a universal constant ε.

Let us now proceed toward the proof of Proposition 7.8. First of all, we have the following:

Lemma 7.10 *There exists a constant $c > 0$ depending on Ω' and the X_i's such that*

$$E_\eta^{S^x}\left(x, Q_\eta(\rho)\right) \subset E_\eta^X\left(x, Q_\eta(c\rho)\right) \quad \forall x \in \Omega_\eta. \tag{7.10}$$

Proof. Let $y \in E_\eta^{S^x}\left(x, Q_\eta(\rho)\right)$; this means that

$$y = E_\eta^{S^x}(x, h) \text{ for some } \|h\|_\eta \leq \rho,$$

and we want to prove that there exists h', with $\|h'\|_\eta \leq c\rho$, such that

$$E_\eta^{S^x}(x, h) = E_\eta^X(x, h').$$

Since, by Theorem 5.1, for any $x \in \Omega'$ the mapping

$$h' \longmapsto E_\eta^X(x, h')$$

is a diffeomorphism (for $|h'| < \delta$), we can invert it, writing

$$h' = \Theta(h) \equiv E_\eta^X(x, \cdot)^{-1} E_\eta^{S^x}(x, h).$$

Moreover, by Remark 7.9, the constants involved in this diffeomorphism are under control.

We want to prove that

$$\|h'\|_\eta \leqslant c\|h\|_\eta \tag{7.11}$$

for some constant $c > 0$ only depending on Ω' and the X_i's. We have

$$\begin{aligned} \|h'\|_\eta &\leqslant c\left(\|h\|_\eta + \|h' - h\|_\eta\right) \\ &\leqslant c\left(\|h\|_\eta + |h' - h|^{1/r}\right) \end{aligned} \tag{7.12}$$

Since $E_\eta^X(x, \cdot)^{-1}$ is a diffeomorphism, we have

$$\begin{aligned} |h' - h| &= \left|E_\eta^X(x, \cdot)^{-1} E_\eta^{S^x}(x, h) - E_\eta^X(x, \cdot)^{-1} E_\eta^X(x, h)\right| \\ &\leqslant c\left|E_\eta^{S^x}(x, h) - E_\eta^X(x, h)\right| \end{aligned} \tag{7.13}$$

We are going to show that

$$\left|E_\eta^{S^x}(x, h) - E_\eta^X(x, h)\right| \leqslant c\|h\|_\eta^r \tag{7.14}$$

which, together with (7.11), (7.12), (7.13), will imply our assertion.

Namely,

$$E_\eta^{S^x}(x,h)=\prod_{j=1}^{M}\exp\left(\left|h_{i_j}\right|^{1/l_j}\sigma_j S_{r_j}^x\right)(x)\,;$$
$$E_\eta^{X}(x,h)=\prod_{j=1}^{M}\exp\left(\left|h_{i_j}\right|^{1/l_j}\sigma_j X_{r_j}\right)(x)$$

(here $\sigma_j=\pm 1$). Set:

$$y_n=\prod_{j=1}^{n}\exp\left(\left|h_{i_j}\right|^{1/l_j}\sigma_j S_{r_j}^x\right)(x)$$
$$\widetilde{y}_n=\prod_{j=1}^{n}\exp\left(\left|h_{i_j}\right|^{1/l_j}\sigma_j X_{r_j}\right)(x)$$

and let us show by induction on n that $|y_n-\widetilde{y}_n|\leqslant c\,\|h\|_\eta^r$. For $n=1$ we can apply directly the argument in the proof of Theorem 3.4, getting

$$|y_1-\widetilde{y}_1|\leqslant c\left(|h_{i_1}|^{1/l_1}\right)^r\leqslant c\,\|h\|_\eta^r\,.$$

Assuming the assertion for n, let us write now:

$$y_{n+1}=\exp\left(\left|h_{i_{n+1}}\right|^{1/l_{n+1}}\sigma_{n+1}S_{r_j}^x\right)(y_n)\,;$$
$$\widetilde{y}_{n+1}=\exp\left(\left|h_{i_{n+1}}\right|^{1/l_{n+1}}\sigma_{n+1}X_{r_j}\right)(\widetilde{y}_n)\,.$$

We can repeat again the argument in the proof of Theorem 3.4. Let

$$y_{n+1}=\varphi(1)\,;\widetilde{y}_{n+1}=\gamma(1)\ \text{ with}$$

$$\begin{cases}\varphi'(\tau)=\left|h_{i_{n+1}}\right|^{1/l_{n+1}}\sigma_{n+1}\left(S_{r_j}^x\right)_{\varphi(\tau)}\\ \varphi(0)=y_n\end{cases}\qquad\begin{cases}\gamma'(\tau)=\left|h_{i_{n+1}}\right|^{1/l_{n+1}}\sigma_{n+1}\left(X_{r_j}\right)_{\gamma(\tau)}\\ \gamma(0)=\widetilde{y}_n\end{cases}$$

Then

$$\begin{aligned}\varphi(s)-\gamma(s)&=y_n-\widetilde{y}_n+\int_0^s\left[\varphi'(\tau)-\gamma'(\tau)\right]d\tau=\\&=y_n-\widetilde{y}_n+\int_0^s\left|h_{i_{n+1}}\right|^{1/l_{n+1}}\sigma_{n+1}\left[\left(S_{r_j}^x\right)_{\varphi(\tau)}-\left(X_{r_j}\right)_{\varphi(\tau)}\right]d\tau+\\&+\int_0^s\left|h_{i_{n+1}}\right|^{1/l_{n+1}}\sigma_{n+1}\left[\left(X_{r_j}\right)_{\varphi(\tau)}-\left(X_{r_j}\right)_{\gamma(\tau)}\right]d\tau\\&=A+B+C.\end{aligned}$$

Now, by inductive assumption,

$$|A|\leq c\,\|h\|_\eta^r$$

while by (3.2)

$$|B| \leq c \int_0^s \left|h_{i_{n+1}}\right|^{1/l_{n+1}} |\varphi(\tau) - x|^{r-1} d\tau \leq$$
$$\leq c \int_0^s \|h\|_\eta \|h\|_\eta^{r-1} d\tau \leq c \|h\|_\eta^r$$

where we used the fact that

$$|\varphi(\tau) - x| \leq |\varphi(\tau) - y_n| + \sum_{k=2}^{n} |y_k - y_{k-1}| + |y_1 - x| \leq c \sum_{k=1}^{n+1} |h_{i_k}|^{1/l_k} \leq c \|h\|_\eta .$$

Finally,

$$|C| \leq c \int_0^s \left|h_{i_{n+1}}\right|^{1/l_{n+1}} |\varphi(\tau) - \gamma(\tau)| d\tau \leq c \|h\|_\eta \int_0^s |\varphi(\tau) - \gamma(\tau)| d\tau.$$

Collecting the previous inequalities, Gronwall's Lemma implies

$$|\varphi(s) - \gamma(s)| \leq c \|h\|_\eta^r ,$$

which for $s = 1$ gives the desired assertion. This ends the proof of (7.14) and hence of (7.10). ■

Next, we need the following:

Lemma 7.11 *For any $y \in \Omega_\eta$ and $h \in Q$, the map*

$$x \longmapsto E_\eta^{S^x}(y, h)$$

is Lipschitz continuous in Ω_η, and its Jacobian satisfies:

$$\left| \frac{\partial E_\eta^{S^x}(y, h)}{\partial x} \right| \leq c |h|^{1/r} \quad \text{for a.e. } x \in \Omega_\eta.$$

Also, for any $y \in \Omega_\eta$ the map $(x, h) \longmapsto E_\eta^{S^x}(y, h)$ is continuous in $\Omega_\eta \times Q$.

Proof. Continuity with respect to (x, h), as well as Lipschitz continuity with respect to x are immediate. Let us prove the bound on derivatives. We have

$$\frac{\partial}{\partial x}\left[E_\eta^{S^x}(y, h)\right] = \frac{\partial}{\partial x}\left[\left(\prod_{j=i}^{M} \exp\left(\left|h_{k_j}\right|^{1/l_{k_j}} \sigma_j S_{r_j}^x\right)\right)(y)\right]$$

with $\sigma_j = \pm 1$. To fix ideas, let us compute the derivative of the composition of two such terms:

$$\frac{\partial}{\partial x}\left[\exp\left(|h_{k_1}|^{1/l_{k_1}} \sigma_1 S_{r_1}^x\right) \exp\left(|h_{k_2}|^{1/l_{k_2}} \sigma_2 S_{r_2}^x\right)(y)\right]$$
$$= \frac{\partial}{\partial x}\left[\exp\left(|h_{k_1}|^{1/l_{k_1}} \sigma_1 S_{r_1}^x\right)(z)\right]_{z=\exp\left(\left|h_{k_2}\right|^{1/l_{k_2}} \sigma_2 S_{r_2}^x\right)(y)} +$$
$$+ \frac{\partial}{\partial z}\left[\exp\left(|h_{k_1}|^{1/l_{k_1}} \sigma_1 S_{r_1}^x\right)(z)\right]_{z=\exp\left(\left|h_{k_2}\right|^{1/l_{k_2}} \sigma_2 S_{r_2}^x\right)(y)} \cdot \frac{\partial}{\partial x}\left[\exp\left(|h_{k_2}|^{1/l_{k_2}} \sigma_2 S_{r_2}^x\right)(y)\right]$$
$$\equiv A(h, x) + B(h, x) \cdot C(h, x) .$$

Let us inspect the term $A(h,x)$. In order to compute

$$\frac{\partial}{\partial x}\left[\exp\left(|h_{k_1}|^{1/l_{k_1}}\,\sigma_1 S_{r_1}^x\right)(z)\right],$$

let us write

$$\exp\left(|h_{k_1}|^{1/l_{k_1}}\,\sigma_1 S_{r_1}^x\right)(z)=\gamma(x,1)$$

with

$$\begin{cases}\frac{d\gamma}{\partial\tau}(x,\tau)=|h_{k_1}|^{1/l_{k_1}}\,\sigma_1\left(S_{r_1}^x\right)_{\gamma(x,\tau)}\\ \gamma(x,0)=z.\end{cases}$$

Then

$$\begin{aligned}|\gamma(x+w,\tau)-\gamma(x,\tau)| &\leq |h_{k_1}|^{1/l_{k_1}}\int_0^\tau\left|\left(S_{r_1}^{x+w}\right)_{\gamma(x+w,s)}-\left(S_{r_1}^x\right)_{\gamma(x,s)}\right|ds\\ &\leq |h_{k_1}|^{1/l_{k_1}}\int_0^\tau\left|\left(S_{r_1}^{x+w}\right)_{\gamma(x+w,s)}-\left(S_{r_1}^x\right)_{\gamma(x+w,s)}\right|ds+\\ &+|h_{k_1}|^{1/l_{k_1}}\int_0^\tau\left|\left(S_{r_1}^x\right)_{\gamma(x+w,s)}-\left(S_{r_1}^x\right)_{\gamma(x,s)}\right|ds\\ &\equiv A_1+A_2.\end{aligned}$$

Now, since the coefficients of the vector field $S_{r_1}^x$ depend in a Lipschitz continuous way on the point x,

$$|A_1|\leq c\,|h_{k_1}|^{1/l_{k_1}}\,|w|\,\tau$$

while

$$|A_2|\leq|h_{k_1}|^{1/l_{k_1}}\int_0^\tau|\gamma(x+w,s)-\gamma(x,s)|\,ds$$

and, by Gronwall's inequality,

$$|\gamma(x+w,\tau)-\gamma(x,\tau)|\leq c\,|h_{k_1}|^{1/l_{k_1}}\,|w|\,\tau,$$

which for $\tau=1$ gives

$$\left|\exp\left(|h_{k_1}|^{1/l_{k_1}}\,\sigma_1 S_{r_1}^{x+w}\right)(z)-\exp\left(|h_{k_1}|^{1/l_{k_1}}\,\sigma_1 S_{r_1}^x\right)(z)\right|\leq c\,|h_{k_1}|^{1/l_{k_1}}\,|w|\,.$$

This shows that $x\mapsto\exp\left(|h_{k_1}|^{1/l_{k_1}}\,\sigma_1 S_{r_1}^x\right)(z)$ is Lipschitz continuous with Lipschitz constant $\leq c\,|h_{k_1}|^{1/l_{k_1}}$. Hence, the L^∞ function

$$x\mapsto\frac{\partial}{\partial x}\exp\left(|h_{k_1}|^{1/l_{k_1}}\,\sigma_1 S_{r_1}^x\right)(z)$$

has L^∞ norm $\leq c\,|h_{k_1}|^{1/l_{k_1}}$.

The term $C(h,x)$ is similar to $A(h,x)$. As to the term

$$B(h,x)=\frac{\partial}{\partial z}\left[\exp\left(|h_{k_1}|^{1/l_{k_1}}\,\sigma_1 S_{r_1}^x\right)(z)\right],$$

note that the exponential is taken with respect to a smooth vector field, and the derivative of an exponential with respect to the initial condition z is bounded in terms of the derivatives of the coefficients of the vector field $|h_{k_1}|^{1/l_{k_1}} \sigma_1 S_{r_1}^x$ which is bounded by $c\,|h_{k_1}|^{1/l_{k_1}}$, uniformly with respect to x. By composition, we can conclude that

$$\left|\frac{\partial}{\partial x}\left[E_\eta^{S^x}(y,h)\right]\right| \leq c\,|h|^{1/r}$$

for a.e. $x \in \Omega_\eta$, any $y \in \Omega_\eta, h \in Q$. ■

Lemma 7.12 *Let*

$$\Theta(x,h) = E_\eta^X(x,\cdot)^{-1} E_\eta^{S^x}(x,h) \;\; \textit{for} \;\; (x,h) \in \Omega_\eta \times Q.$$

Then, the mapping $(x,h) \mapsto \Theta(x,h)$ *is Lipschitz continuous in* $\Omega_\eta \times Q$. *Moreover, the Jacobian* $J_{\Theta(x,h)}$ *of the map* $x \mapsto \Theta(x,h)$ *satisfies*

$$\left\|J_{\Theta(\cdot,h)}\right\|_{L^\infty(\Omega_\eta)} \leq \omega(h) \quad \forall h \in Q,$$

where $\omega(h) \to 0$ *as* $h \to 0$.

Proof. The function

$$E_\eta^{S^x}(y,h)$$

is C^1 with respect to (y,h), locally uniformly with respect to x and, by Lemma 7.11, is Lipschitz continuous with respect to x; hence

$$(x,h) \mapsto E_\eta^{S^x}(x,h)$$

is Lipschitz continuous. Let us show that the function

$$(x,y) \mapsto E_\eta^X(x,\cdot)^{-1}(y)$$

is C^1 in the joint variables; this will allow to conclude that $(x,h) \mapsto \Theta(x,h)$ is Lipschitz continuous in $\Omega_\eta \times Q$.

Let us consider the function

$$G(x,y,h) = y - E_\eta^X(x,h).$$

Since the function $(x,h) \mapsto E_\eta^X(x,h)$ is C^1 in the joint variables, $G(x,x,0) = 0$ and $\frac{\partial G}{\partial h}(x,x,0)$ has maximal rank, by the implicit function theorem there exists a unique C^1 function $h = F(x,y)$ such that

$$G(x,y,F(x,y)) = 0.$$

This F is exactly $(x,y) \mapsto E_\eta^X(x,\cdot)^{-1}(y)$, so we are done.

We now want to prove the bound on the Jacobian of Θ.

Since $\Theta(x,0) = 0\ \forall x$, for $h = 0$ the Jacobian of $\Theta(\cdot,0)$ vanishes. We have therefore to prove that $J_{\Theta(x,h)}$ is continuous at $h = 0$, uniformly with respect to x. We have, with the obvious meaning of symbols,

$$\begin{aligned} J_{\Theta(x,h)} &= \left(J_{x\mapsto E_\eta^X(x,\cdot)^{-1}(y)}\right)_{y=E_\eta^{S^x}(x,h)} + \left(J_{y\mapsto E_\eta^X(x,\cdot)^{-1}(y)}\right)_{y=E_\eta^{S^x}(x,h)} \cdot J_{x\mapsto E_\eta^{S^x}(x,h)} \\ &= \left(J_{x\mapsto E_\eta^X(x,\cdot)^{-1}(y)}\right)_{y=E_\eta^{S^x}(x,h)} + \left(J_{E_\eta^X(x,\cdot)}\right)^{-1}(\Theta(x,h)) \cdot J_{x\mapsto E_\eta^{S^x}(x,h)}. \end{aligned}$$

Differentiating with respect to x the identity

$$F\left(x, E_\eta^X(x,h')\right) = h'$$

we get

$$\frac{\partial}{\partial x} F\left(x, E_\eta^X(x,h')\right) + \frac{\partial}{\partial y} F\left(x, E_\eta^X(x,h')\right) \cdot J_{E_\eta^X(\cdot,h')} = 0$$

that is

$$\begin{aligned} \left(\frac{\partial}{\partial x} F(x,y)\right)_{/y=E_\eta^{S^x}(x,h)} &= -\left(\frac{\partial}{\partial y} F(x,y)\right)_{/y=E_\eta^{S^x}(x,h)} \cdot J_{x\mapsto E_\eta^X(x,h')} = \\ &= -\left(J_{E_\eta^X(x,\cdot)}\right)^{-1}(h') \cdot J_{x\mapsto E_\eta^X(x,h')}. \end{aligned}$$

Hence

$$\begin{aligned} J_{\Theta(\cdot,h)} &= -\left(J_{E_\eta^X(x,\cdot)}\right)^{-1}(h') \cdot J_{x\mapsto E_\eta^X(x,h')} + \left(J_{E_\eta^X(x,\cdot)}\right)^{-1}(h') \cdot J_{x\mapsto E_\eta^{S^x}(x,h)} \\ &\equiv A(x,h) + B(x,h) \end{aligned}$$

Now, since $E_\eta^X(x,h')$ is C^1 in the joint variables, $E_\eta^X(x,\cdot)$ is a diffeomorphism, and $h' = \Theta(x,h)$ is Lipschitz continuous, we conclude that $A(x,h)$ is h-continuous, uniformly with respect to x.

The same is true for the term $\left(J_{E_\eta^X(x,\cdot)}\right)^{-1}(h')$ appearing in $B(x,h)$. It remains to check that $h \mapsto J_{x\mapsto E_\eta^{S^x}(x,h)}$ is continuous *at least at* $h = 0$. This can be seen as follows:

$$\frac{\partial}{\partial x} E_\eta^{S^x}(x,h) = \frac{\partial}{\partial x}\left[E_\eta^{S^x}(y,h)\right]_{/y=x} + \frac{\partial}{\partial x}\left[E_\eta^{S^z}(x,h)\right]_{/z=x}.$$

The first term is continuous at $h = 0$, by Lemma 7.11, while the second term is continuous in h, because for fixed z the vector fields S^z are smooth, and $E_\eta^{S^z}(x,h)$ is a C^1 function in the joint variables (x,h). This ends the proof. ■

We can come, at last, to the

Proof of Proposition 7.8. We have to prove that the map $E_\eta^{S^x}(x,h)$ is X-controllable with hitting time $T \le \alpha\rho$. Actually, it is enough to prove X-controllability with a hitting time $T \le c\rho$ (with c possibly larger than α), because if condition (1) in the statement of Theorem 7.4 holds for some α then it still holds for a larger one, and if we replace α with a larger one we can still fulfil condition (iii) replacing β with a larger one. We have therefore to prove that there exists a function $\gamma : \Omega_\eta \times Q \times [0, T] \to \mathbb{R}^p$ satisfying conditions (C1)-(C2'), which we will recall here.

(C1) For any $(x, h) \in \Omega_\eta \times Q_\eta(\rho)$, $t \mapsto \gamma(x, h, t)$ is a X-subunit path connecting x and $E_\eta^{S^x}(x, h)$, that is

$$\begin{cases} \frac{d}{dt}\gamma(x,h,t) = \sum a_j(t)(X_j)_{\gamma(x,h,t)} \text{ for suitable } a_j \text{ with } \sum |a_j(t)|^2 \leq 1 \\ \gamma(x,h,0) = x; \gamma(x,h,T(x,h)) = E_\eta^{S^x}(x,h) \end{cases}$$

for a suitable $T(x, h) \leq T$.

To prove this, let $y = E_\eta^{S^x}(x, h)$ for some $h \in Q_\eta(\rho)$. By Lemma 7.10 (and its proof),

$$\begin{aligned} y &= E_\eta^X(x, h') \text{ for some } h' \in Q_\eta(c\rho), \text{ namely} \\ h' &= \Theta(x, h) \equiv E_\eta^X(x, \cdot)^{-1} E_\eta^{S^x}(x, h). \end{aligned}$$

(Note that this $\Theta(x, h)$ is the one studied in Lemma 7.12). We can build an admissible curve $t \mapsto \widetilde{\gamma}(x, h', t)$ connecting x to $y = E_\eta^X(x, h')$ in time $T \leq c\|h'\|_\eta \leq c\rho$ moving along the curve which define the quasiexponential map $E_\eta^X(x, \cdot)$, as suggested in [33, pp.336-7]: namely, if

$$E_\eta^X(x, h') = \prod_{j=1}^M \exp\left(\left|h'_{k_j}\right|^{1/l_{k_j}} \sigma_j X_{r_j}\right)(x)$$

then one easily checks that any map

$$(x, h') \mapsto \exp\left(\left|h'_{k_j}\right|^{1/l_{k_j}} \sigma_j X_{r_j}\right)(x)$$

is X-controllable with hitting time $\left|h'_{k_j}\right|^{1/l_{k_j}}$; by composition, Lemma 4.2 in [33] implies that $E_\eta^X(x, h')$ is X-controllable with hitting time

$$T \leq c \sup_{h \in Q} \|h\|_\eta \leq c\rho,$$

and in particular there exists a curve $\widetilde{\gamma}$ with the properties required by (C1). Next, we have to check:

(C2') For any $(h, t) \in Q \times [0, T]$, $x \mapsto \gamma(x, h, t)$ is a one-to-one bilipschitz map having jacobian determinant bounded away from zero, i.e.

$$b \equiv \inf_{O \times Q \times [0,T]} \left|\frac{\partial \gamma}{\partial x}\right| > 0.$$

Namely, we have to compute the x-derivative of the composed function

$$\gamma(x, h, t) = \widetilde{\gamma}(x, \Theta(x, h), t),$$

that is

$$\frac{\partial}{\partial x}\gamma(x,h,t)=\frac{\partial\widetilde{\gamma}}{\partial x}(x,\Theta(x,h),t)+\frac{\partial\widetilde{\gamma}}{\partial h'}(x,\Theta(x,h),t)\cdot\frac{\partial}{\partial x}\left[E_{\eta}^{X}(x,\cdot)^{-1}E_{\eta}^{S^{x}}(x,h)\right]. \tag{7.15}$$

First, let us recall that $\Theta(x,0)=0$ and $x\mapsto\widetilde{\gamma}(x,h',t)$ has the smoothness of $x\mapsto E_{\eta}^{X}(x,h')$ hence is $C^{r-1,1}$. Moreover, $\widetilde{\gamma}(x,0,t)=x$, hence

$$\frac{\partial\widetilde{\gamma}}{\partial x}(x,0,t)=I$$

(identity matrix) and, by continuity, $\frac{\partial\widetilde{\gamma}}{\partial x}(x,\Theta(x,h),t)$ is close to the identity matrix for h small enough.

Second, we know that $(x,h')\mapsto\widetilde{\gamma}(x,h',t)$ has the smoothness of $(x,h')\mapsto E_{\eta}^{X}(x,h')$, hence is C^{1}. Therefore

$$\frac{\partial\widetilde{\gamma}}{\partial h'}(x,\Theta(x,h),t)\ \text{ is bounded.}$$

Finally, by Lemma 7.12

$$\left|\frac{\partial}{\partial x}\left[E_{\eta}^{X}(x,\cdot)^{-1}E_{\eta}^{S^{x}}(x,h)\right]\right|\leq c\omega(h)\quad\to 0\text{ as }h\to 0.$$

By (7.15), these facts imply that $\frac{\partial\gamma}{\partial x}$ is a small perturbation of the identity, for small h.

Summarizing, the situation is the following:

$$\begin{aligned}&\gamma(x_1,h,t)-\gamma(x_2,h,t)=\\&=[\widetilde{\gamma}(x_1,\Theta(x_1,h),t)-\widetilde{\gamma}(x_2,\Theta(x_1,h),t)]+[\widetilde{\gamma}(x_2,\Theta(x_1,h),t)-\widetilde{\gamma}(x_2,\Theta(x_2,h),t)]\\&\equiv A+B.\end{aligned}$$

Since, for small h', the map $x\longmapsto\widetilde{\gamma}(x,h',t)$ is a diffeomorphism,

$$|A|\geq c\,|x_1-x_2|\,.$$

On the other hand, by Lemma 7.12,

$$|B|\leq|\Theta(x_1,h)-\Theta(x_2,h)|\leq c\omega(h)\,|x_1-x_2|$$

Hence for h small enough $x\longmapsto\gamma(x,h,t)$ is a bilipschitz map, with Jacobian determinant bounded away from zero. Note that, asking h small enough amounts to diminishing the constant r_0 in Theorem 7.6, which is allowed, as we have already noted. This completes the proof of Proposition 7.8, and therefore of Theorem 7.2. ■

Remark 7.13 *If we assume that our vector fields X_i's only belong to $C^{r-1}\left(\overline{\Omega}\right)$, instead of $C^{r-1,1}(\Omega)$, the theory developed in sections 2-5 allows to derive a rougher version of Poincaré's inequality. Let us sketch it here. For a fixed point*

$x_0 \in \Omega'$, let us consider the smooth approximating vector fields $S_i^{x_0}$. Since these are smooth Hörmander's vector fields, they satisfy a Poincaré's inequality

$$\int_{B\times B} |u(y) - u(x)|\, dydx \leq c\rho\, |B| \int_{\lambda B} |S^{x_0} u(y)|\, dy \tag{7.16}$$

where $B = B_1^{S^{x_0}}(x_0, \rho)$; by Theorem 5.9, a similar inequality also holds with $B = B_1^X(x_0, \rho)$, and possibly a larger number λ. Now, let us recall that by Proposition 3.1

$$S_i^{x_0} = X_i + \sum_{j=1}^{n} c_{ij}(x)\, \partial_{x_j} \text{ with } c_{ij}(x) = o\left(|x - x_0|^{r-1}\right) \text{ as } x \to x_0.$$

Hence (7.16) rewrites as

$$\int_{B\times B} |u(y) - u(x)|\, dydx \leq c\rho\, |B| \int_{\lambda B} |Xu(y)|\, dy + |B|\, o(\rho^r) \int_{\lambda B} |\nabla u(y)|\, dy$$

(where ∇ is the Euclidean gradient), or

$$\int_{B} |u(y) - u_B(x)|\, dydx \leq c\rho \int_{\lambda B} |Xu(y)|\, dy + o(\rho^r) \int_{\lambda B} |\nabla u(y)|\, dy.$$

8 Applications

There is a large literature dealing with relations between Poincaré's inequality and other results about both Sobolev spaces and solutions to second order PDEs, both in the Euclidean (elliptic) context and in the subelliptic one. We refer to Hajlasz-Koskela's monograph [25] for a good exposition and a rich source of further references on this area of research. Some of these results have been established in great generality, as axiomatic theories. For instance, it is well-known that, roughly speaking, the validity of the doubling condition and a Poincaré's inequality imply a Sobolev embedding. This fact has been proved, at different levels of generality, by Saloff-Coste [49], Garofalo-Nhieu [23], Franchi-Lu-Wheeden [21], Hajlasz-Koskela [25]. In turn, the doubling condition, Poincaré and Sobolev inequalities allow to reply Moser's iteration technique, and prove a Harnack inequality and a Hölder continuity result for local solutions to (elliptic or subelliptic) variational second order equations. In this section we want to point out, for convenience of the reader, some precise statements of this kind, which describe a few consequences of the results we have proved so far, which can be easily derived from the aforementioned general theories, and constitute new results, in our general setting.

8.1 Sobolev embedding and p-Poincaré's inequality

Here we keep Assumptions (D) stated at the beginning of § 7. We start noting that (7.2) implies, by Hölder's inequality,

$$\frac{1}{|B|} \int_{B} |u(y) - u_B|\, dy \leq c\rho \left(\frac{1}{|\lambda B|} \int_{\lambda B} |Xu(y)|^p\, dy \right)^{1/p} \text{ for any } p > 1. \tag{8.1}$$

Then, applying Theorem 13.1 in [25] we have the following strong result:

Theorem 8.1 *For any $\Omega' \Subset \Omega$, $p \geq 1$, there exist $c, r_0 > 0$, such that:*
(i) (Sobolev inequality) There exists a constant $k > 1$ such that

$$\left(\frac{1}{|B|}\int_B |\varphi(x)|^{kp}\,dx\right)^{1/kp} \leq c\rho\left(\frac{1}{|B|}\int_B |X\varphi(x)|^p\,dx\right)^{1/p} \tag{8.2}$$

for any $\varphi \in C_0^\infty(B)$, with $B = B(x,\rho)$, $\rho \leq r_0$, $x \in \Omega'$, and the balls are taken with respect to the distance $d_{X,1}$.
(ii) (Poincaré's p-p inequality)

$$\left(\frac{1}{|B|}\int_B |\varphi(x) - \varphi_B|^p\,dx\right)^{1/p} \leq c\rho\left(\frac{1}{|B|}\int_B |X\varphi(x)|^p\,dx\right)^{1/p} \tag{8.3}$$

for any $\varphi \in C^\infty(B)$, B as above.

Note that, quite surprisingly, in (8.3) a ball of the *same radius* appears at both sides of the inequality; this fact, instead, is natural in (8.2), where the function φ is assumed compactly supported in B. This theorem is proved in [25] exploiting a set of assumptions which, in our context of nonsmooth Hörmander's vector fields, we have proved in the previous sections, namely:
(a) Poincaré's inequality (Theorem 7.2 and in particular (8.1));
(b) the doubling condition for metric balls with respect to the distance d_1(5.10);
(c) the equivalence of the Euclidean topology with d_1-topology, which follows from Proposition 5.8.

8.2 Moser's iteration for variational second order operators

Let us consider a linear second order variational operator of the kind

$$Lu \equiv \sum_{i,j=1}^{n} X_i^*\left(a_{ij}(x)\,X_j u\right) \tag{8.4}$$

where $X_1, ..., X_n$ is our set of nonsmooth Hörmander's vector fields, X_i^* denotes the transposed operator of X_i, and $\{a_{ij}\}_{i,j=1}^n$ is a symmetric uniformly positive definite matrix of $L^\infty(\Omega)$ functions:

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j \leq \lambda^{-1}|\xi|^2$$

for some $\lambda > 0$, any $\xi \in \mathbb{R}^n$, a.e. $x \in \Omega$. We say that u is a local solution to the equation $Lu = 0$ in Ω if

$$u \in W_{X,loc}^{1,2}(\Omega) = \left\{u \in L_{loc}^2(\Omega) : X_i u \in L_{loc}^2(\Omega) \text{ for } i = 1, 2, ..., n\right\}$$

and

$$\int_{\Omega}\sum_{i,j=1}^{n}a_{ij}X_iuX_j\varphi dx=0 \text{ for any } \varphi\in C_0^{\infty}(\Omega).$$

In this context, Theorem 8.1 (with $p=2$) gives the tools to settle the classical Moser's iterative method, and prove the facts collected in the following:

Theorem 8.2 *Let u be a local solution to $Lu=0$ in Ω. Then:*

(i) u is locally bounded, with

$$\|u\|_{L^{\infty}(B)}\leq c\left(\frac{1}{|2B|}\int_{2B}|u(x)|^2dx\right)^{1/2}$$

for any $2B\subset\Omega$, with c depending on the coefficients a_{ij} only through the number λ.

(ii) If u is positive in Ω, then it satisfies a Harnack's inequality:

$$\sup_B u\leq c\inf_B u$$

for any $2B\subset\Omega$, with c depending on the coefficients a_{ij} only through the number λ.

(iii) u is Hölder continuos (in the usual, Euclidean sense) of some exponent $\alpha\in(0,1)$, on any subset $\Omega'\Subset\Omega$:

$$|u(x)-u(y)|\leq c|x-y|^{\alpha}$$

for any $x,y\in\Omega'$, with c,α depending on Ω' and depending on the coefficients a_{ij} only through the number λ, and c also depending on $\|u\|_{L^2(\Omega)}$.

The above theorem follows, for instance, applying the general theory developed by Sawyer-Wheeden in [52] (see in particular Theorem 8 in [52]). To check the assumptions of this theory one needs to exploit Theorem 8.1 and the facts (b), (c) recalled in the previous subsection. Actually, the Hölder continuity result which follows from Theorem 8 in [52] is much more general than the one we have stated in (iii): it holds for local solutions to a nonhomogeneous equation, also involving lower order terms. With the terminology introduced in [52], one can say that the operator L in (8.4) is L^q*-subelliptic*. We do not state this result in its full generality for the sake of simplicity.

Clearly, the local Hölder continuity result can be applied also to local solutions for nonlinear operators of the kind:

$$Pu=\sum_{i=1}^{n}X_i^*\left(a_{ij}(x,u(x))X_ju\right).$$

For smooth Hörmander's vector fields, the results contained in Theorem 8.2 follow from Nagel-Stein-Wainger's doubling condition and Jerison's Poincaré inequality (see [43], [28]). Analogous results, in a weighted context, have been proved by Lu in [34].

We also point out that operators (8.4) structured on nonsmooth Hörmander's vector fields can be seen also as particular instances of X-elliptic operators, in the sense of Lanconelli-Kogoj [32], with the same consequences already described.

9 Appendix: some known results about O.D.E.'s

1. Gronwall's Lemma

We state the version of Gronwall's Lemma that we use throughout this paper. For a proof, see for instance [10, p.625].

Lemma 9.1 *Let $\phi : [0,1] \to \mathbb{R}$ be a nonnegative continuous function such that*

$$\phi(t) \leq c \int_0^t \phi(s)\, ds + K \tag{9.1}$$

for any $t \in [0,1]$ and two positive constants c, K. Then there exists a constant $c_1 > 0$, only depending on c, such that

$$\phi(t) \leq c_1 K.$$

2. Existence results and uniformity matters

Theorem 9.2 (Carathéodory's existence theorem) *Let $F(t,x)$ be a function defined for $t \in (-T,T)$, $x \in \mathbb{R}^p$, F continuous in x for fixed t and measurable in t for fixed x. Assume that*

$$|F(t,x)| \leq M(t)$$

with $M \in L^1(a,b)$ for any $[a,b] \subset (-T,T)$. Then, for every $x_0 \in \mathbb{R}^p$ there exists an absolutely continuous function $\phi : (-T,T) \to \mathbb{R}^p$ solution to the problem

$$\begin{cases} \phi'(t) = F(t, \phi(t)) \ \text{ for a.e. } t \in (-T,T) \\ \phi(0) = x_0 \end{cases}$$

exists.

For the proof, see e.g. [51, p.140]. In the proof of Theorem 3.4 we apply this theorem to

$$F(t,x) = \sum_{|I| \leq r} a_I(t) \left(X_{[I]}\right)_x$$

where $a_I(\cdot)$ are bounded measurable functions on $[0,1]$, and the vector fields X_i are $C^{r-1}\left(\overline{\Omega}\right)$. Now, for any fixed $\Omega' \Subset \Omega'' \Subset \Omega$, we can find a function $\widetilde{F}(t,x)$ satisfying the assumptions of the Carathéodory's theorem and agreeeing with

$F(t,x)$ for $x \in \Omega''$. When $x_0 \in \Omega'$ and $|a_I(t)| \leq \delta^{|I|}$ with δ small enough, there exists a solution ϕ to

$$\begin{cases} \phi'(t) = \widetilde{F}(t, \phi(t)) \text{ for } t \in [0,1] \\ \phi(0) = x_0 \end{cases}$$

such that $\phi(t) \in \Omega''$ for $t \in [0,1]$ and therefore ϕ solves

$$\begin{cases} \phi'(t) = \sum_{|I| \leq r} a_I(t) \left(X_{[I]}\right)_{\phi(t)} \text{ for } t \in [0,1] \\ \phi(0) = x_0. \end{cases}$$

Note that this δ depends on Ω and Ω', but not on x_0.

Theorem 9.3 (Cauchy's existence and uniqueness theorem) *Let X be a Lipschitz continuous vector field defined in some domain $\Omega \subset \mathbb{R}^p$, and $\Omega' \Subset \Omega$. There exists a number $\delta > 0$, depending on X, Ω, Ω', such that for every $x_0 \in \Omega'$, a unique C^1 solution $\phi : [-\delta, \delta] \to \Omega$ to the problem*

$$\begin{cases} \phi'(t) = X_{\phi(t)} \text{ for } t \in [-\delta, \delta] \\ \phi(0) = x_0 \end{cases} \tag{9.2}$$

exists.

For the proof, see [44]. We stress the fact that the number δ can be chosen independently of x_0, at least when x_0 ranges in a compact subset of Ω. This uniformity property has been implicitly used in this paper.

3. Discussion about the dependence of the constants on the smooth vector fields in the results proved by Nagel-Stein-Wainger [43]

Here we want to justify Claim 3.3 stated in §3. We have checked in detail this Claim, revising the whole argument of [43]. Here we cannot repeat the whole reasoning, but limit ourself to some remarks which stress the points to be kept in mind, in order to understand the quantitative dependence of the constants. What follows is intended to be read keeping at hand the paper [43]: we will use their notations without any explanation.

1. We apply the construction of [43], Chapter II, §1, assuming that the vector fields Y_i are *all* the commutators $X_{[I]}$ of our smooth vector fields $X_0, X_1, ..., X_n$, with $|I| \leq m$. If $Y_i = X_{[I]}$, we will set $d_i = |I|$. It is not difficult to see that, by the Jacobi identity, for any multiindices I, J we can write

$$\left[X_{[I]}, X_{[J]}\right] = \sum_{|K|=|I|+|J|} b_{IJ}^K X_{[K]}$$

where b_{IJ}^K are universal constants only depending on I, J, K (this fact is stated for instance in [27]). Therefore we can write equation (1) of [43] as

$$[Y_j, Y_k] = \sum_{d_l \leq d_j + d_k} c_{jk}^l(x) Y_l$$

where the "functions" $c_{jk}^{l}(x)$ are actually *universal constants*.

2. Let us call "admissible function" any function which can be obtained, starting from the coefficients of the smooth vector fields X_i, by linear combination and a finite number of operations of sums, products, and derivatives; moreover, it is allowed to divide by the quantity

$$\det(Y_{i_1}, Y_{i_2}, ..., Y_{i_N})$$

where $Y_{i_1}, Y_{i_2}, ..., Y_{i_N}$ is a fixed basis in an open subset $\Omega_I \subset \Omega$. Clearly, admissible functions belong to $C^\infty(\Omega_I)$.

Let $a_j^l(x)$ have the meaning explained in [43, p.116]. A key role is played in [43] by the modules of functions A_s^p, defined as the $C^\infty(\Omega)$ submodule of $C^\infty(\Omega_I)$ generated by all the functions of the form

$$a_{j_1}^{l_1} \cdot a_{j_2}^{l_2} \cdot ... a_{j_k}^{l_k}$$

where the indices satisfy suitable conditions. Now, we claim that, keeping in mind our remark 1 and revising the whole reasoning of Chapter II, §1 in [43], one can check that all the arguments and statements of that section remain true if we redefine the classes of functions A_s^p as the modules generated by the functions $a_{j_1}^{l_1} \cdot a_{j_2}^{l_2} \cdot ... a_{j_k}^{l_k}$ taking as "scalars" not all the functions in $C^\infty(\Omega)$, but only *admissible functions*.

This fact is crucial because, whenever we prove that a function belongs to a class A_s^p, this implies a quantitative estimate in terms of the quantities allowed by our Claim 3.3.

3. Revising the whole reasoning of the following sections of Chapter II in [43], then, one can check that most of the arguments do not involve new forms of dependence of the relevant constants on the vector fields X_i. The points that require a more careful inspection are those involving the Baker-Campbell-Hausdorff formula (henceforth, BCH formula), since this identity, in principle, involves infinitely many derivatives. So, our next remark is devoted to BCH formula.

4. We need the following finite BCH formula with a remainder:

for any $\Omega' \Subset \Omega$, given two positive integers k_0, j_0 there exist $r_0 > 0$ and $C > 0$ such that, if $|s|, |t| < r_0$ then

$$\exp(sX)\exp(tY)(x) = \exp\left(\sum_{k+j\geq 1, k\leq k_0, j\leq j_0} s^k t^j C_{k,j}\right)(x) + O\left(s^{k_0+1}\right) + O\left(t^{j_0+1}\right) \tag{9.3}$$

for any $x \in \Omega'$, where:

(i) $C_{k,j}$ denotes a finite linear combination of commutators of X, Y, with universal coefficients, where every commutator contains k times X and j times Y;

(ii) the remainders satisfy the estimates

$$\left|O\left(s^{k_0+1}\right)\right| \leq Cs^{k_0+1}, \left|O\left(t^{j_0+1}\right)\right| \leq Ct^{j_0+1}$$

where the constants r_0, C only depend on a finite number of $C^k(\Omega')$ norms of the coefficients of X, Y.

Although the above fact is probably well known, we have not been able to find a precise reference for the last statement about the dependence of the constants r_0 and C. However, revising the proof of this formula given for instance in [3], one can check that this is actually the case.

The identity (9.3) is applied several times in §§3-5 of Chapter II of [43], taking as X, Y suitable commutators of our vector fields; thanks to the above remark, the dependence of the constants satisfies also in this case the desired control.

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