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L^p ESTIMATES FOR NONVARIATIONAL HYPOELLIPTIC OPERATORS WITH VMO COEFFICIENTS

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ABSTRACT. Let $X_1, X_2, \ldots, X_q$ be a system of real smooth vector fields, satisfying Hörmander's condition in some bounded domain $\Omega \subset \mathbb{R}^n$ $(n > q)$. We consider the differential operator

$$\mathcal{L} = \sum_{i=1}^{q} a_{ij}(x) X_i X_j,$$

where the coefficients $a_{ij}(x)$ are real valued, bounded measurable functions, satisfying the uniform ellipticity condition:

$$\mu|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}(x)\xi_i\xi_j \leq \mu^{-1}|\xi|^2$$

for a.e. $x \in \Omega$, every $\xi \in \mathbb{R}^q$, some constant μ. Moreover, we assume that the coefficients a_{ij} belong to the space *VMO* ("Vanishing Mean Oscillation"), defined with respect to the subelliptic metric induced by the vector fields $X_1, X_2, \ldots, X_q$. We prove the following local $\mathcal{L}^p$-estimate:

$$\|X_i X_j f\|_{\mathcal{L}^p(\Omega')} \leq c \left\{ \|\mathcal{L}f\|_{\mathcal{L}^p(\Omega)} + \|f\|_{\mathcal{L}^p(\Omega)} \right\}$$

for every $\Omega' \subset\subset \Omega$, $1 < p < \infty$. We also prove the local Hölder continuity for solutions to $\mathcal{L}f = g$ for any $g \in \mathcal{L}^p$ with p large enough. Finally, we prove $\mathcal{L}^p$-estimates for higher order derivatives of f, whenever g and the coefficients a_{ij} are more regular.

0. Introduction and main results

Let $X_0, X_1, \ldots, X_q$ be a system of C^∞ real vector fields defined in $\mathbb{R}^n$ $(n \geq q+1)$, that is,

$$X_i = \sum_{j=1}^{n} b_{ij}(x) \frac{\partial}{\partial x_j} \quad \text{with} \quad b_{ij} \in C^\infty(\mathbb{R}^n)$$

$(i = 0, 1, \ldots, q;\ j = 1, 2, \ldots, n)$, and let

$$L = \sum_{i=1}^{q} X_i^2 + X_0. \tag{0.1}$$

Recall that a linear differential operator P with C^∞ coefficients is said to be hypoelliptic in an open set Ω if, whenever the equation $Pu = f$ is satisfied, in the distributional sense, in the neighborhood of a point in Ω and f is C^∞ in that neighborhood, then u is also C^∞ in that neighborhood. A famous theorem proved by

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Hörmander in '67 (see [24]) states that the operator (0.1) is hypoelliptic whenever the X_i's satisfy at every point of Ω the so-called "Hörmander's condition": the vector fields X_i, their commutators $[X_i, X_j] = X_iX_j - X_jX_i$, the commutators of the X_i's with their commutators, and so on until a certain step s, generate $\mathbb{R}^n$ (as a vector space). Therefore, this condition gives a strong regularity property for the differential operator L.

In '69 Bony [2] proved that the Dirichlet problem for operators (0.1) satisfying Hörmander's condition can be solved, in classical sense, at least for suitable classes of domains.

A further problem is that of proving a priori estimates, in suitable Sobolev or Hölder spaces, for solutions to the equation $Lu = f$. For instance, one would like to have a bound on the $\mathcal{L}^p$ norm ($1 < p < \infty$) of X_iX_ju in terms of the $\mathcal{L}^p$ norms of Lu and u. A first result of this kind was achieved in '75 by Folland [18], assuming the existence of an underlying structure of homogeneous group for which the operator L is left-invariant and homogeneous of degree two. Later, in '76, Rothschild and Stein [35] proved a similar result for any system of vector fields satisfying Hörmander's condition. This deep generalization was made possible by a technique consisting in lifting the vector fields X_i to a higher dimensional space, obtaining new vector fields $\tilde{X}_i$ which are free up to some step s, and can be locally approximated by other vector fields Y_i which fit the assumptions of Folland's quoted paper.

The $\mathcal{L}^p$-estimates proved both in [18] and in [35] are local, so that an $\mathcal{L}^p$ theory for the Dirichlet problem for general Hörmander's operators of kind (0.1) is still lacking. Also, note that the operators considered in the above papers have always C^∞ coefficients, in contrast with the fact that $\mathcal{L}^p$-estimates for linear PDEs do not require intrinsically a high regularity of the coefficients.

A useful point of view in the study of these operators, started in '81 with the paper [17] by Fefferman-Phong. This paper contains the definition of a "subelliptic metric", induced by the vector fields X_i. Roughly speaking, the integral curves of the vector fields are, by definition, the geodesics of the induced metric. It turns out that the metric balls are well shaped to describe geometrical properties related to the operator. For instance, the fundamental solution of L can be estimated pointwise through the size of these metric balls (see Sanchez-Calle, '84, [37], and Nagel-Stein-Weinger, '85, [33]). Moreover, these metric balls define a structure of homogeneous space, in the sense of Coifman-Weiss, '71, [15], so that many tools from singular integrals and real-variable theory can be naturally employed in the study of hypoelliptic operators.

Let us turn now to the problem of proving a priori estimates for classes of operators more general than (0.1), and in particular allowing nonsmooth coefficients. A first possible extension is that modeled on variational elliptic operators:

$$L = \sum_{i,j=1}^{q} X_i^* \left(a_{ij}(x) X_i\right) \tag{0.2}$$

where X_i^* denotes the formal adjoint of X_i and the matrix $a_{ij}(x)$ satisfies an ellipticity condition (in uniform or degenerate sense). For instance, in '92 G. Lu [29],[30] studied operators of kind (0.2) where X_i are smooth Hörmander's vector fields, while the coefficients $a_{ij}(x)$ are measurable functions satisfying the degenerate ellipticity condition:

$$c^{-1}w(x)|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}(x)\xi_i\xi_j \leq cw(x)|\xi|^2$$

for every $\xi \in \mathbb{R}^q$, where w is an A_2-weight, in the sense of Muckenhoupt, with respect to the subelliptic metric induced by the X_i's. In this case the bilinear form

$$a(u,v) = \sum_{i,j=1}^{q} \int a_{ij}(x) X_i u(x)\, X_i v(x)\, dx + \int (uvw)(x)\, dx$$

allows us to give a natural definition of weak solution to the equation $Lu = f$, and to prove $\mathcal{L}^2$-estimates for the first order derivatives $X_i u$ in terms of Lu and u. In this setting, several results have been proved, including Harnack inequality and existence and size estimates for the Green's function. See also the paper of Franchi-Lu-Wheeden [20], and references therein, for related results. Recently, Franchi-Serapioni-Serra Cassano [21] have proved that a Harnack inequality for operators of kind (0.2) holds whenever the X_i's are Lipschitz continuous vector fields satisfying suitable geometric assumptions (which in particular hold for smooth Hörmander's vector fields).

A different direction of research is that of studying the non-variational analog of operators (0.1), namely:

$$\mathcal{L} = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j \tag{0.3}$$

or

$$\mathcal{L} = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j + X_0 \tag{0.4}$$

with $a_{ij}(x)$, again, satisfying some ellipticity condition.

Some regularity results of Schauder ($C^{2,\alpha}$) type for operators (0.3) have been obtained in '92 by C.-J. Xu, [40].

On the other hand, some classes of operators of kind (0.4), modeled on Kolmogorov-Fokker-Planck operators, have been studied in '93–'94 by Lanconelli and Polidoro (see [27],[34],[28]): in this case $X_i = \partial_{x_i}$, and X_0 has a particular form, which makes $X_0, X_1, \dots, X_q$ a system of Hörmander's vector fields. In these papers the fundamental solution of the operator L is studied, and a Harnack inequality is proved, assuming the coefficients a_{ij} constant [28] or Hölder continuous [34]. In '96, Bramanti-Cerutti-Manfredini [8] proved local $\mathcal{L}^p$-estimates for the same class of operators, assuming the coefficients a_{ij} in the class *VMO*, with respect to the subelliptic metric. The class *VMO* ("Vanishing Mean Oscillation", see Sarason [36]) appeared in the study of $\mathcal{L}^p$-estimates for non-variational uniformly elliptic equations in '91–'93, with the papers of Chiarenza-Frasca-Longo [11], [12]. Since *VMO* functions can have some kind of discontinuities, this theory extended the classical one, by Agmon-Douglis-Nirenberg [1]. The analog $\mathcal{L}^p$ theory for parabolic operators with *VMO* coefficients was developed in '93 by Bramanti-Cerutti [5].

Recently (see [4]), the authors proved $\mathcal{L}^p$-estimates for operators of kind (0.4), where the X_i's are a system of smooth Hörmander's vector fields, satisfying Folland's assumptions of translation invariance and homogeneity, while the coefficients

a_{ij} are *VMO*, with respect to the subelliptic metric, and satisfy the ellipticity condition:

$$\mu|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}(x)\xi_i\xi_j \leq \mu^{-1}|\xi|^2. \tag{0.5}$$

The aim of this paper is to prove these results, for operators of type (0.3) in the general case of a system of Hörmander's vector fields, that is without assuming the underlying structure of homogeneous group. Explicitly, our main result is the following:

Theorem 0.1. *Let*

$$\mathcal{L} = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j$$

where $X_1, \ldots, X_q$ form a system of C^∞ real vector fields defined in some bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq q)$, satisfying Hörmander's condition (see §1.1 for the definition). The coefficients $a_{ij}(x)$ are real valued bounded measurable functions defined in Ω, belonging to the class VMO(Ω), defined with respect to the subelliptic metric induced by the vector fields X_i (see §1.4); the matrix $\{a_{ij}(x)\}$ (not necessarily symmetric) is uniformly elliptic on $\mathbb{R}^q$:

$$\mu\,|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}(x)\ \xi_i\xi_j \leq \mu^{-1}\,|\xi|^2 \qquad \text{for every } \xi \in \mathbb{R}^q, \text{ a.e. } x \in \Omega,$$

for some positive constant μ. Then, for every $p \in (1, \infty)$, any $\Omega' \subset\subset \Omega$, there exists a constant c depending on the vector fields X_i, the numbers n, q, p, μ, the VMO moduli of the coefficients a_{ij} (see §1.4), Ω, Ω' such that for every u belonging to the Sobolev space $S_X^{2,p}(\Omega)$ (see §1.1 for definition)

$$\|\,u\|_{S_X^{2,p}(\Omega')} \leq c\Big\{\|\mathcal{L}u\|_{\mathcal{L}^p(\Omega)} + \|u\|_{\mathcal{L}^p(\Omega)}\Big\}.$$

The above result generalizes the $\mathcal{L}^p$-estimates proved by Rothschild-Stein [35], since our operators do not belong to the class considered there, whenever the a_{ij} are not C^∞ (in fact, they can be discontinuous). Moreover, we prove the following regularity result:

Theorem 0.2. *Under the same assumptions of Theorem 0.1, the following estimate holds:*

$$\|\,u\|_{S_X^{2+k,p}(\Omega')} \leq c\Big\{\|\mathcal{L}u\|_{S_X^{k,p}(\Omega)} + \|u\|_{\mathcal{L}^p(\Omega)}\Big\}$$

for every positive integer k such that $a_{ij} \in S_X^{k,\infty}(\Omega)$ and $\mathcal{L}u \in S_X^{k,p}(\Omega)$.

From the estimate of Theorem 0.1, we also deduce the local Hölder continuity for solutions to the equation $\mathcal{L}u = f$, when $f \in \mathcal{L}^p(\Omega)$ with p large enough, as well as an extension of this result to higher order derivatives:

Theorem 0.3. *Under the same assumptions of Theorem 0.1, if $f \in S_X^{2,p}(\Omega)$ for some $p \in (1, \infty)$ and $\mathcal{L}f \in \mathcal{L}^s(\Omega)$ for some $s > Q$ (Q is the "homogeneous dimension" which will be defined in §1.2), then f belongs to some Hölder space $\Lambda_X^\alpha(\Omega')$ (see §1.4 for definition) and*

$$\|f\|_{\Lambda_X^\alpha(\Omega')} \leq c\left\{\|\mathcal{L}f\|_{\mathcal{L}^r(\Omega)} + \|f\|_{\mathcal{L}^p(\Omega)}\right\}$$

for $r = max(p, s)$.

Theorem 0.4. *Under the same assumptions of Theorem 0.1, if* $a_{ij} \in S_X^{k,\infty}(\Omega)$, $f \in S_X^{2,p}(\Omega)$ *for some* $p \in (1, \infty)$ *and* $\mathcal{L}f \in S^{k,s}(\Omega)$ *for some positive integer* k, *some* $s > Q$, *then*

$$\|f\|_{\Lambda_X^{k,\alpha}(\Omega')} \leq c\left\{\|\mathcal{L}f\|_{S^{k,r}(\Omega)} + \|f\|_{\mathcal{L}^p(\Omega)}\right\}$$

for $r = max(p, s)$.

The line of the proof of Theorem 0.1 follows, as close as possible, that of the analog result in [4], which, in turn, was inspired by [11], [5], [8]. To explain the main ideas and techniques which are involved, it is worthwhile, as a starting point, to summarize the steps which form the original idea of Chiarenza-Frasca-Longo [11], in the case of a nonvariational uniformly elliptic operator $\mathcal{L}$.

1. The coefficients a_{ij} of the operator $\mathcal{L}$ are frozen at some point, getting a constant coefficient operator $\mathcal{L}_0$, for which an explicit fundamental solution is known. Then any test function u can be represented as a convolution of this fundamental solution with $\mathcal{L}_0 u$.

2. From this representation formula another one follows, assigning the second order derivatives of any test function u as the sum of two kinds of objects: a singular integral operator applied to $\mathcal{L}u$, and the commutator of this singular integral with the multiplication for the coefficients a_{ij}, applied to the second order derivatives of u.

3. Next, one takes the $\mathcal{L}^p$ norm of each term in the above representation formula; singular integrals and their commutators can be estimated, respectively, by the classical Calderón-Zygmund theory, and Coifman-Rochberg-Weiss' theorem on the commutator of a CZ operator with a BMO function (see [14]). A standard technique of expansion in spherical harmonics is also employed, to reduce singular integrals "with variable kernel" to convolution-type singular integrals.

4. The assumption of VMO coefficients allows us "to make small" the BMO seminorm of the coefficients. Therefore, the norm of the commutators can be made small, and their contribution taken to the left-hand side. So we get a local $\mathcal{L}^p$ estimate on the derivatives of u, in terms of $\mathcal{L}u$, for any test function whose support is small enough. Passing from this result to the local estimate for any solution to the equation $\mathcal{L}u = f$ in a domain is now routine.

We now sketch how the above idea can be settled in our context.

1. We apply Rothschild-Stein's technique of "litfing and approximation" ([35]), as well as Folland's results for the homogeneous situation ([18]): the operator $\mathcal{L}$ is "lifted" to an operator $\tilde{\mathcal{L}}$ depending also on extra variables; the coefficients a_{ij} are then frozen; the resulting operator can be locally approximated by a left invariant homogeneous operator, for which a homogeneous fundamental solution exists. These facts allow us to write an "explicit" representation formula for $X_i X_j u$ in terms of singular intergals and commutators of singular integrals.

2. The theory of singular integrals and commutators of singular integrals on homogeneous spaces is then the natural context where $\mathcal{L}^p$ estimates can be carried out. We need the results of Coifman-Weiss [15], and the extension to homogeneous spaces of the commutator theorem for singular and fractional integrals: these results have been proved by Bramanti-Cerutti [6], [7] and Bramanti [3].

3. The expansion of the singular kernels in series of spherical harmonics can be repeated in our context, thanks to an estimate that we have proved in [4] on the derivatives of the fundamental solution corresponding to the frozen operator. Note that we need a uniform estimate for a family of fundamental solutions depending on a parameter in a possible discontinuous way.

4. The above steps lead to a local $\mathcal{L}^p$ estimate on the derivatives of u, in terms of $\tilde{\mathcal{L}}u$, for any test function whose support is small enough. In order to prove, from this result, the local estimate for any solution to the equation $\tilde{\mathcal{L}}u = f$ in a domain, we need some properties of the Sobolev spaces induced by the vector fields, which only in part have been already studied. Finally, from the estimates for the lifted operator $\tilde{\mathcal{L}}$, estimates for $\mathcal{L}$ follow immediately.

By the way, we note that the results of [37], [33] about subelliptic metrics, balls, etc., play a minor role in this paper: these are used only in §1.4, to formulate the *VMO* assumption on the coefficients in terms of the "natural" metric (instead of the metric which is "natural" for the proof).

To have a first idea of the proof, the reader who is already familiar with the paper of Rothschild-Stein [35] can give a glance to §§2.1–2.2 and then pass to §3.1, to see how the theory of singular integrals on homogeneous spaces is involved.

1. Preliminaries and known results

1.1. **Systems of vector fields satisfying Hörmander's condition. Sobolev spaces.** Let $X_1, \dots, X_q$ be C^∞ real vector fields on a domain $\Omega \subset \mathbb{R}^n$ $(q \le n)$. For every multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ with $1 \le \alpha_i \le q$, we define

$$X_\alpha = \left[X_{\alpha_d}, \left[X_{\alpha_{d-1}}, \dots [X_{\alpha_2}, X_{\alpha_1}] \dots\right]\right],$$

and $|\alpha| = d$. We call X_α a commutator of the X_i's of length d. We say that $X_1, \dots, X_q$ satisfy Hörmander's condition of step s at some point $x_0 \in \mathbb{R}^n$ if $\{X_\alpha(x_0)\}_{|\alpha| \le s}$ spans $\mathbb{R}^n$.

Let $\mathcal{G}(s,q)$ be the free Lie algebra of step s on q generators, that is, the quotient of the free Lie algebra with q generators by the ideal generated by the commutators of length at least $s+1$. We say that the vector fields $X_1, \dots, X_q$, which satisfy Hörmander's condition of step s at some point $x_0 \in \mathbb{R}^n$, are free up to order s at x_0 if $n = \dim \mathcal{G}(s,q)$, as a vector space (note that inequality $\le$ always holds).

Example 1.1. Consider the vector fields defining the Lie algebra of the Heisenberg group in $\mathbb{R}^3$:

$$X = \frac{\partial}{\partial x} + 2y\,\frac{\partial}{\partial t}; \quad Y = \frac{\partial}{\partial y} - 2x\,\frac{\partial}{\partial t}; \quad [X,Y] = -4\,\frac{\partial}{\partial t} = -4T$$

for $(x,y,t) \in \mathbb{R}^3$. With the above notation, we would write:

$$X_1 = X; \quad X_2 = Y; \quad X_{(2,1)} = -4T; \quad |1| = 1; \quad |2| = 1; \quad |(2{,}1)| = 2.$$

X_1, X_2 satisfy Hörmander's condition of step 2 at every point of $\mathbb{R}^3$, and they are free up to step 2, because $3 = \dim \mathcal{G}(2,2)$.

On the other hand, if we consider

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\,\frac{\partial}{\partial y},$$

for $(x,y) \in \mathbb{R}^2$, then X_1, X_2 satisfy Hörmander's condition of step 2 at every point of $\mathbb{R}^2$, but they are not free up to step 2, because $2 < \dim \mathcal{G}(2,2)$. □

Theorem 1.2 (Lifting Theorem, Rothschild-Stein [35], p. 272)**.** *Let* $X_1, \dots, X_q$ *be* C^∞ *real vector fields on a domain* $\Omega \subset \mathbb{R}^n$ *satisfying Hörmander's condition of step* s *at some point* $x_0 \in \Omega$. *Then in terms of new variables,* $t_{n+1}, \dots, t_N$, *there exist smooth functions* $\lambda_{il}(x,t)$ $(1 \le i \le q,\ n+1 \le l \le N)$ *defined in a neighborhood* $\tilde{U}$ *of* $\xi_0 = (x_0, 0) \in \Omega \times \mathbb{R}^{N-n} = \tilde{\Omega}$ *such that the vector fields* $\tilde{X}_i$ *given by*

$$\tilde{X}_i = X_i + \sum_{l=n+1}^{N} \lambda_{il}(x,t) \frac{\partial}{\partial t_l}, \qquad i = 1, \dots, q,$$

satisfy Hörmander's condition of step s *and are free up to step* s *at every point in* $\tilde{U}$.

In this paper we will use several kinds of Sobolev spaces:

Definition 1.3. If $Z = (Z_1, Z_2, \dots, Z_q)$ is any system of smooth vector fields in $\mathbb{R}^n$, we can define the Sobolev spaces $S_Z^{k,p}(\mathbb{R}^n)$ of $\mathcal{L}^p$-functions with k derivatives, with respect to the vector fields Z_i's, in $\mathcal{L}^p$ ($1 \le p \le \infty$, k positive integer). Explicitly,

$$\|f\|_{S_Z^{k,p}(\mathbb{R}^n)} = \|f\|_{\mathcal{L}^p(\mathbb{R}^n)} + \sum_{h=1}^{k} \sum_{j_i=1}^{q} \|Z_{j_1} Z_{j_2} \dots Z_{j_h} f\|_{\mathcal{L}^p(\mathbb{R}^n)}.$$

Analogous definitions can be given for function spaces defined on a bounded domain Ω. Also, we can define as usual the spaces of functions vanishing at the boundary, $S_{0,Z}^{k,p}(\Omega)$.

In particular, we will denote by $S_X^{k,p}$, $S_{\tilde{X}}^{k,p}$ the Sobolev spaces generated, respectively, by the original vector fields $(X_1, \dots, X_q)$ and the lifted ones $\left(\tilde{X}_1, \dots, \tilde{X}_q\right)$.

1.2. **Homogeneous groups and left invariant vector fields.** If $e_1, \dots, e_q$ are generators of the free Lie algebra $\mathcal{G}(q,s)$ and

$$e_\alpha = \left[e_{\alpha_d}, \left[e_{\alpha_{d-1}}, \dots \left[e_{\alpha_2}, e_{\alpha_1}\right] \dots\right]\right],$$

then there exists a set A of multi-indices α so that $\{e_\alpha\}_{\alpha \in A}$ is a basis of $\mathcal{G}(q,s)$ as a vector space. This allows us to identify $\mathcal{G}(q,s)$ with $\mathbb{R}^N$. Note that $\operatorname{Card} A = N$ while $\max_{\alpha \in A} |\alpha| = s$, the step of the Lie algebra.

The Campbell-Hausdorff series defines a multiplication in $\mathbb{R}^N$ (see e.g. [37]):

$$\left(\sum_{\alpha \in A} u_\alpha e_\alpha\right) \circ \left(\sum_{\alpha \in A} v_\alpha e_\alpha\right) = \sum_{\alpha \in A} (u_\alpha + v_\alpha)\, e_\alpha + \frac{1}{2}\left[\sum_{\alpha \in A} u_\alpha e_\alpha, \sum_{\alpha \in A} v_\alpha e_\alpha\right] + \dots$$

that makes $\mathbb{R}^N$ the group $N(q,s)$, that is, the simply connected Lie group associated to $\mathcal{G}(q,s)$. We can naturally define dilations in $N(q,s)$ by

$$D(\lambda)\Big((u_\alpha)_{\alpha \in A}\Big) = \left(\lambda^{|\alpha|} u_\alpha\right)_{\alpha \in A}. \tag{1.1}$$

These are automorphisms of $N(q,s)$, which is therefore a homogeneous group, in the sense of Stein (see [38], pp. 618–622). We will call it G, when the numbers q, s are implicitly understood. (Actually, throughout the paper the numbers q, s are fixed once and for all.)

We can define in G a homogeneous norm $\|\cdot\|$ as follows. For any $u \in G$, $u \ne 0$, set

$$\|u\| = \rho \quad \Leftrightarrow \quad \left|D(\rho^{-1})u\right| = 1,$$

where $|\cdot|$ denotes the Euclidean norm; also, let $\|0\| = 0$. Then:

(1.2a) $\|D(\lambda)u\| = \lambda \|u\|$ for every $u \in G, \lambda > 0$;

(1.2b) the set $\{u \in G : \|u\| = 1\}$ coincides with the Euclidean unit sphere $\sum_N$;

(1.2c) the function $u \mapsto \|u\|$ is smooth outside the origin;

there exists $c(G) \geq 1$ such that for every $u, v \in G$

(1.2d) $\|u \circ v\| \leq c\,(\|u\| + \|v\|)$ and $\left\|u^{-1}\right\| \leq c\,\|u\|$;

(1.2e) $\frac{1}{c}\,|v| \leq \|v\| \leq c\,|v|^{1/s}$ if $\|v\| \leq 1$.

The above definition of norm is taken from [16]. This norm is equivalent to that defined in [38], but in addition satisfies (1.2.b), a property we shall use in the proof of Theorem 2.11 (to expand a kernel in series of spherical harmonics). The properties (1.2.a-b-c) are immediate while (1.2.d) is proved in [38], p.620 and (1.2.e) is Lemma 1.3 of [18].

In view of the above properties, it is natural to define the "quasidistance" d by

$$d(u,v) = \left\|v^{-1} \circ u\right\|.$$

For d the following hold:

$$d(u,v) \geq 0 \quad \text{and} \quad d(u,v) = 0 \text{ if and only if } u = v; \tag{1.3a}$$

$$\frac{1}{c}\,d(v,u) \leq d(u,v) \leq c\,d(v,u); \tag{1.3b}$$

$$d(u,v) \leq c\left(d(u,z) + d(z,v)\right) \tag{1.3c}$$

for every $u,\ v,\ z \in \mathbb{R}^N$ and some positive constant $c(G) \geq 1$. We also define the balls with respect to d as

$$B(u,r) \equiv B_r(u) \equiv \left\{v \in \mathbb{R}^N\colon\ d(u,v) < r\right\}.$$

Note that $B(0,r)=D(r)B(0,1)$. It can be proved (see [38], p. 619) that the Lebesgue measure in $\mathbb{R}^N$ is the Haar measure of G. Therefore, by (1.1),

$$|B(u,r)| = |B(0,1)|\ r^Q, \tag{1.4}$$

for every $u \in G$ and $r > 0$, where $Q = \sum_{\alpha \in A} |\alpha|$. We will call Q the homogeneous dimension of G.

Next, we define the convolution of two functions in G as

$$(f * g)(x) = \int_{\mathbb{R}^N} f(x \circ y^{-1})\, g(y)\, dy = \int_{\mathbb{R}^N} g(y^{-1} \circ x)\, f(y)\, dy,$$

for every couple of functions for which the above integrals make sense.

Let τ_u be the left translation operator acting on functions: $(\tau_u f)(v) = f(u \circ v)$. We say that a differential operator P on G is left invariant if $P(\tau_u f) = \tau_u(Pf)$ for every smooth function f. From the above definition of convolution we read that if P is any left invariant differential operator,

$$P(f * g) = f * Pg$$

(provided the integrals converge).

We say that a differential operator P on G is homogeneous of degree $\delta > 0$ if

$$P\left(f\left(D(\lambda)u\right)\right) = \lambda^{\delta}\,(Pf)(D(\lambda)u)$$

for every test function f, $\lambda > 0$, $u \in \mathbb{R}^N$. Also, we say that a function f is homogeneous of degree $\delta \in \mathbb{R}$ if

$$f\left(D(\lambda)u\right) = \lambda^{\delta}\, f\left(u\right) \quad \text{for every } \lambda > 0,\ u \in \mathbb{R}^N.$$

Clearly, if P is a differential operator homogeneous of degree δ_1 and f is a homogeneous function of degree δ_2, then Pf is a homogeneous function of degree $\delta_2 - \delta_1$, while fP is a differential operator, homogeneous of degree $\delta_1 - \delta_2$. For example, $u_\alpha \frac{\partial}{\partial u_\beta}$ is homogeneous of degree $|\beta| - |\alpha|$.

Denote by Y_j ($j = 1, \ldots, q$) the left-invariant vector field on G which agrees with $\frac{\partial}{\partial u_j}$ at 0. Then Y_j is homogeneous of degree 1 and, for every multi-index α, Y_α is homogeneous of degree $|\alpha|$.

1.3. **The Rothschild-Stein approximation theorem and the metric induced by the lifted fields.**

Definition 1.4. A differential operator on G is said to have local degree less than or equal to ℓ if, after taking the Taylor expansion at 0 of its coefficients, each term obtained is homogeneous of degree $\leq \ell$.

Remark 1.5. It is useful to make more explicit the above definition, in the case of vector fields, that is, first order differential operators. If $f(u)$ is a C^∞-function, not necessarily analytic, we can write, for every fixed integer K, its Taylor expansion for $u \to 0$:

$$f(u) = \sum_{k=0}^{K} \sum_{j=1}^{J_k} c_{kj}\, p_{kj}(u) + o\left(|u|^K\right),$$

where the error term $o\left(|u|^K\right)$ is a C^∞-function, the c_{kj}'s are constants and $\left\{p_{kj}\right\}_{j=1}^{J_k}$ denote all the monomials of degree k (in the usual sense). Now, any p_{kj} is also homogeneous of degree α_{kj}, with $k \leq \alpha_{kj} \leq ks$ (s being the step of the Lie algebra).

Therefore, for any integer K we can also rewrite the Taylor expansion of f with respect to the homogeneity, in the following way:

$$f(u) = \sum_{k=0}^{K} \sum_{j=1}^{J_k} \tilde{c}_{kj}\, \tilde{p}_{kj}(u) + o\left(\|u\|^K\right),$$

where the error term $o\left(\|u\|^K\right)$ is a C^∞-function, the $\tilde{c}_{kj}$'s are constants and the $\tilde{p}_{kj}$'s are homogeneous monomials of degree k. Now let

$$P = \sum_{\alpha \in A} c_\alpha(u)\, \frac{\partial}{\partial u_\alpha}$$

be a vector field of local degree $\leq \ell$: this means that we can write, for every integer K,

$$P = \sum_{\alpha \in A} \left(\sum_{k=0}^{K} \sum_{j=1}^{J_k} c_{\alpha kj}\, p_{kj}(u)\, \frac{\partial}{\partial u_\alpha} \right) + g(u)\, \frac{\partial}{\partial u_\alpha}$$

where $g(u) = o\left(\|u\|^K\right)$, $g \in C^\infty$, the $c_{\alpha kj}$'s are constants, the $p_{kj}(u)$'s are homogeneous monomials of degree k and every term $p_{kj}(u)\,\frac{\partial}{\partial u_\alpha}$ appearing in the expansion is a differential operator homogeneous of degree $\leq \ell$, i.e. $|\alpha| - k \leq \ell$. □

Let $\tilde{X}_1, \dots, \tilde{X}_q$ be the vector fields constructed in Theorem 1.2. Let $\{\tilde{X}_\alpha(\xi)\}_{\alpha\in A}$ be a basis for $\mathbb{R}^N$ for every $\xi \in \tilde{U}$. A remarkable result by Rothschild-Stein states that we can locally approximate these vector fields with the left invariant vector fields Y_i, defined at the end of §1.2.

For $\xi, \eta \in \tilde{U}$, define the map

$$\Theta_\xi(\eta) = (u_\alpha)_{\alpha\in A}$$

with

$$\eta = \exp\Big(\sum_{\alpha\in A} u_\alpha\,\tilde{X}_\alpha\Big)\,\xi.$$

We now state Rothschild-Stein's approximation theorem (see [35], p. 273). Our formulation is taken from [37], p. 146.

Theorem 1.6. *Let $\tilde{X}_1, \dots, \tilde{X}_q$ be as in Theorem 1.2. Then there exist open neighborhoods U of 0 and V, W of ξ_0 in $\mathbb{R}^N$, with $W \subset\subset V$ such that:*

(a) $\Theta_\xi \mid V$ is a diffeomorphism onto the image, for every $\xi \in V$;

(b) $\Theta_\xi(V) \supseteq U$ for every $\xi \in W$;

(c) $\Theta\colon V \times V \to \mathbb{R}^N$, defined by $\Theta(\xi,\eta) = \Theta_\xi(\eta)$ is $C^\infty(V\times V)$;

(d) In the coordinates given by Θ_ξ, we can write $\tilde{X}_i = Y_i + R_i^\xi$ on U, where R_i^ξ is a vector field of local degree ≤ 0 depending smoothly on $\xi \in W$. Explicitly, this means that for every $f \in C_0^\infty(G)$:

$$\tilde{X}_i\left(f(\Theta_\xi(\cdot))\right)(\eta) = \left(Y_i f + R_i^\xi f\right)(\Theta_\xi(\eta)). \tag{1.5}$$

(e) More generally, for every $\alpha \in A$ we can write

$$\tilde{X}_\alpha = Y_\alpha + R_\alpha^\xi$$

with R_α^ξ a vector field of local degree $\leq |\alpha| - 1$ depending smoothly on ξ.

Other important properties of the map Θ are stated in the next theorem (see Rothschild-Stein [35], pp. 284–287):

Theorem 1.7. *Let V be as in Theorem 1.6. For $\xi, \eta \in V$, define*

$$\rho(\xi,\eta) = \|\Theta(\xi,\eta)\|,$$

where $\|\cdot\|$ is the homogeneous norm defined above. Then:

(a) $\Theta(\xi,\eta) = \Theta(\eta,\xi)^{-1} = -\Theta(\eta,\xi)$;

(b) whenever $\rho(\xi,\eta)$ and $\rho(\xi,\zeta)$ are both ≤ 1,

$$\|\Theta(\xi,\eta) - \Theta(\zeta,\eta)\| \leq c\left(\rho(\xi,\zeta) + \rho(\xi,\zeta)^{1/s}\rho(\xi,\eta)^{1-1/s}\right) \tag{b1}$$

where s is the step of the Lie algebra;

$$\rho(\zeta,\eta) \leq c\left(\rho(\xi,\zeta) + \rho(\eta,\xi)\right). \tag{b2}$$

(c) Under the change of coordinates $u = \Theta_\xi(\eta)$, the measure element becomes:

$$d\eta = c(\xi)\cdot(1 + O(\|u\|))\,du,$$

where $c(\xi)$ is a smooth function, bounded and bounded away from zero in V. The same is true for the change of coordinates $u = \Theta_\eta(\xi)$.

Proof of (c). Observe that this point is not exactly the same stated in [35]: in fact, they choose a particular measure $d\eta$, in order to have $c \equiv 1$. For us $d\eta$ always denotes the Lebesgue measure.

We will compute the Jacobian determinant of the inverse mapping $\eta = \Theta_\xi^{-1}(u)$. To do this, set

$$\tilde{X}_\alpha = \sum_{k=1}^{N} c_{\alpha k}(\eta) \frac{\partial}{\partial \eta_k} \quad \text{for every } \alpha \in A$$

and rewrite the left-hand side of (1.5) as

$$\sum_k c_{\alpha k}(\eta) \sum_j \frac{\partial f}{\partial u_j} \left(\Theta_\xi(\eta)\right) \frac{\partial}{\partial \eta_k} \left[\left(\Theta_\xi(\eta)\right)_j\right].$$

Then (1.5), evaluated at $\eta = \xi$, becomes:

$$\sum_k c_{\alpha k}(\xi) \sum_j \frac{\partial f}{\partial u_j}(0) \frac{\partial}{\partial \eta_k} \left[\left(\Theta_\xi(\eta)\right)_j\right]_{\eta=\xi} = \left(Y_\alpha f + R_\alpha^\xi f\right)(0) = \left(Y_\alpha f\right)(0).$$

Since $|A| = N$, changing for a moment our notation we can assume that the multi-index α is actually an index ranging from 1 to N. Choosing $f(u) = u_h$ ($h = 1, \ldots, N$),

$$\sum_k c_{\alpha k}(\xi) \frac{\partial}{\partial \eta_k} \left[\left(\Theta_\xi(\eta)\right)_h\right]_{\eta=\xi} = Y_\alpha\left[u_h\right](0) = \delta_{\alpha h}$$

where the last equality follows recalling that

$$Y_\alpha[f](0) = \frac{d}{dt} f\left(\exp tY_\alpha\right)_{/t=0}$$

and that $\exp tY_\alpha$ equals, in local coordinates, $(0, \ldots, t, \ldots, 0)$ with t in the α-th position.

Defining the square matrix

$$C(\xi) = \left\{c_{\alpha k}(\xi)\right\}_{\alpha k}$$

and letting $J(\xi)$ be the Jacobian determinant of the mapping $u = (\Theta_\xi(\eta))$ at $\eta = \xi$, we get

$$\operatorname{Det}\left[C(\xi)\right] \cdot J(\xi) = 1.$$

Hence the Jacobian determinant of the mapping $\eta = \Theta_\xi^{-1}(u)$ at $u = 0$ equals $\operatorname{Det}[C(\xi)] \equiv c(\xi)$. Since the determinant of $\eta = \Theta_\xi^{-1}(u)$ is a smooth function in u, it equals

$$c(\xi) \cdot \left(1 + O\left(\|u\|\right)\right).$$

The analog result for the change of coordinates $u = \Theta_\eta(\xi)$ follows point (a) and the smoothness of the map $u \mapsto u^{-1}$ in G. □

We now need the following

Definition 1.8 (Homogeneous space, see Coifman-Weiss, [15])**.** Let S be a set and $d\colon S\times S\rightarrow[0,\infty)$. We say that d is a quasidistance if it satisfies properties (1.3a), (1.3b), (1.3c). The balls defined by d induce a topology in S; we assume that the balls are open sets, in this topology. Moreover, we assume there exists a regular Borel measure μ on X, such that the "doubling condition" is satisfied:

$$\mu\left(B_{2r}(x)\right)\leq c\cdot\mu\left(B_r(x)\right)$$

for every $r>0$, $x\in S$, some constant c. Then we say that (S,d,μ) is a homogeneous space.

Actually, the definition of homogeneous space in [15] requires d to be symmetric, and not only to satisfy (1.3b). However, the results about homogeneous spaces that we will use still hold under these more general assumptions.

Theorem 1.7 implies the following

Proposition 1.9. *(i) For every $\xi_0\in\tilde{\Omega}$, there exists a neighborhood $\tilde{S}=S\times I$ of ξ_0, where $S\subset\Omega$ and I is a box in $\mathbb{R}^{N-n}$ such that $\tilde{S}$, with the quasidistance ρ and the Lebesgue measure, is a homogeneous space.*

(ii) The measure of a ρ-ball $B_r(\xi)$ is equivalent to r^Q, when r is small enough, uniformly in $\xi\in\tilde{S}$.

Note that $\left(\tilde{S},\rho,d\xi\right)$ is a bounded homogeneous space.

Proof. Theorem 1.7 (b2) implies that ρ, restricted to $\tilde{S}$, is actually a quasidistance. Theorem 1.7 (c) and (1.4) imply (ii). Therefore the Lebesgue measure is doubling, and $\left(\tilde{S},\rho,d\xi\right)$ is a homogeneous space. □

Remark 1.10. Since all our results are local, we shall mainly work with homogeneous spaces like $\left(\tilde{S},\rho,d\xi\right)$. We point out that, in what follows, it will be necessary to shrink the neighborhood $\tilde{S}$ in order for some results to hold. For simplicity, we shall keep the same notation $\tilde{S}$ to denote possibly different sets.

1.4. **Subelliptic metrics, *VMO* spaces and Hölder spaces.** The spaces *BMO* and *VMO* can be defined in any homogeneous space (S,d,μ), as in the Euclidean case. For any locally integrable function f, any $r>0$, set

$$\eta_f(r)=\sup_{t\leq r}\sup_{B_t}\frac{1}{\mu(B_t)}\int_{B_t}\left|f(x)-f_{B_t}\right|d\mu(x),$$

where the inner sup is taken over all the "balls" B_t of radius t, and $f_B=\fint_B f=\mu(B)^{-1}\int_B f$. We say that $f\in$*BMO* if η_f is bounded, and set

$$\|f\|_*=\sup_r\eta_f(r).$$

We say that $f\in$*VMO* if $f\in$*BMO* and $\eta_f(r)\rightarrow0$ as $r\rightarrow0$. The function η_f will be called "*VMO* modulus of f".

By the way we note that, in view of John-Nirenberg's Theorem, proved in homogeneous spaces by Burger (see [9]), we can equivalently define *BMO* and *VMO* spaces replacing

$$\fint_B\left|f(x)-f_B\right|\,d\mu(x)\quad\text{with}\quad\fint_B\left|f(x)-f_B\right|^2d\mu(x).$$

This has the advantage that we can write

$$\fint_B |f(x) - f_B|^2 \, d\mu(x) \le \fint_B |f(x) - c|^2 \, d\mu(x) \quad \text{for every } c \in \mathbb{R}. \tag{1.6}$$

In the proof of Theorem 0.1 we will mainly work with the lifted vector fields $\tilde{X}_i$. As a consequence, the constant in the final $\mathcal{L}^p$-estimate will depend on the coefficients $a_{ij}(x)$ through the "ellipticity constant" μ and the *VMO* moduli of the "lifted coefficients" $\tilde{a}_{ij}(x,t) = a_{ij}(x)$, with respect to the structure of homogeneous space that we have in $\mathbb{R}^N$, the space of the lifted variables. Since our final goal is to study functions and differential operators defined on $\mathbb{R}^n$, it is desirable to express also the *VMO* assumption on the coefficients in terms of the structure induced in $\mathbb{R}^n$ by the original fields X_i. This is possible in view of the results of [37], [33], who have studied the relationship between the geometry of balls defined by the X_i's in $\mathbb{R}^n$ and the $\tilde{X}_i$'s in $\mathbb{R}^N$. We now recall some known results on this subject.

First of all, to every system of Hörmander's vector fields $X = (X_1, X_2, \dots, X_q)$, we can naturally associate a subelliptic metric d_X (see [17], [32], [37]), in the following way.

Definition 1.11. Let Ω be a bounded domain in $\mathbb{R}^n$. We say that an absolutely continuous curve $\gamma : [0,T] \to \Omega$ is a sub-unit curve with respect to the system of smooth vector fields $X_1, X_2, \dots, X_q$ if for any $\xi \in \mathbb{R}^n$

$$\langle \gamma'(t), \xi \rangle^2 \le \sum_{j=1}^{q} \langle X_j(\gamma(t)), \xi \rangle^2$$

for a.e. $t \in [0,T]$. For any $x_1, x_2 \in \Omega$, we define

$$d_X(x_1,x_2) = \inf \{ T : \exists \text{ a sub-unit curve } \gamma : [0,T] \to \Omega, \gamma(0) = x_1, \gamma(T) = x_2 \}.$$

It can be proved that, if the vector fields satisfy Hörmander's condition, the above set is nonempty, so that $d_X(x_1,x_2)$ is finite for every pair of points. Moreover, d_X is a distance in Ω, and it is continuous with respect to the usual topology of $\mathbb{R}^n$. We will call d_X the subelliptic distance (associated to X).

So we have a metric d_X in $\mathbb{R}^n$, induced by the original vector fields X_i, and a metric $d_{\tilde{X}}$ in $\mathbb{R}^N$, induced by the lifted vector fields $\tilde{X}_i$. This last metric is in fact equivalent to the Rothschild-Stein quasidistance ρ we have introduced in Theorem 1.7. Namely, the following holds:

Lemma 1.12 (Sanchez-Calle, [37], Lemma 7, p. 153)**.** *With the above notation,*

$$c\rho(\xi,\eta) \le d_{\tilde{X}}(\xi,\eta) \le C\rho(\xi,\eta) \quad \textit{for every } \xi, \eta \in \tilde{S}.$$

The important property proved for these subelliptic metrics is that, whenever X is a system of Hörmander's vector fields, d_X is locally doubling, with respect to the Lebesgue measure (see [17], or Lemma 5, p. 151 in [37]). In particular, if S is any bounded domain of $\mathbb{R}^n$, (S, d_X, dx) is a homogeneous space, and we want to study the relationship between the *VMO* space defined over this homogeneous space, and the *VMO* space that we have already introduced. Now, quoting Sanchez-Calle,

> "The problem is that there is no clear relation between d_X and $d_{\tilde{X}}$. In particular, a ball for $d_{\tilde{X}}$ does not look, in general, like the product of a ball for d_X and some ball in $(N-n)$-space. However, and this is the key point, the volume of a ball for $d_{\tilde{X}}$ is, essentially,

the volume of the ball of the same radius for d_X, times the volume of some ball in $(N-n)$-space".
(See [37], p. 144.)

Theorem 1.13 (see Sanchez-Calle [37], Theorem 4, p. 151 and Lemma 7, p. 153). *Let X_i $(i=1,\dots,q)$ be smooth Hörmander's vector fields in $\mathbb{R}^n$, and $\tilde{X}_i$ be their lifted vector fields in $\mathbb{R}^N$. Then there exist constants $r_0, c, C > 0$ such that*

$$\begin{aligned} &\frac{1}{C} vol\left(B_{\tilde{X}}\left((x,h),r\right)\right) \\ &\quad \leq vol\left(B_X\left(x,r\right)\right) \cdot vol\left\{h' \in \mathbb{R}^{N-n} : (z,h') \in B_{\tilde{X}}\left((x,h),r\right)\right\} \\ &\quad \leq C\, vol\left(B_{\tilde{X}}\left((x,h),r\right)\right) \end{aligned} \tag{1.7}$$

for every $z \in B_X(x,cr)$ and $r \leq cr_0$. (Here "vol" stands for the Lebesgue measure in the appropriate dimension, x denotes a point in $\mathbb{R}^n$ and h a point in $\mathbb{R}^{N-n}$.) Moreover:

$$d_{\tilde{X}}\left((x,h),(x',h')\right) \geq d_X\left(x,x'\right). \tag{1.8}$$

We are now ready to prove our result:

Theorem 1.14. *The function $f(x)$ belongs to the space BMO (or VMO) over the homogeneous space (S, d_X, dx) if and only if the function $\tilde{f}(\xi) = \tilde{f}(x,t) = f(x)$ belongs to the space BMO (or VMO, respectively) over the homogeneous space $\left(\tilde{S}, \rho, d\xi\right)$.*

Proof. By (1.6) we have (with c a constant to be chosen later):

$$⨍_{B_{\tilde{X}}((x,t),r)} |\tilde{f}(y,s) - \tilde{f}_{B_{\tilde{X}}((x,t),r)}|^2 dy\, ds \leq ⨍_{B_{\tilde{X}}((x,t),r)} |f(y)-c|^2 dy\, ds$$

(by (1.8))

$$\begin{aligned} &\leq \frac{1}{\mathrm{vol}\left(B_{\tilde{X}}\left((x,t),r\right)\right)} \int_{B_X(x,r)} \int_{\left\{s \in \mathbb{R}^{N-n} : (y,s) \in B_{\tilde{X}}((x,t),r)\right\}} \left|f(y)-c\right|^2 dy\, ds \\ &= \frac{1}{\mathrm{vol}\left(B_{\tilde{X}}\left((x,t),r\right)\right)} \int_{B_X(x,r)} \mathrm{vol}\Big\{s : (y,s) \in B_{\tilde{X}}\left((x,t),r\right)\Big\} \left|f(y)-c\right|^2 dy \end{aligned}$$

(by (1.7), assuming $r \leq cr_0^2$)

$$\leq C ⨍_{B_X(x,r)} \left|f(y)-c\right|^2 dy = C ⨍_{B_X(x,r)} \left|f(y) - f_{B_X(x,r)}\right|^2 dy$$

where we have finally chosen $c = f_{B_X(x,r)}$.

The above inequality shows that if $f(x)$ belongs to *BMO* (or *VMO*) with respect to the homogeneous space (S, d_X, dx), then $\tilde{f}(\xi)$ belongs to *BMO* (or *VMO*, respectively) with respect to the homogeneous space $\left(\tilde{S}, d_{\tilde{X}}, d\xi\right)$. The converse can be proved analogously. Finally, by Lemma 1.12 and the doubling property of the Lebesgue measure, we can see that the *BMO* and *VMO* spaces defined over $\left(\tilde{S}, d_{\tilde{X}}, d\xi\right)$ and $\left(\tilde{S}, \rho, d\xi\right)$ coincide. So the theorem is proved. □

The notion of subelliptic metric allows us to define Hölder spaces in a natural way:

Definition 1.15 (Hölder spaces)**.** If $Z = (Z_1, Z_2, \dots, Z_q)$ is any system of smooth Hörmander's vector fields in $\mathbb{R}^n$, d_Z is the subelliptic metric induced by Z (see §1.4), and Ω is a bounded domain in $\mathbb{R}^n$, we define the Hölder spaces $\Lambda_Z^{k,\alpha}(\Omega)$, for $\alpha \in (0,1)$, k nonnegative integer, setting

$$|f|_{\Lambda^\alpha(\Omega)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x)-f(y)|}{d_Z(x,y)^\alpha}$$

and

$$\|f\|_{\Lambda_Z^{k,\alpha}(\Omega)} = \sum_{j_i=1}^{q} |Z_{j_1}Z_{j_2}\dots Z_{j_k}f|_{\Lambda^\alpha(\Omega)} + \sum_{h=0}^{k}\sum_{j_i=1}^{q} \|Z_{j_1}Z_{j_2}\dots Z_{j_h}f\|_{\mathcal{L}^\infty(\Omega)}.$$

In particular, we will denote by $\Lambda_X^{k,\alpha}$, $\Lambda_{\tilde{X}}^{k,\alpha}$, the Hölder spaces generated, respectively, by the original vector fields $(X_1, \dots, X_q)$ and the lifted ones $\left(\tilde{X}_1, \dots, \tilde{X}_q\right)$.

2. Differential operators and singular integrals

2.1. **Differential operators and fundamental solutions.** We now define various differential operators that we will handle in the following. First of all, our main interest is to study

$$\mathcal{L} = \sum_{i,j=1}^{q} a_{ij}(x)X_iX_j,$$

where the X_i's are smooth real vector fields satisfying Hörmander's condition of step s in some domain $\Omega \subset \mathbb{R}^n$, and the matrix $a_{ij}(x)$ satisfies (0.5). Applying Theorem 1.2 to these fields, we obtain the vector fields $\tilde{X}_i$ which are free up to order s and satisfy Hörmander's condition of step s, in some domain $\tilde{\Omega} \subset \mathbb{R}^N$. For $\xi = (x,t) \in \tilde{\Omega}$, set $\tilde{a}_{ij}(x,t) = a_{ij}(x)$ and let

$$\tilde{\mathcal{L}} = \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi)\tilde{X}_i\tilde{X}_j$$

be the lifted operator, defined in $\tilde{\Omega}$. Next, we freeze $\tilde{\mathcal{L}}$ at some point $\xi_0 \in \tilde{\Omega}$, and consider the frozen lifted operator:

$$\tilde{\mathcal{L}}_0 = \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi_0)\tilde{X}_i\tilde{X}_j.$$

In view of Theorem 1.6, to study $\tilde{\mathcal{L}}_0$, we will consider the approximating operator, defined on the group G:

$$\mathcal{L}_0^* = \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi_0)Y_iY_j.$$

The following proposition, proved in [4], summarizes some of the properties of these operators.

Proposition 2.1. *Let X_i be smooth vector fields satisfying Hörmander's condition of step s in some domain $\Omega \subset \mathbb{R}^n$, and let the matrix $a_{ij}(x)$ be uniformly elliptic on $\mathbb{R}^q$. Then:*

(i) if the coefficients $a_{ij}(x)$ are $C^\infty(\Omega)$, then $\mathcal{L}$ can be rewritten in the form

$$\mathcal{L} = \sum_{k=1}^{q} Z_k^2 \ + \ Z_0$$

where the vector fields Z_k are linear functions of the X_i's and their commutators of length 2. Therefore the Z_k's satisfy Hörmander's condition of step $s+1$, $\mathcal{L}$ is hypoelliptic in Ω and the local $\mathcal{L}^p$ estimates for $\mathcal{L}$ follow from [35].

(ii) For every $\xi_0 \in \tilde{\Omega}$, the operator $\mathcal{L}_0^*$ is hypoelliptic, left invariant and homogeneous of degree two with respect to the structure of homogeneous group G in $\mathbb{R}^N$; moreover, the transpose $\mathcal{L}^T$ of $\mathcal{L}$ is hypoelliptic, too.

We will apply to $\mathcal{L}_0^*$ the results contained in the following two theorems. The first is due to Folland (see Theorem 2.1 and Corollary 2.8 in [18]):

Theorem 2.2. *Existence of a homogeneous fundamental solution. Let $\mathcal{L}$ be a left invariant differential operator homogeneous of degree two on a homogeneous group G, such that $\mathcal{L}$ and $\mathcal{L}^T$ are both hypoelliptic. Moreover, assume $Q \geq 3$. Then there is a unique fundamental solution Γ such that:*
(a) $\Gamma \in C^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$;
(b) Γ *is homogeneous of degree* $(2-Q)$;
(c) for every distribution τ,

$$\mathcal{L}(\tau * \Gamma) = (\mathcal{L}\tau) * \Gamma = \tau.$$

Note that the only possibility in order for the condition $Q \geq 3$ not to hold is $Q = N = n = 2$; in this case the Lie algebra generated by X_1, X_2 is abelian, and this means that $\mathcal{L}$ is a uniformly elliptic operator in two variables, for which a complete $\mathcal{L}^p$ theory is known to hold, whenever the coefficients a_{ij} are bounded (see Talenti, [39]).

The second result is due to Folland-Stein (see Proposition 8.5 in [19], Proposition 1.8 in [18]):

Theorem 2.3. *Let K_h be a kernel which is $C^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$ and homogeneous of degree $(h-Q)$, for some integer h with $0 < h < Q$; let T_h be the operator*

$$T_h f = f * K_h$$

and let P^h be a left invariant differential operator homogeneous of degree h. Then

$$P^h T_h f = P.V.\Big(f * P^h K_h\Big) + \alpha f$$

for some constant α depending on P^h and K_h. The function $P^h K_h$ is $C^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$ and homogeneous of degree $-Q$. It also satisfies the vanishing property:

$$\int_{r<\|x\|<R} P^h K_h\ (x)dx = 0 \qquad \text{for } 0 < r < R < \infty.$$

The singular integral operator

$$f \mapsto P.V.\Big(f * P^h K_h\Big)$$

is continuous on $\mathcal{L}^p$ for $1 < p < \infty$.

By Proposition 2.1, the operator $\mathcal{L}_0^*$ satisfies the assumptions of Theorem 2.2; therefore, it has a fundamental solution with pole at the origin which is homogeneous of degree $(2-Q)$. Let us denote it by $\Gamma(\xi_0;\cdot)$, to indicate its dependence on the frozen coefficients $\tilde{a}_{ij}(\xi_0)$. Also, set for $i,j=1,\dots,q$,

$$\Gamma_{ij}(\xi_0;u) = Y_iY_j\ [\Gamma(\xi_0;\cdot)]\,(u).$$

The next theorem summarizes the properties of $\Gamma(\xi_0;\cdot)$ and $\Gamma_{ij}(\xi_0;\cdot)$ that we will need in the following. All of them follow from Theorems 2.2–2.3.

Theorem 2.4. *For every* $\xi_0 \in \mathbb{R}^N$*:*
(a) $\Gamma(\xi_0,\cdot) \in C^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$;
(b) $\Gamma(\xi_0,\cdot)$ *is homogeneous of degree* $(2-Q)$;
(c) for every test function f *and every* $v \in \mathbb{R}^N$,

$$f(v) = \left(\mathcal{L}_0^* f * \Gamma(\xi_0;\cdot)\right)(v) = \int_{\mathbb{R}^N} \Gamma\left(\xi_0;u^{-1}\circ v\right)\mathcal{L}_0^* f(u)\,du;$$

moreover, for every $i,j=1,\dots,q$, *there exist constants* $\alpha_{ij}(\xi_0)$ *such that*

$$Y_iY_j\,f(v) = P.V.\int_{\mathbb{R}^N} \Gamma_{ij}\left(\xi_0;u^{-1}\circ v\right)\mathcal{L}_0^* f(u)du + \alpha_{ij}(\xi_0)\cdot\mathcal{L}_0^* f(v); \tag{2.1}$$

(d) $\Gamma_{ij}(\xi_0;\cdot) \in C^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$;
(e) $\Gamma_{ij}(\xi_0;\cdot)$ *is homogeneous of degree* $-Q$;
(f)

$$\int_{r<\|u\|<R} \Gamma_{ij}(\xi_0;u)\,du = \int_{\|u\|=1} \Gamma_{ij}(\xi_0;u)\,d\sigma(u) = 0 \quad \textit{for every } R>r>0.$$

The above properties hold for any fixed ξ_0. We also need some uniform bound for Γ, with respect to ξ_0. The next theorem, proved by the authors in [4], contains this kind of result.

Theorem 2.5. *For every multi-index* β, *there exists a constant* $c_1 = c_1(\beta, G, \mu)$ *such that*

$$\sup_{\substack{\|u\|=1\\ \xi\in\mathbb{R}^N}} \left|\left(\frac{\partial}{\partial u}\right)^\beta \Gamma_{ij}(\xi;u)\right| \le c_1,$$

for any $i,j=1,\dots,q$. *Moreover, for the* α_{ij}*'s appearing in (2.1), the uniform bound*

$$\sup_{\xi\in\mathbb{R}^N} |\alpha_{ij}(\xi)| \le c_2$$

holds for some constant $c_2 = c_2(G,\mu)$.

2.2. **Operators of type ℓ.** We are going to define suitable kernels, adapted to the operator $\tilde{\mathcal{L}}_0$ that we have introduced in §2.1, which will induce singular and fractional integrals on the homogeneous space described in Proposition 1.9. Here we follow Rothschild-Stein [35], with slight changes to handle the frozen variable ξ_0. Let $\Gamma(\xi_0;\cdot)$ be as in §2.1, and let

$$k_0(\xi_0;\xi,\eta) = a(\xi)\Gamma(\xi_0;\Theta(\eta,\xi))b(\eta)$$

be the Rothschild-Stein-like parametrix (see [35], p.297) where a,b are cutoff functions fixed once and for all. Roughly speaking, a singular kernel is a function with the same properties as $\left[\tilde{X}_i\tilde{X}_j\, k_0(\xi_0;\cdot,\eta)\right](\xi)$. Let us compute these derivatives using the Approximation Theorem (Theorem 1.6) and the definition of differential operator of local degree $\leq \ell$, and taking into account the dependence on the frozen point ξ_0:

$$\tilde{X}_i\left[a(\cdot)\Gamma(\xi_0;\Theta\,(\eta,\cdot))\,b(\eta)\right](\xi)$$

$$=\tilde{X}_i a(\xi)\left[\Gamma(\xi_0;\Theta\,(\eta,\xi))b(\eta)\right]+a(\xi)b(\eta)\left[Y_i\Gamma(\xi_0;\cdot)\right](\Theta\,(\eta,\xi))$$

$$+a(\xi)b(\eta)\left[R_i^{\eta}\Gamma(\xi_0;\cdot)\right](\Theta\,(\eta,\xi))\,.$$

The third term can be expanded, for every positive integer K, (see Remark 1.5) as

$$a(\xi)b(\eta)\sum_{\alpha\in A}\left\{\sum_{k=0}^{K}\sum_{j=1}^{J_k}c_{i\alpha kj}\,(\eta)\left(p_{kj}(\cdot)\,\frac{\partial\Gamma(\xi_0;\cdot)}{\partial u_\alpha}\right)(\Theta\,(\eta,\xi))\right.$$

$$\left.+c_{i\alpha}(\eta)\left(g_{i\alpha}(\cdot)\,\frac{\partial\Gamma(\xi_0;\cdot)}{\partial u_\alpha}\right)(\Theta\,(\eta,\xi))\right\}$$

where $g_{i\alpha}(u)=o\left(\|u\|^K\right)$, $c_{i\alpha}(\eta)$, $c_{i\alpha kj}\,(\eta)\in C^\infty$, $p_{kj}(u)$ are homogeneous monomials of degree k and $k\geq|\alpha|$ for every k and α appearing in the sum.

The above computation motivates the following

Definition 2.6. We say that $k(\xi_0;\xi,\eta)$ is a frozen kernel of type ℓ, for some nonnegative integer ℓ, if for every positive integer m we can write

$$k(\xi_0;\xi,\eta)=\sum_{i=1}^{H_m}a_i(\xi)b_i(\eta)\left[D_i\Gamma(\xi_0;\cdot)\right](\Theta(\eta,\xi))+a_0(\xi)b_0(\eta)\left[D_0\Gamma(\xi_0;\cdot)\right](\Theta(\eta,\xi))$$

where a_i, b_i $(i=0,1,\dots,H_m)$ are test functions, D_i are differential operators such that: for $i=1,\dots,H_m$, D_i is homogeneous of degree $\leq 2-\ell$ (so that $D_i\Gamma(\xi_0;\cdot)$ is a homogeneous function of degree $\geq \ell-Q$), and D_0 is a differential operator such that $D_0\Gamma(\xi_0;\cdot)$ has m derivatives with respect to the vector fields Y_i $(i=1,\dots,q)$.

We say that $T(\xi_0)$ is a frozen operator of type $\ell\geq 1$ if $k(\xi_0;\xi,\eta)$ is a frozen kernel of type ℓ and

$$T(\xi_0)f(\xi)=\int k(\xi_0;\xi,\eta)\,f(\eta)\,d\eta;$$

we say that $T(\xi_0)$ is a frozen operator of type 0 if $k(\xi_0;\xi,\eta)$ is a frozen kernel of type 0 and

$$T(\xi_0)f(\xi)=P.V.\int k(\xi_0;\xi,\eta)\,f(\eta)\,d\eta+\alpha\,(\xi_0,\xi)\,f\,(\xi)\,,$$

where α is a bounded function. If $k(\xi_0;\xi,\eta)$ is a frozen kernel of type ℓ, we say that $k(\xi;\xi,\eta)$ is a variable kernel of type ℓ, and

$$Tf(\xi)=\int k(\xi;\xi,\eta)f(\eta)\,d\eta$$

is a variable operator of type ℓ (if $\ell=0$, the integral must be taken in principal value sense and the term $\alpha\,(\xi,\xi)\,f\,(\xi)$ must be added).

Remark 2.7. Note that, since $\mathcal{L}_0^*$ is selfadjoint, $\Gamma(\xi_0; u) = \Gamma(\xi_0; u^{-1})$. Recalling also that $\Theta(\xi, \eta) = \Theta(\eta, \xi)^{-1}$, we easily see that if $k(\xi_0; \xi, \eta)$ is a frozen kernel of type ℓ, then also $k(\xi_0; \eta, \xi)$ is a frozen kernel of type ℓ.

The computation carried out before Definition 2.6 essentially proves the following

Lemma 2.8. *If $k(\xi_0; \xi, \eta)$ is a frozen kernel of type $\ell \geq 1$, then $\left(\tilde{X}_i k\right)(\xi_0; \cdot, \eta)(\xi)$ is a frozen kernel of type $\ell - 1$.*

For instance, if $k_0(\xi_0; \xi, \eta) = a(\xi)\,\Gamma(\xi_0; \Theta\,(\eta,\xi))b(\eta)$, then k_0, $\tilde{X}_i k_0\,(\xi_0; \cdot, \eta)\,(\xi)$, $\tilde{X}_i \tilde{X}_j k_0\,(\xi_0; \cdot, \eta)\,(\xi)$ (for $i, j = 1, \ldots, q$) are, respectively, frozen kernels of type 2, 1, 0.

Moreover, $a(\xi)b(\eta)\,[R_i^\eta \Gamma(\xi_0; \cdot)]\,(\Theta\,(\eta, \xi))$ and $a(\xi)b(\eta)\,[Y_j R_i^\eta \Gamma(\xi_0; \cdot)]\,(\Theta\,(\eta, \xi))$ are, respectively, frozen kernels of type 2, 1. We point out explicitly these examples because we will use them several times in the following.

Lemma 2.9. *If $T(\xi_0)$ is a frozen operator of type $\ell \geq 1$, then $\tilde{X}_i T(\xi_0)$ is a frozen operator of type $\ell - 1$.*

Proof. For $\ell \geq 2$ this follows from the previous lemma. For $\ell = 1$ we have to show how the differentiation of the integral can be carried out.

In view of the result for $\ell \geq 2$ we can assume, without loss of generality, that the kernel of $T(\xi_0)$ is of the kind

$$k_1(\xi_0; \xi, \eta) = a\,(\xi)\, Y_j \Gamma\,(\xi_0; \Theta\,(\eta, \xi))\, b\,(\eta)\,.$$

We introduce a regularized version of k_1, say $k_{1,\varepsilon}$, defined replacing the fundamental solution Γ with its regularized version Γ^ε:

$$\Gamma^\varepsilon(\xi_0, u) = \frac{1}{(\|u\| + \epsilon)^{Q-2}}\;\Gamma\left(\xi_0, D(\|u\|^{-1})u\right).$$

Then, in distributional sense,

$$\begin{aligned}\tilde{X}_i \left[\int k_1(\xi_0; \xi, \eta)\, f(\eta)\, d\eta\right] &= \lim_{\varepsilon \to 0} \left[\int \tilde{X}_i(k_{1,\varepsilon}(\xi_0; \cdot, \eta))(\xi)\, f(\eta)\, d\eta\right] \\ &= \lim_{\varepsilon \to 0} \left[\int a(\xi) Y_i Y_j \Gamma^\varepsilon(\xi_0; \Theta(\eta, \xi)) b(\eta) f(\eta)\, d\eta + \ldots\right]\end{aligned}$$

where the dots stand for nonsingular terms; now changing variable and denoting by $J_\xi\,(u)$ the Jacobian determinant we get

$$\begin{aligned}&= \lim_{\varepsilon \to 0} \left[\int a\,(\xi)\, Y_i Y_j \Gamma^\varepsilon\,(\xi_0; u)\,(bf)\left(\Theta_\xi^{-1}\,(u)\right) J_\xi\,(u)\; du + \ldots\right] \\ &= \lim_{\varepsilon \to 0} \left[\int a\,(\xi)\, Y_j \Gamma^\varepsilon\,(\xi_0; u)\, Y_i^T \left((bf)\left(\Theta_\xi^{-1}\,(u)\right) J_\xi\,(u)\right) du + \ldots\right] \\ &= \left[\int a\,(\xi)\, Y_j \Gamma\,(\xi_0; u)\, Y_i^T \left((bf)\left(\Theta_\xi^{-1}\,(u)\right) J_\xi\,(u)\right) du + \ldots\right].\end{aligned}$$

Therefore

$$\tilde{X}_i \left[\int k_1\,(\xi_0; \xi, \eta)\; f\,(\eta)\; d\eta\right] = a\,(\xi)\left\langle Y_j \Gamma\,(\xi_0; \cdot)\,, Y_i^T \left((bf)\left(\Theta_\xi^{-1}\,(\cdot)\right) J_\xi\,(\cdot)\right)\right\rangle + \ldots\,.$$

The last expression is the distributional derivative Y_i of the homogeneous distribution $Y_j\Gamma(\xi_0;\cdot)$. By Theorem 2.3, this equals

$$P.V.\int a(\xi)\,Y_iY_j\Gamma(\xi_0;u)\,(bf)\left(\Theta_\xi^{-1}(u)\right)J_\xi(u)\,du+\alpha(\xi_0)\,(bf)(\xi)\,J_\xi(0)+\dots$$
$$=P.V.\int a(\xi)\,Y_iY_j\Gamma(\xi_0;\Theta(\xi,\eta))\,(bf)(\eta)\,d\eta+\alpha(\xi_0,\xi)\,f(\xi)+\dots$$

where the dots stand for a frozen operator of type one, applied to f, $\alpha(\xi_0)$ is bounded by Theorem 2.3, and therefore $\alpha(\xi_0,\xi)=\alpha(\xi_0)\,(bf)(\xi)\,J_\xi(0)$ is bounded in ξ,ξ_0. □

A deeper property relating frozen operators of type ℓ with differentiation by $\tilde{X}_i$ is contained in the following:

Theorem 2.10. *If $T(\xi_0)$ is a frozen operator of type $\ell\geq 0$, then*

$$\tilde{X}_iT(\xi_0)=\sum_{k=1}^{q}T_k(\xi_0)\,\tilde{X}_k+T_0(\xi_0)$$

for suitable frozen operators $T_k(\xi_0)$ of type ℓ ($k=0,1,\dots,q$).

The above property is analogous to property (14.8) in [35], and can be proved with the same computation.

The following sections of this part will be devoted to prove:

Theorem 2.11. *Let T be a variable operator of type 0. Then for every $p\in(1,\infty)$ there exists $c=c(p,T)$ such that:*

(i) $\|Tf\|_p\leq c\,\|f\|_p$ *for every $f\in\mathcal{L}^p$,*

(ii) $\|T(af)-a\cdot Tf\|_p\leq c\,\|a\|_*\,\|f\|_p$ *for every $f\in\mathcal{L}^p$, $a\in BMO$.*

(iii) Moreover, for every $a\in VMO$, every $\epsilon>0$, there exists $r=r(p,T,\eta_a,\epsilon)>0$ such that for every $\xi_0\in\mathbb{R}^N$ and every $f\in\mathcal{L}^p$ with sprt $f\subseteq B_r(\xi_0)$,

$$\|T(af)-a\cdot Tf\|_p\leq\epsilon\,\|f\|_p\,.$$

(Recall that η_a denotes the VMO modulus of a, defined in §1.4.)

Henceforth we will denote by $[T,a]$ the operator $f\mapsto T(af)-a\cdot Tf$, that is, the commutator of T with the operator of multiplication by a.

Point (iii) follows from (ii) by the definition of *VMO*, using a localization technique which was first employed in [11]. So, we will prove (i)–(ii). To do this we need to apply several results about integral operators on homogeneous spaces, which we recall here in a form which is suitable for our purposes.

2.3. **Some known results about singular integrals in homogeneous spaces.**

Theorem 2.12 (Singular integrals and their commutators)**.** *Assume (S,d,μ) is a bounded homogeneous space (see Definition 1.8). Let $k:S\times S\setminus\{x=y\}\to\mathbb{R}$ be a kernel satisfying:*

(i) the growth condition: there exists a constant c_1 such that, for every $x,y\in S$,

$$|k(x,y)|\leq\frac{c_1}{\mu\,(B\,(x,d(x,y)))};$$

(ii) the mean value inequality: there exist constants $c_2 > 0$, $\beta > 0$, $M > 1$ *such that for every* $x_0 \in S$, $r > 0$, $x \in B_r(x_0)$, $y \notin B_{Mr}(x_0)$,

$$|k(x_0, y) - k(x, y)| \; + |k(y, x_0) - k(y, x)| \leq \frac{c_2}{\mu\left(B\left(x_0, d(x_0, y)\right)\right)} \cdot \frac{d(x_0, x)^\beta}{d(x_0, y)^\beta};$$

(iii) the cancellation property: there exists $c_3 > 0$ *such that for every* r, R, $0 < r < R < \infty$, *a.e.* x

$$\left|\int_{r<d(x,y)<R} k(x,y)\, d\mu(y)\right| \; + \; \left|\int_{r<d(x,z)<R} k(z,x)\, d\mu(z)\right| \leq \; c_3. \tag{2.2}$$

(iv) the convergence condition: for a.e. $x \in S$ *there exists*

$$\lim_{\varepsilon \to 0} \int_{\varepsilon<d(x,y)<1} k(x,y)\, d\mu(y). \tag{2.3}$$

For $f \in \mathcal{L}^p$, $p \in (1, \infty)$, *set*

$$K_\epsilon f(x) = \int_{\epsilon<d(x,y)<1/\epsilon} k(x,y)\, f(y)\, d\mu(y).$$

Then $K_\epsilon f$ *converges (strongly) in* $\mathcal{L}^p$ *for* $\epsilon \to 0$ *to an operator* Kf *satisfying*

$$\|Kf\|_p \; \leq \; c\,\|f\|_p \quad \text{for every } f \in \mathcal{L}^p. \tag{2.4}$$

Moreover, for the operator K *the commutator estimate*

$$\|[K, a]\, f\|_p \; \leq \; c\,\|a\|_*\, \|f\|_p \tag{2.5}$$

holds for every $f \in \mathcal{L}^p$, $a \in$*BMO. Finally, the constant* c *in* (2.4)–(2.5) *has the following form:*

$$c(p, S, \beta, M) \cdot (c_1 + c_2 + c_3).$$

The estimate (2.5) has been proved in [6], and the estimate (2.4) is a consequence of results in [15] and [13] (see [6] for details).

Theorem 2.13 (Fractional integrals and their commutators)**.** *Assume* (S, d, μ) *is a homogeneous space, and let*

$$k_\alpha\,(x,y) = \mu\,\left(B\left(x, d(x,y)\right)\right)^{\alpha-1} \quad \text{for some } \alpha \in (0,1).$$

Set

$$I_\alpha f(x) = \int_{S\setminus\{x\}} k_\alpha\,(x,y)\; f(y)\, d\mu(y)$$

and, for any $a \in$*BMO, set*

$$T_\alpha^a f(x) = \int_{S\setminus\{x\}} k_\alpha\,(x,y)\; |a(x) - a(y)|\; f(y)\, d\mu(y).$$

Then for every $p \in (1, 1/\alpha)$, $1/q = 1/p - \alpha$ *we have*

$$\|I_\alpha f\|_q \leq c\,\|f\|_p\,, \tag{2.6}$$

$$\|T_\alpha^a f\|_q \leq c\,\|a\|_*\,\|f\|_p\,. \tag{2.7}$$

In particular,

$$\|[I_\alpha, a]\, f\|_q \; \leq \; c\,\|a\|_*\,\|f\|_p\,. \tag{2.8}$$

The constant c *in* (2.6)–(2.8) *has the form* $c = c(p, S, \alpha)$.

Estimates (2.6), (2.7) have been proved, respectively, in [23] and [7].

Theorem 2.14 (Commutators of operators with positive kernels, see [3]). *Assume (S,d,μ) is a homogeneous space and let $k : S \times S \setminus \{x = y\} \to \mathbb{R}$ be a positive kernel satisfying conditions (i)–(ii) of Theorem 2.12. Moreover, assume that the integral operator*

$$Tf(x) = \int_{S\setminus\{x\}} k(x,y)\, f(y)\, d\mu(y)$$

is well defined and continuous on $\mathcal{L}^p$ for every $p \in (1,\infty)$. For any $a \in$BMO, set

$$T^a f(x) = \int_{S\setminus\{x\}} k\,(x,y)\ |a(x) - a(y)|\, f(y)\, d\mu(y).$$

Then

$$\|T^a f\|_p \le c\, \|a\|_* \, \|f\|_p \,. \tag{2.9}$$

In particular,

$$\|[T,a]\, f\|_p \ \le\ c\, \|a\|_* \, \|f\|_p \,. \tag{2.10}$$

The constant c in (2.9)–(2.10) *has the following form:*

$$c(p,S,\beta,M)\cdot(c_1+c_2)$$

where the symbols have the same meaning as in Theorem 2.12.

Estimates (2.7) and (2.9) imply the following

Corollary 2.15 (Commutators of operators with equivalent kernels, see [3], [7])**.** *Under the same assumptions of Theorem 2.14, let $k'(x,y)$ be a positive measurable kernel such that*

$$k'(x,y) \le c_4 k(x,y),$$

and let T' be the integral operator with kernel k'. Then T' satisfies the commutator estimate (2.10) *with the constant c replaced by $c\cdot c_4$.*

Analogously, if $k'(x,y)$ is a positive measurable kernel such that

$$k'(x,y) \le c_5 k_\alpha(x,y),$$

with k_α as in Theorem 2.13, and T' is the integral operator with kernel k', then T' satisfies the commutator estimate (2.8) *with the constant c replaced by $c\cdot c_5$.*

2.4. **Proof of Theorem 2.11.**

First part of the proof. The multiplicative part of the variable operator T, that is, $f(\xi) \mapsto \alpha(\xi,\xi)\, f(\xi)$, obviously satisfies the required estimates. By Definition 2.6, if $k(\xi;\xi,\eta)$ is a variable kernel of type 0, we can split it into its singular and regular parts:

$$k(\xi;\xi,\eta) \equiv S(\xi;\xi,\eta) + R(\xi;\xi,\eta)$$
$$\equiv \left(\sum_{i=1}^{H} a_i(\xi) b_i(\eta)\ [D_i\Gamma(\xi_0;\cdot)]\ (\Theta(\eta,\xi))\ + a_0(\xi) b_0\,(\eta)\ [D_0\Gamma\,(\xi_0;\cdot)]\,(\Theta(\eta,\xi)) \right)_{/\xi_0=\xi}$$

where a_i, b_i $(i = 0,1,\dots,H)$ are test functions, D_i $(i = 1,\dots,H)$ are differential operators homogeneous of degree ≤ 2 (so that $D_i\Gamma(\xi_0;\cdot)$ is a homogeneous function of degree $\geq -Q$), and D_0 is a differential operator such that $D_0\Gamma(\xi_0;\cdot)$ is locally bounded. (Here "singular" and "regular" just mean "unbounded" and "bounded".)

Let us handle first the regular part. By Theorem 2.5,

$$|R(\xi;\xi,\eta)| \leq c\,|a_0(\xi)b_0\,(\eta)| \equiv c\cdot K(\xi,\eta).$$

The kernel $K(\xi,\eta)$ clearly maps $\mathcal{L}^p$ into $\mathcal{L}^p$ continuously, hence the same is true for R. As to the commutator estimate, by Corollary 2.15 it is enough to prove it for K. But this is trivial, because $[K,\alpha]\,f = |a_0|\,[1,\alpha]\,(|b_0|\,f)$ (where we denote by 1 the operator with kernel 1), and the kernel 1 obviously satisfies the assumptions of Theorem 2.14 (recall that our homogeneous space is bounded). □

To handle the singular part, we apply Calderón-Zygmund's technique of expansion in spherical harmonics (see [10]). Let

$$\Big\{O_{km}\Big\}_{\substack{m=0,1,2,\dots\\ k=1,\dots,g_m}}$$

be a complete orthonormal system of spherical harmonics in $\mathcal{L}^2(\Sigma_N)$; we denote by m the degree of the polynomial and by g_m is the dimension of the space of spherical harmonics of degree m in $\mathbb{R}^N$. It is known that

$$g_m \leq c(N)\cdot m^{N-2} \quad \text{for every } m=1,2,\dots. \tag{2.11}$$

For any fixed $\xi \in \mathbb{R}^N$, $u \in \Sigma_N$, we can write the expansion:

$$\Gamma(\xi;u) = \sum_{m=0}^{\infty}\ \sum_{k=1}^{g_m} c^{km}\,(\xi)\ O_{km}\,(u)\,. \tag{2.12}$$

If $u \in \mathbb{R}^N$, let $u' = D\left(\|u\|^{-1}\right)u$; recall that, by (1.2.b), $u' \in \Sigma_N$. By (2.12) and homogeneity of $\Gamma(\xi;\cdot)$ we have

$$\Gamma(\xi;u) = \sum_{m=0}^{\infty}\ \sum_{k=1}^{g_m} c^{km}\,(\xi)\ \frac{O_{km}\,(u')}{\|u\|^{Q-2}}\,.$$

Observe that, whenever $f \in C^\infty(\Sigma_N)$ and

$$b_{km} \;=\; \int_{\Sigma_N} f(u)\,O_{km}(u)\,d\sigma(u)$$

are the Fourier coefficients of f with respect to $\{O_{km}\}$, by known properties of spherical harmonics, for every positive integer h there exists c_h such that

$$|b_{km}| \;\leq\; c_h\cdot m^{-2h}\sup_{\substack{|\beta|=2r\\ u\in\Sigma_N}}\left|\left(\frac{\partial}{\partial u}\right)^{\beta} f(u)\right|. \tag{2.13}$$

In view of Theorem 2.5, we get from (2.13) the following bound on the coefficients $c^{km}(\xi)$ appearing in the expansion (2.12): for every positive integer r there exists a constant $c = c(r,G,\mu)$ such that

$$\sup_{\xi\in\mathbb{R}^N}\left|c^{km}\,(\xi)\right| \leq c(h,G,\mu)\cdot m^{-2h} \tag{2.14}$$

for every $m = 1,2,\dots$; $k = 1,\dots,g_m$.

Now, let

$$H_{km}(u) = \frac{O_{km}(u')}{\|u\|^{Q-2}}\ . \tag{2.15}$$

We will need the following:

Lemma 2.16. *For every multi-index β,*

$$\sup_{\Sigma_N}\left|\left(\frac{\partial}{\partial u}\right)^{\beta} H_{km}(u)\right| \leq c(\beta)\cdot \max_{|\alpha|\leq|\beta|}\sup_{\Sigma_N}\left|\left(\frac{\partial}{\partial u}\right)^{\alpha} O_{km}(u)\right|.$$

Proof. Recall that $u' \equiv P(u) \equiv D(1/\|u\|)\, u$, and $P(u)$ is smooth outside the origin. A direct computation gives the result for $|\beta| = 1$; iteration yields the general case. □

We will also use the following bounds for spherical harmonics:

$$\left|\left(\frac{\partial}{\partial u}\right)^{\beta} O_{km}(u)\right| \ \leq\ c(N)\cdot m^{\frac{N-2}{2}+|\beta|} \tag{2.16}$$

for $u \in \Sigma_N$, $k = 1, \dots, g_m$, $m = 1, 2, \dots$.

Now, if

$$K(\xi;\xi,\eta) = a(\xi)\,[D\Gamma(\xi;\cdot)]\,(\Theta(\eta,\xi))\ b(\eta)$$

is any term in the finite sum defining the singular part of a variable kernel of type 0, we can rewrite it as

$$\begin{aligned} K(\xi;\xi,\eta) &= \sum_{m=0}^{\infty}\ \sum_{k=1}^{g_m} c^{km}(\xi)\ \{a(\xi)\ [DH_{km}(\cdot)]\ (\Theta(\eta,\xi))\ b(\eta)\} \\ &\equiv \sum_{m=0}^{\infty}\ \sum_{k=1}^{g_m} c^{km}(\xi)\ K_{km}(\xi,\eta)\,, \end{aligned} \tag{2.17}$$

where $K_{km}(\xi,\eta)$ are constant kernels of type ≥ 0. More precisely, K_{km} will be called "constant kernel of type ℓ" if the differential operator D appearing in (2.17) is homogeneous of degree $2-\ell$, so that DH_{km} is homogeneous of degree $\ell - Q$. We can take $\ell < Q$; otherwise the term involving DH_{km} belongs to the "regular" part of the variable kernel of type 0.

We will now study in detail constant kernels of type ℓ. The basic result is contained in the following

Proposition 2.17. *Let W be a function defined on G, smooth outside the origin and homogeneous of degree $\ell - Q$, and let*

$$K(\xi,\eta) = a(\xi)W(\Theta(\eta,\xi))b(\eta),$$

where a, b are fixed test functions. Then

(i) K satisfies the growth condition: there exists a constant c such that

$$|K(\xi,\eta)| \leq c\cdot\sup_{\Sigma_N}|W(u)|\cdot\rho(\xi,\eta)^{\ell-Q}.$$

(ii) K satisfies the mean value inequality: there exist constants $c > 0$, $M > 1$ such that for every ξ_0, ξ, η with $\rho(\eta,\xi_0) \geq M\,\rho(\xi,\xi_0)$,

$$\begin{aligned} &|K(\xi_0,\eta) - K(\xi,\eta)| + |K(\eta,\xi_0) - K(\eta,\xi)| \\ &\leq c\left\{\sup_{\Sigma_N}|W| + \sup_{\Sigma_N}|\nabla W|\right\}\cdot\frac{\rho(\xi_0,\xi)^{\beta}}{\rho(\xi_0,\eta)^{\beta+Q-\ell}}\ , \end{aligned} \tag{2.18}$$

where $\beta = 1/s$ (s is as in Theorem 1.7).

(iii) If $\ell = 0$ and W satisfies the vanishing property

$$\int_{r<\|u\|<R} W(u)\,du = 0 \quad \text{for every } R > r > 0,$$

then K satisfies the cancellation property

$$\left|\int_{r<\rho(\xi,\eta)<R} K(\xi,\eta)\,d\eta\right| \leq c \cdot \sup_{\Sigma_N} |W| \cdot (R-r) \quad \text{for every } R > r > 0. \tag{2.19}$$

All the constants c in (i)-(ii)-(iii) are independent of the function W.

Proof. (i) is trivial, by the homogeneity of W. To prove (ii), we recall an analogous result proved in [4] (Proposition 2.10) in the homogeneous context:

$$|W(u) - W(u+v)| \leq c \cdot \sup_{\Sigma_N} |\nabla W| \cdot \frac{\|v\|}{\|u\|^{Q-\ell+1}} \qquad \text{if} \quad \|v\| \leq \frac{1}{M}\|u\|. \tag{2.20}$$

Let us write:

$$\begin{aligned} |K(\eta,\xi_0) - K(\eta,\xi)| \leq & |a(\eta)b(\xi)|\ |W(\Theta(\xi_0,\eta)) - W(\Theta(\xi,\eta))| \\ & + |W(\Theta(\xi_0,\eta))|\ |a(\eta)[b(\xi) - b(\xi_0)]| \\ \equiv & \mathrm{I} + \mathrm{II}. \end{aligned}$$

By (2.20),

$$\mathrm{I} \leq |a(\eta)b(\xi)|\ c \cdot \sup_{\Sigma_N} |\nabla W| \cdot \frac{\|\Theta(\xi_0,\eta) - \Theta(\xi,\eta)\|}{\|\Theta(\xi_0,\eta)\|^{Q-\ell+1}} \tag{2.21}$$

provided that

$$\|\Theta(\xi_0,\eta) - \Theta(\xi,\eta)\| \leq \frac{1}{M}\|\Theta(\xi_0,\eta)\|. \tag{2.22}$$

Let

$$\rho(\xi,\xi_0) \leq \epsilon\rho(\xi_0,\eta) \tag{2.23}$$

with ϵ to be chosen later. Then, using Theorem 1.7 (b1) and the fact that $\rho(\xi_0,\eta)$ and $\rho(\xi,\eta)$ are equivalent,

$$\begin{aligned} \|\Theta(\xi_0,\eta) - \Theta(\xi,\eta)\| & \leq c\left(\rho(\xi,\xi_0) + \rho(\xi,\xi_0)^{1/s}\rho(\xi,\eta)^{1-1/s}\right) \\ & \leq c\,\rho(\xi_0,\eta)\left(\varepsilon + \varepsilon^{1/s}\right) \leq \frac{1}{M}\rho(\xi_0,\eta) \end{aligned} \tag{2.24}$$

for a suitable choice of ε. Hence (2.23) implies (2.22) and therefore (2.21). Moreover, (2.21) and (2.24) imply that

$$\mathrm{I} \leq c \cdot \sup_{\Sigma_N} |\nabla W| \cdot \frac{\rho(\xi_0,\xi)^{1/s}}{\rho(\xi_0,\eta)^{1/s+Q-\ell}}. \tag{2.25}$$

Now,

$$\begin{aligned} \mathrm{II} \leq & c \sup_{\Sigma_N} |W| \frac{1}{\rho(\xi_0,\eta)^{Q-\ell}} \cdot |a(\eta)|\ |b(\xi) - b(\xi_0)| \\ \leq & c \sup_{\Sigma_N} |W| \frac{1}{\rho(\xi_0,\eta)^{Q-\ell}} \cdot |\xi - \xi_0| \\ \leq & c \sup_{\Sigma_N} |W| \frac{1}{\rho(\xi_0,\eta)^{Q-\ell}} \cdot \|\Theta(\xi_0,\eta) - \Theta(\xi,\eta)\| \end{aligned}$$

where the first inequality follows by homogeneity, the second because b is smooth, the third because Θ is a diffeomorphism. Reasoning as above, this implies

$$\mathrm{II} \le c \cdot \sup_{\Sigma_N} |W| \cdot \frac{\rho(\xi_0,\xi)^{1/s}}{\rho(\xi_0,\eta)^{1/s+Q-\ell-1}}. \tag{2.26}$$

From (2.25)–(2.26) we get that $|K(\eta,\xi_0) - K(\eta,\xi)|$ is bounded by the right-hand side of (2.18), whenever $\rho(\eta,\xi_0) \ge M\rho(\xi,\xi_0)$. To get the analogous bound for $|K(\xi_0,\eta) - K(\xi,\eta)|$, it's enough to apply the previous estimate to the function $\tilde{K}(\xi,\eta) = K(\eta,\xi) = a(\eta)\tilde{W}(\Theta(\eta,\xi))b(\xi)$, with $\tilde{W}(u) = W(u^{-1})$, noting that $\tilde{W}(u)$ is still homogeneous of degree $\ell - Q$ and smooth outside the origin. This completes the proof of (ii).

To prove (iii), let us write

$$\int_{r<\rho(\xi,\eta)<R} a(\xi)W(\Theta(\eta,\xi))b(\eta)\,d\eta$$

$$= a(\xi)\int_{r<\rho(\xi,\eta)<R} W(\Theta(\eta,\xi))\,[b(\eta)-b(\xi)]\;d\eta$$

$$+a(\xi)b(\xi)\int_{r<\rho(\xi,\eta)<R} W(\Theta(\eta,\xi))\,d\eta \equiv \mathrm{I}+\mathrm{II}.$$

By the change of variable $u = \Theta(\eta,\xi)$, Theorem 1.7 (c) gives

$$\mathrm{II} = a(\xi)b(\xi)c(\xi)\int_{r<\|u\|<R} W(u)\;(1+O(\|u\|))\;du$$

by the vanishing property of W

$$= a(\xi)\,b(\xi)\,c(\xi)\int_{r<\|u\|<R} W(u)\,O(\|u\|)\,du.$$

Then

$$|\mathrm{II}| \le c\cdot\int_{r<\|u\|<R} |W(u)|\,\|u\|\;du \le c\cdot\sup_{\Sigma_N}|W|\cdot\int_{r<\|u\|<R} \|u\|^{1-Q}\,du$$

$$\le c\cdot\sup_{\Sigma_N}|W|\cdot(R-r).$$

On the other hand,

$$|\mathrm{I}| \le c\cdot\sup_{\Sigma_N}|W|\cdot\int_{r<\rho(\xi,\eta)<R} |\xi-\eta|\;\rho(\xi,\eta)^{-Q}\;d\eta.$$

Since Θ is a diffeomorphism,

$$|\xi-\eta| \le c\rho(\xi,\eta),$$

and we can conclude the estimate as in case II. □

We are now ready for the

Second part of the proof of Theorem 2.11. Let us come back to expansion (2.17), which represents any term in the finite sum defining the singular part of a variable kernel of type 0 as a series of constant kernels of type ≥ 0:

$$S(\xi;\xi,\eta) = \sum_{m=0}^{\infty} \sum_{k=1}^{g_m} c^{km}(\xi) \{a(\xi) [DH_{km}(\cdot)] (\Theta(\eta,\xi)) b(\eta)\}$$
$$\equiv \sum_{m=0}^{\infty} \sum_{k=1}^{g_m} c^{km}(\xi) K_{km}(\xi,\eta).$$

Our next goal is to obtain $\mathcal{L}^p$ estimates for the constant operators of type ℓ:

$$T_{km} f(\xi) = \int K_{km}(\xi,\eta) f(\eta) d\eta$$

and their commutators, with some information on the dependence of the operator norms on the integers k, m. If K_{km} is a constant kernel of type $\ell \geq 0$,

$$|K_{km}(\xi,\eta)| \leq c\rho(\xi,\eta)^{\ell-Q} \cdot \sup_{\Sigma_N} |DH_{km}(\cdot)| \tag{2.27}$$

with D differential operator homogeneous of degree $(2-\ell)$. By Lemma 2.16 and (2.16), this implies

$$|K_{km}(\xi,\eta)| \leq c\rho(\xi,\eta)^{\ell-Q} \cdot c(\ell,N)\, m^{(N-2)/2+2-\ell} \leq c\rho(\xi,\eta)^{\ell-Q} \cdot m^{N/2+1}. \tag{2.28}$$

We now consider separately the cases $\ell > 0$, $\ell = 0$.

If $\ell > 0$, by (2.28), Theorem 2.13 and Corollary 2.15, the operator T_{km} satisfies estimates like (2.6) and (2.8) with constant $c \cdot m^{N/2+1}$, for $p \in (1, Q/\ell)$, $1/q = 1/p - \ell/Q$. Hence, since the space is bounded, the operator T_{km} satisfies estimates (2.4)–(2.5) for $p \in (1, Q/\ell)$. However, it is easy to see that also the transpose of T_{km} satisfies (2.4)–(2.5) for $p \in (1, Q/\ell)$; hence, by duality, T_{km} satisfies (2.4)–(2.5) also for $p \in (Q/(Q-\ell), \infty)$. Finally, interpolation gives the boundedness in the full range $(1, \infty)$.

If $\ell = 0$, by Proposition 2.17 (ii), Lemma 2.16 and (2.16), the kernel $K_{km}(\xi,\eta)$ satisfies mean value inequality (2.18) with constant $c \cdot m^{N/2+2}$. By Theorem 2.3, the function $DH_{km}(\cdot)$ satisfies the vanishing property required by *(iii)* of Proposition 2.17, hence the kernel $K_{km}(\xi,\eta)$ satisfies the cancellation property (2.19), with constant $c \cdot m^{N/2+1}$. Note that this implies (2.2), with constant $c \cdot m^{N/2+1}$ (recall that the space is bounded) as well as condition (2.3). Therefore we can apply Theorem 2.12 and conclude that the operator T_{km} satisfies estimates (2.4)–(2.5) for $p \in (1, \infty)$, with constant $c \cdot m^{N/2+2}$.

We now recall the bound (2.14) on the coefficients $c^{km}(\xi)$ of the expansion (2.17) and inequality (2.11). Putting together all these facts we get estimates (2.4)–(2.5), for every $p \in (1, \infty)$, for the operator with kernel $S(\xi;\xi,\eta)$. This concludes the proof of Theorem 2.11. □

3. Representation formulas, Sobolev spaces and local estimates

3.1. Representation formulas and local estimates in $\mathcal{L}^p$-spaces.

Theorem 3.1 (Parametrix for $\tilde{\mathcal{L}}_0$)**.** *For every test function a, every ξ_0, there exist two frozen operators of type two, $P(\xi_0)$, $P^*(\xi_0)$, and $2q^2$ frozen operators of type* *one,* $S_{ij}(\xi_0)$, $S^*_{ij}(\xi_0)$ $(i,j=1,\dots,q)$ *such that for every test function* f*,*

$$\tilde{\mathcal{L}}_0 P(\xi_0) f(\xi) = a(\xi) f(\xi) + \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi_0)\, S_{ij}(\xi_0) f(\xi);$$

$$P^*(\xi_0)\tilde{\mathcal{L}}_0 f(\xi) = a(\xi) f(\xi) + \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi_0)\, S^*_{ij}(\xi_0) f(\xi). \tag{3.1}$$

Proof. We start by fixing a test function b, such that $ab = a$, and defining:

$$P(\xi_0) f(\xi) = \frac{a(\xi)}{c(\xi)} \int \Gamma(\xi_0; \Theta(\eta,\xi))\, b(\eta)\, f(\eta)\, d\eta,$$

where $c(\xi)$ is, as in Theorem 1.7, (c), the Jacobian determinant of the mapping $\eta = \Theta_\xi^{-1}(u)$ at $u = 0$. $P(\xi_0)$ is a frozen operator of type 2. Let us compute $\tilde{\mathcal{L}}_0 P(\xi_0) f$. By Theorem 1.6, we can write, for every smooth function g,

$$\begin{aligned}&\tilde{\mathcal{L}}_0\ [g(\Theta(\eta,\cdot))]\,(\xi)\\ &\quad= (\mathcal{L}_0^* g)\,(\Theta(\eta,\xi)) + \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi_0) \left(\left(Y_i R_j^\eta + Y_j R_i^\eta + R_i^\eta R_j^\eta\right) g\right) (\Theta(\eta,\xi)).\end{aligned}$$

Since $\mathcal{L}_0^*\Gamma(\xi_0;\cdot) = \delta_0$, we can write, formally,

$$\begin{aligned}&\frac{a(\xi)}{c(\xi)}\,\tilde{\mathcal{L}}_0 \left(\int \Gamma(\xi_0;\Theta(\eta,\cdot))\, b(\eta)\, f(\eta)\, d\eta\right)(\xi)\\ &= \frac{a(\xi)}{c(\xi)} \int \delta_0(\Theta(\eta,\xi))\, b(\eta)\, f(\eta)\, d\eta\\ &\quad+ \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi_0)\, \frac{a(\xi)}{c(\xi)} \int \left(\left(Y_i R_j^\eta + Y_j R_i^\eta + R_i^\eta R_j^\eta\right)\Gamma(\xi_0;\cdot)\right)(\Theta(\eta,\xi))\, b(\eta)\, f(\eta)\, d\eta.\end{aligned} \tag{3.2}$$

With the change of variables $u = \Theta_\xi(\eta)$, the first integral becomes

$$a(\xi)\int \delta_0(u)\ (bf)\,(\Theta_\xi^{-1}(u))\ (1 + O\,(\|u\|))\ du = a(\xi) b(\xi) f(\xi),$$

since $\Theta(\eta,\xi) = 0$ if and only if $\xi = \eta$. Therefore

$$\frac{a(\xi)}{c(\xi)}\,\tilde{\mathcal{L}}_0 \left(\int \Gamma(\xi_0;\Theta(\eta,\cdot))\, b(\eta)\, f(\eta)\, d\eta\right)(\xi) = a(\xi) f(\xi) + \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi_0)\, S_{ij}(\xi_0) f(\xi),$$

where $S_{ij}(\xi_0)$ are frozen operators of type 1, as noted after Lemma 2.8.

To complete the computation of $\tilde{\mathcal{L}}_0 P(\xi_0) f$, we write:

$$\tilde{\mathcal{L}}_0 P(\xi_0) f\,(\xi) = \Big(\tilde{\mathcal{L}}_0\left(\frac{a}{c}\right)\Big)(\xi)\cdot\Big(\int \Gamma(\xi_0;\Theta(\eta,\xi))\,b(\eta)\,f(\eta)\,d\eta\Big)$$

$$+\sum_{i,j=1}^{q}[\tilde{a}_{ij}(\xi_0)+\tilde{a}_{ji}(\xi_0)]\Big(\tilde{X}_i\left(\frac{a}{c}\right)\Big)(\xi)\int\Big(\big(Y_j+R_j^{\eta}\big)\,\Gamma(\xi_0;\cdot)\Big)(\Theta(\eta,\xi))b(\eta)f(\eta)d\eta$$

$$+a(\xi)f(\xi)+\sum_{i,j=1}^{q}\tilde{a}_{ij}(\xi_0)\,S_{ij}(\xi_0)f(\xi)$$

$$= a(\xi)f(\xi)+\sum_{i,j=1}^{q}\tilde{a}_{ij}(\xi_0)\,S'_{ij}(\xi_0)f(\xi),$$

where $S'_{ij}(\xi_0)$ are still frozen operator of type 1.

Now we apply the above reasoning to the transpose of $\tilde{\mathcal{L}}_0$:

$$\tilde{\mathcal{L}}_0^T = \sum_{i,j=1}^{q}\tilde{a}_{ij}(\xi_0)\tilde{X}_i^T\tilde{X}_j^T.$$

Note that the approximation theorem can be transposed, writing:

$$\tilde{X}_j^T\ [f(\Theta(\eta,\cdot))]\,(\xi) = \Big(\Big(Y_j^T+R_j^{\eta\,T}\Big)\,f\Big)\,(\Theta(\eta,\xi)),$$

where $R_j^{\eta\,T}$ is still a differential operator of local degree ≤ 0. Recalling also that $\mathcal{L}_0^*$ is selfadjoint, we get:

$$\tilde{\mathcal{L}}_0^T P(\xi_0)f\,(\xi) = a(\xi)f(\xi)+\sum_{i,j=1}^{q}\tilde{a}_{ij}(\xi_0)\,S''_{ij}(\xi_0)f(\xi),$$

for suitable frozen operators of type 1. Finally, transposing the previous identity,

$$P^T(\xi_0)\,\tilde{\mathcal{L}}_0 f(\xi) = a(\xi)f(\xi)+\sum_{i,j=1}^{q}\tilde{a}_{ij}(\xi_0)\,S'''_{ij}(\xi_0)f(\xi),$$

where $S'''_{ij}(\xi_0)$ are suitable frozen operators of type 1. Note that

$$P^T(\xi_0)g(\xi) = b(\xi)\int\Gamma(\xi_0;\Theta(\xi,\eta))\,a(\eta)\,g(\eta)\,d\eta$$

is a frozen operator of type 2 (see Remark 2.7).

To justify the formal computation (3.2), we can reason as Folland-Stein do in the proof of Proposition 16.2 in [19]: let

$$\Gamma^{\varepsilon}(\xi_0,u) = \frac{1}{(\|u\|+\varepsilon)^{Q-2}}\ \Gamma\left(\xi_0, D(\|u\|^{-1})u\right)$$

be the regularized fundamental solution of $\mathcal{L}_0^*$, and define:

$$P^{\varepsilon}(\xi_0)f(\xi) = a(\xi)\int\Gamma^{\varepsilon}(\xi_0;\Theta(\eta,\xi))\,b(\eta)\,f(\eta)\,d\eta.$$

Then

$$\begin{aligned}&\|P^{\varepsilon}(\xi_0)f-P(\xi_0)f\|_{\infty}\\&\quad\le\|f\|_{\infty}\int|a(\xi)\left\{\Gamma^{\varepsilon}(\xi_0;\Theta(\eta,\xi))-\Gamma(\xi_0;\Theta(\eta,\xi))\right\}b(\eta)|\,d\eta\to 0\ \text{ for }\ \varepsilon\to 0.\end{aligned}$$

Therefore $P^{\varepsilon}(\xi_0)f\to P(\xi_0)f$ as a distribution, hence $\tilde{\mathcal{L}}_0 P^{\varepsilon}(\xi_0)f\to\tilde{\mathcal{L}}_0 P(\xi_0)f$. So we can rewrite the above proof with $P(\xi_0)$ replaced by $P^{\varepsilon}(\xi_0)$, and finally pass to the limit: then $\tilde{\mathcal{L}}_0$ can be taken under the integral sign, and we are done. □

Theorem 3.2. *Let $p\in(1,\infty)$. There exists r_0 such that for every test function f with sprtf $\subseteq B_{r_0}$,*

$$\|f\|_{S^{2,p}}\le c\left\{\left\|\tilde{\mathcal{L}}f\right\|_p+\|f\|_p\right\}.$$

Proof. Let us apply $\tilde{X}_k\tilde{X}_h$ to both sides of (3.1):

$$\tilde{X}_k\tilde{X}_h P^{*}(\xi_0)\tilde{\mathcal{L}}_0 f(\xi)=\tilde{X}_k\tilde{X}_h\ [a(\xi)f(\xi)]+\sum_{i,j=1}^{q}\tilde{a}_{ij}(\xi_0)\,\tilde{X}_k\tilde{X}_h S^{*}_{ij}(\xi_0)f(\xi).$$

By Lemma 2.9 and Theorem 2.10,

$$\tilde{X}_k\tilde{X}_h S^{*}_{ij}(\xi_0)f(\xi)=\sum_{l=1}^{q}T_l^{ij}(\xi_0)\tilde{X}_l f(\xi)+T_0^{ij}(\xi_0)f\,(\xi)$$

where $T_l^{ij}(\xi_0)$ $(l=0,1,\dots,q)$ are suitable frozen operators of type 0. Therefore,

$$\begin{aligned}&\tilde{X}_k\tilde{X}_h\ [a(\xi)f(\xi)]\\&\quad=T(\xi_0)\,\tilde{\mathcal{L}}_0 f(\xi)-\sum_{i,j=1}^{q}\tilde{a}_{ij}(\xi_0)\left(\sum_{l=1}^{q}T_l^{ij}(\xi_0)\tilde{X}_l f(\xi)+T_0^{ij}(\xi_0)f(\xi)\right),\end{aligned}\tag{3.3}$$

where $T(\xi_0)$ is a frozen operator of type 0. The above formula holds for every ξ, ξ_0 in a neighborhood of 0. Now let us write:

$$\tilde{\mathcal{L}}_0 f(\xi)=\tilde{\mathcal{L}}f(\xi)+(\tilde{\mathcal{L}}_0-\tilde{\mathcal{L}})f(\xi)=\tilde{\mathcal{L}}f(\xi)+\sum_{i,j=1}^{q}\left[\tilde{a}_{ij}(\xi_0)-\tilde{a}_{ij}(\xi)\right]\ \tilde{X}_i\tilde{X}_j f(\xi).$$

Then

$$T(\xi_0)\,\tilde{\mathcal{L}}_0 f(\xi)=T(\xi_0)\,\tilde{\mathcal{L}}f(\xi)+T(\xi_0)\left(\sum_{i,j=1}^{q}\left[\tilde{a}_{ij}(\xi_0)-\tilde{a}_{ij}(\cdot)\right]\ \tilde{X}_i\tilde{X}_j f\right)(\xi)\,.$$

Finally letting $\xi_0=\xi$, (3.3) becomes:

$$\begin{aligned}\tilde{X}_k\tilde{X}_h\,f(\xi)&=T\,\tilde{\mathcal{L}}f(\xi)-\sum_{i,j=1}^{q}\left[T\left(\tilde{a}_{ij}(\cdot)\tilde{X}_i\tilde{X}_j f\right)(\xi)-\tilde{a}_{ij}(\xi)\,T\left(\tilde{X}_i\tilde{X}_j f\right)(\xi)\right]\\&\quad-\sum_{i,j=1}^{q}\tilde{a}_{ij}(\xi)\left(\sum_{l=1}^{q}T_l^{ij}\,\tilde{X}_l f(\xi)+T_0^{ij}\,f\,(\xi)\right),\end{aligned}$$

for every test function f supported where $a \equiv 1$. Here T, T_l^{ij} are variable operators of type 0. Note that the term in brackets is the commutator of T with the multiplication by $\tilde{a}_{ij}$, applied to the function $\tilde{X}_i\tilde{X}_j f$. We now apply Theorem 2.11: for every $p \in (1,\infty)$, every fixed $\varepsilon > 0$, every test function f with support small enough (depending on ε), we can write

$$\left\|\tilde{X}_k\tilde{X}_h f\right\|_p \leq c\left\|\tilde{\mathcal{L}}f\right\|_p + \varepsilon\cdot\sum_{i,j=1}^{q}\left\|\tilde{X}_i\tilde{X}_j f\right\|_p + c\sum_{l=1}^{q}\left\|\tilde{X}_l f\right\|_p + c\left\|f\right\|_p,$$

that is,

$$\|f\|_{S^{2,p}} \leq c\left\{\left\|\tilde{\mathcal{L}}f\right\|_p + \|f\|_{S^{1,p}}\right\}. \tag{3.4}$$

Note that the constant c depends on the *VMO* moduli of $\tilde{a}_{ij}$, which are controlled, by Theorem 1.14, by the *VMO* moduli of a_{ij} (with respect to the subelliptic metric induced in $\mathbb{R}^n$ by the X_i's). Now, let us come back to (3.1) and take only one derivative $\tilde{X}_h$ to both sides. Reasoning as above we find:

$$\tilde{X}_h f(\xi) = T\,\tilde{\mathcal{L}}f(\xi)$$

$$+\sum_{i,j=1}^{q}\left[\tilde{a}_{ij}(\xi)\,T\left(\tilde{X}_i\tilde{X}_j f\right) - T\left(\tilde{a}_{ij}(\cdot)\tilde{X}_i\tilde{X}_j f\right)\right] - \sum_{i,j=1}^{q}\tilde{a}_{ij}(\xi)\,T^{ij}f(\xi),$$

where T is a variable operator of type 1, and T^{ij} are variable operators of type 0. Therefore:

$$\left\|\tilde{X}_h f\right\|_p \leq c\left\|\tilde{\mathcal{L}}f\right\|_p + \varepsilon\cdot\sum_{i,j=1}^{q}\left\|\tilde{X}_i\tilde{X}_j f\right\|_p + c\,\|f\|_p. \tag{3.5}$$

Substituting in (3.4) we get

$$\|f\|_{S^{2,p}} \leq c\left\{\left\|\tilde{\mathcal{L}}f\right\|_p + \|f\|_p\right\}.$$

□

The next step is to remove from Theorem 3.2 the restriction on the support of f. To this aim, we need three standard tools: suitable cutoff functions, approximation theorems and interpolation inequalities for Sobolev norms.

First we construct a suitable family of cutoff functions. Given two concentric ρ-balls B_{r_1}, B_{r_2} and a function $\varphi \in C_0^\infty(\mathbb{R}^N)$, we write $B_{r_1} \prec \varphi \prec B_{r_2}$ to say that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on B_{r_1} and $\mathrm{sprt}\varphi \subseteq B_{r_2}$.

Lemma 3.3 (Radial cutoff functions)**.** *For any $\sigma \in (0,1)$, $r > 0$, $\xi \in \mathbb{R}^N$ there exists $\varphi \in C_0^\infty\left(\mathbb{R}^N\right)$ with the following properties:*

$$B_{\sigma r}(\xi) \prec \varphi \prec B_{\sigma' r}(\xi) \qquad \text{with } \sigma' = (1+\sigma)/2;$$

$$\left|\tilde{X}_i\varphi\right| \leq \frac{c}{(1-\sigma)\,r}, \qquad \left|\tilde{X}_i\tilde{X}_j\varphi\right| \leq \frac{c}{\sigma(1-\sigma)^2 r^2} \qquad \text{for } i,j = 1,\dots,q.$$

Proof. Pick a function $f\colon[0,r)\to[0,1]$ satisfying:

$$f\equiv 1 \text{ in } [0,\sigma r),\ f\equiv 0 \text{ in } [\sigma' r,r),\ f\in C^{\infty}(0,r),$$

$$|f'|\leq\frac{c}{(1-\sigma)r}\ ,\ \ |f''|\leq\frac{c}{(1-\sigma)^2r^2}\ .$$

Setting $\varphi(\eta)=f(\|\Theta(\xi,\eta)\|)$, we can compute:

$$\tilde{X}_i\varphi(\eta)=f'(\|\Theta(\xi,\eta)\|)\tilde{X}_i(\|\Theta(\xi,\cdot)\|)(\eta);$$

$$\begin{aligned}\tilde{X}_i\tilde{X}_j\varphi(\eta)=&f''(\|\Theta(\xi,\eta)\|)\,\tilde{X}_i(\|\Theta(\xi,\cdot)\|)(\eta)\ \tilde{X}_j(\|\Theta(\xi,\cdot)\|)(\eta)\\&+f'(\|\Theta(\xi,\eta)\|)\,\tilde{X}_i\tilde{X}_j(\|\Theta(\xi,\cdot)\|)(\eta).\end{aligned}$$

Next, we use the approximation theorem:

$$\tilde{X}_i(\|\Theta(\xi,\cdot)\|)(\eta)=\left(\left(Y_i+R_i^{\xi}\right)(\|\cdot\|)\right)(\Theta(\xi,\eta)).$$

By homogeneity of the norm, $Y_i(\|u\|)$ is bounded and, since R_i^{ξ} has local degree ≤ 0, $R_i^{\xi}(\|u\|)$ is also uniformly bounded. Analogously,

$$\left|\tilde{X}_i\tilde{X}_j(\|\Theta(\xi,\cdot)\|)(\eta)\right|\leq c/\|\Theta(\xi,\eta)\|,\quad \text{for } \|\Theta(\xi,\eta)\| \text{ small enough.}$$

Since $f'(\|u\|)\neq 0$ for $\|u\|>\sigma r$, we get the result. □

Now let $Z=(Z_1,Z_2,\dots,Z_q)$ be any system of smooth vector fields in $\mathbb{R}^n$.We recall the following approximation result, proved by Franchi-Serapioni-Serra Cassano (see [21], Proposition 1.4), for the Sobolev space $S_Z^{1,p}$.

Theorem 3.4. *Let $1\leq p<\infty$, and let $J_\varepsilon(\xi)=\varepsilon^{-N}J(\varepsilon^{-1}|\xi|)$ be a spherically symmetric mollifier supported in the Euclidean ball $\left\{\xi\in\mathbb{R}^N:|\xi|<\varepsilon\right\}$, such that $\int_{\mathbb{R}^N}J(|\xi|)\,d\xi=1$. If $f\in S_Z^{1,p}(\Omega)$ and $\Omega'\subset\subset\Omega$, then*

$$\lim_{\varepsilon\to 0}\|J_\varepsilon * f-f\|_{S_Z^{1,p}(\Omega')}=0, \tag{3.6}$$

where $*$ denotes the standard Euclidean convolution. Moreover,

$$\lim_{\varepsilon\to 0}\|(Z_if)*J_\varepsilon-Z_i(f*J_\varepsilon)\|_{\mathcal{L}^p(\Omega')}=0,\quad \textit{for } i=1,\dots,q. \tag{3.7}$$

(Actually, the result proved in [21] is more general, since they consider weighted Sobolev spaces, and their vector fields are supposed to have Lipschitz coefficients.) In [21], (3.7) is a key point in the proof of (3.6). Since the vector fields Z_i are not assumed to be translation invariant, (3.7) tells us how to handle the commutator of this derivative with the convolution operator. The next result is an extension of (3.6) to second order derivatives.

Proposition 3.5. *With the notation of Theorem 3.4, if $f\in S_Z^{2,p}(\Omega)$ and $\Omega'\subset\subset\Omega$, then*

$$\lim_{\varepsilon\to 0}\|J_\varepsilon * f-f\|_{S_Z^{2,p}(\Omega')}=0.$$

Moreover, if $\varphi\in C_0^{\infty}(\Omega)$, then $f\varphi\in S_{0,Z}^{2,p}(\Omega)$.

Proof.

$$(Z_i Z_j f) * J_\varepsilon - Z_i Z_j (f * J_\varepsilon)$$
$$= \{(Z_i Z_j f) * J_\varepsilon - Z_i [(Z_j f) * J_\varepsilon]\} + \{Z_i [(Z_j f) * J_\varepsilon] - Z_i Z_j (f * J_\varepsilon)\} = \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon.$$

Applying (3.6) to $g = Z_j f$, we see that $\mathrm{I}_\varepsilon \to 0$ in $\mathcal{L}^p(\Omega')$ for $\varepsilon \to 0$. To handle II_ε, we write explicitly

$$Z_i = \sum_k b_{ik} \partial_{\xi_k},$$

so that

$$\begin{aligned}
\mathrm{II}_\varepsilon &= \sum_{k,h} \{b_{ik}\partial_{\xi_k} [(b_{jh}\partial_{\xi_h} f) * J_\varepsilon] - b_{ik}\partial_{\xi_k} [b_{jh}\partial_{\xi_h} (f * J_\varepsilon)]\} \\
&= \sum_{k,h} b_{ik}(\xi)\, \partial_{\xi_k} \left\{ \int [b_{jh}(\xi - \eta) - b_{jh}(\xi)]\, \partial_{\xi_h} f(\xi - \eta)\, J_\varepsilon(\eta)\, d\eta \right\} \\
&= \sum_{k,h} b_{ik}(\xi) \left\{ \int [\partial_{\xi_k} b_{jh}(\xi - \eta) - \partial_{\xi_k} b_{jh}(\xi)]\, \partial_{\xi_h} f(\xi - \eta)\, J_\varepsilon(\eta)\, d\eta \right. \\
&\quad \left. + \int [b_{jh}(\xi - \eta) - b_{jh}(\xi)]\, \partial^2_{\xi_k \xi_h} f(\xi - \eta)\, J_\varepsilon(\eta)\, d\eta \right\} = A_\varepsilon + B_\varepsilon.
\end{aligned}$$

Since $|\eta| < \varepsilon$, if $\xi \in \Omega'$, then $(\xi - \eta)$ belongs to a compact set, on which b_{jh} and $\partial_{\xi_k} b_{jh}$ are uniformly continuous. Hence for every fixed $\delta > 0$, if ε is small enough,

$$|A_\varepsilon| + |B_\varepsilon| \le c\delta \left\{ |\partial_{\xi_h} f| * J_\varepsilon + \left|\partial^2_{\xi_k \xi_h} f\right| * J_\varepsilon \right\},$$

and so

$$\|A_\varepsilon\|_p + \|B_\varepsilon\|_p \le c\delta \left\{ \|\partial_{\xi_h} f\|_p + \left\|\partial^2_{\xi_k \xi_h} f\right\|_p \right\}$$

where all the $\mathcal{L}^p$-norms are taken on Ω'. Therefore $\mathrm{II}_\varepsilon \to 0$ in $\mathcal{L}^p(\Omega')$ for $\varepsilon \to 0$, and we are done. □

Theorem 3.6 (Interpolation inequality for Sobolev norms)**.** *For every $R > 0$, $p \in (1, \infty)$, $\varepsilon > 0$, there exists a positive constant $c(\varepsilon)$ such that if $f \in C_0^\infty(B_R(0))$, then*

$$\left\|\tilde{X}_i f\right\|_p \le \varepsilon \|\Delta f\|_p + c(\varepsilon) \|f\|_p$$

for every $i = 1, \dots, q$, where $\Delta f \equiv \sum_{i=1}^q \tilde{X}_i^2 f$.

Proof. Let $\Delta^* \equiv \sum_{i=1}^q Y_i^2$ and let Γ be the fundamental solution of Δ^*, homogeneous of degree $(2 - Q)$. Write (3.1) with $\tilde{\mathcal{L}}_0$ replaced by Δ, a a test function equal to 1 on $B_R(0)$ and $f \in C_0^\infty(B_R(0))$. We get:

(3.8) $$f(\xi) = P_2 \Delta f(\xi) + S_1 f(\xi)$$

where P_2, S_1 are, respectively, constant operators of type 2, 1 (more precisely, they satisfy the definition of "frozen operators", with $\Gamma(\xi_0; \cdot)$ replaced by Γ). Applying $\tilde{X}_i$ to both sides of (3.8), we get

$$\tilde{X}_i f(\xi) = P_1 \Delta f(\xi) + S_0 f(\xi),$$

where P_1, S_0 are, respectively, a constant operator of type 1 and an operator of type 0 (in the same sense). Hence the result will follow if we prove that

$$\|P_1 \Delta f\|_p \leq \varepsilon \|\Delta f\|_p + c(\varepsilon) \|f\|_p .$$

Let $k(\xi,\eta)$ be the kernel of P_1, φ_ε be a cutoff function (as in Lemma 3.3) with $B_{\varepsilon/2}(\xi) \prec \varphi_\varepsilon \prec B_\varepsilon(\xi)$, and let us split:

$$P_1 \Delta f(\xi) = \int_{\rho(\xi,\eta)>\varepsilon/2} k(\xi,\eta) \left[1-\varphi_\varepsilon(\eta)\right] \Delta f(\eta)\, d\eta$$

$$+ \int_{\rho(\xi,\eta)\leq\varepsilon} k(\xi,\eta)\, \Delta f(\eta)\, \varphi_\varepsilon(\eta)\, d\eta = \mathrm{I} + \mathrm{II}.$$

Then,

$$\mathrm{I} = \int_{\rho(\xi,\eta)>\varepsilon/2} \Delta^T \left(\left[1-\varphi_\varepsilon(\cdot)\right] k(\xi,\cdot)\right)(\eta)\, f(\eta)\, d\eta,$$

so that

$$|\mathrm{I}| \leq c(\varepsilon) \int |f(\eta)|\, d\eta \leq c(\varepsilon) \|f\|_p .$$

To bound II, we use the following

Claim 3.7. Let

$$F_\varepsilon(\xi) = \int_{\rho(\xi,\eta)\leq\varepsilon} \frac{|g(\eta)|}{\rho(\xi,\eta)^{Q-1}}\, d\eta.$$

Then

$$F_\varepsilon(\xi) \leq c\varepsilon\, \mathcal{M}g(\xi)$$

where $\mathcal{M}g$ is the Hardy-Littlewood maximal function (in the homogeneous space defined in Proposition 1.9), and therefore (see [15], Theorem 2.1, p.71)

$$\|F_\varepsilon\|_p \leq c\varepsilon \|g\|_p \quad \text{for every } p \in (1,\infty).$$

Proof of the claim.

$$\begin{aligned} F_\varepsilon(\xi) &= \sum_{n=0}^{\infty} \int_{\frac{\varepsilon}{2^{n+1}} < \rho(\xi,\eta) \leq \frac{\varepsilon}{2^n}} \frac{|g(\eta)|}{\rho(\xi,\eta)^{Q-1}}\, d\eta \\ &\leq \sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{\varepsilon}\right)^{Q-1} \int_{B_{\varepsilon/2^n}(\xi)} |g(\eta)|\, d\eta \\ &\leq \sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{\varepsilon}\right)^{Q-1} c \left(\frac{\varepsilon}{2^n}\right)^{Q} \fint_{B_{\varepsilon/2^n}(\xi)} |g(\eta)|\, d\eta \leq c\varepsilon \mathcal{M}g(\xi). \end{aligned}$$

□

Now we apply the claim with $g = \Delta f$, assuming, as we can do, $|k(\xi,\eta)| \leq c\rho(\xi,\eta)^{1-Q}$, and we are done. □

We need a version of the above interpolation inequality, for functions not necessarily vanishing at the boundary of the domain:

Theorem 3.8. *For any $u \in S^{2,p}_{\tilde{X}}(B_r)$, $p \in [1,\infty)$, $r > 0$, define the following quantities:*

$$\Phi_k = \sup_{1/2<\sigma<1} \left((1-\sigma)^k r^k \left\| D^k u \right\|_{\mathcal{L}^p(B_{r\sigma})} \right) \quad \text{for } k = 0,1,2.$$

Then for any $\delta > 0$ (small enough)

$$\Phi_1 \le \delta\, \Phi_2 + c\,(\delta)\ \Phi_0.$$

Proof. The result follows from Lemma 3.3, Proposition 3.5 and Theorem 3.6, as in the proof of Theorem 4.5 in [4]. □

Theorem 3.9 (Local $\mathcal{L}^p$-estimates for $\tilde{\mathcal{L}} f = F$ in a domain)**.** *Let Ω be a bounded domain of $\mathbb{R}^N$ and $\Omega' \subset\subset \Omega$. If $f \in S^{2,p}_{\tilde{X}}(\Omega)$, then*

$$\| f \|_{S^{2,p}_{\tilde{X}}(\Omega')} \le c \left\{ \left\| \tilde{\mathcal{L}} f \right\|_{\mathcal{L}^p(\Omega)} + \|f\|_{\mathcal{L}^p(\Omega)} \right\}$$

where $c = c\,(p, X, \mu, \eta, \Omega, \Omega')$.

Proof. The result follows from Theorem 3.2, Lemma 3.3, Proposition 3.5 and Theorem 3.8, as in the proof of Theorem 1.5 in [4]. □

So far, we have proved local estimates for the lifted operator $\tilde{\mathcal{L}}$: the last step consists in coming back to our original operator $\mathcal{L}$. This finally leads to the

Proof of Theorem 0.1. Let Ω be a bounded domain in $\mathbb{R}^n$, $\Omega \subset\subset \Omega'$, $f \in S^{2,p}_{X}(\Omega)$. Let I be a (bounded) box in $\mathbb{R}^{N-n}$, $I' \subset\subset I$, $\tilde{\Omega} = \Omega \times I$, $\tilde{\Omega}' = \Omega' \times I'$, and, for $\xi = (x,t) \in \Omega \times I$, set

$$\tilde{f}(\xi) = \tilde{f}(x,t) = f(x).$$

Then

$$\tilde{X}_i \tilde{f}\,(x,t) = X_i f\,(x)\,; \quad \tilde{\mathcal{L}} \tilde{f}\,(x,t) = \mathcal{L} f(x)$$

and

$$\left\| \tilde{f} \right\|_{\mathcal{L}^p(\tilde{\Omega})} = \mid I \mid^{1/p} \|f\|_{\mathcal{L}^p(\Omega)}$$

where $|I|$ denotes the $(N-n)$ Lebesgue measure of I. Therefore, applying Theorem 3.9 to $\tilde{f}$ we can write:

$$\|f\|_{S^{2,p}_{X}(\Omega')} = c \left\| \tilde{f} \right\|_{S^{2,p}_{\tilde{X}}(\tilde{\Omega}')} \le c \left\{ \left\| \tilde{\mathcal{L}} \tilde{f} \right\|_{\mathcal{L}^p(\tilde{\Omega})} + \left\| \tilde{f} \right\|_{\mathcal{L}^p(\tilde{\Omega})} \right\}$$

$$= c \left\{ \|\mathcal{L} f\|_{\mathcal{L}^p(\Omega)} + \|f\|_{\mathcal{L}^p(\Omega)} \right\}.$$

□

Proof of Theorem 0.2. For the sake of simplicity, we will prove the theorem in the case $k = 1$. The same reasoning can be naturally iterated.

First step. We prove the analog of Theorem 3.2, for $\|f\|_{S^{3,p}}$.

We start by differentiating with respect to $\tilde{X}_i$ both sides of (3.1). Applying Theorem 2.10, we have:

$$\begin{aligned}\tilde{X}_h\,(af)\,(\xi) &= \tilde{X}_h\left(P^*(\xi_0)\tilde{\mathcal{L}}_0 f(\xi)\right) - \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi_0)\tilde{X}_h\left(S^*_{ij}(\xi_0)f\right)(\xi)\\ &= \sum_{k=1}^{q} P_k(\xi_0)\tilde{X}_k\tilde{\mathcal{L}}_0 f(\xi) + P_0(\xi_0)\,\tilde{\mathcal{L}}_0 f(\xi)\\ &\quad + \sum_{i,j,k=1}^{q} \tilde{a}_{ij}(\xi_0)\,S^k_{ij}(\xi_0)\tilde{X}_k f(\xi) + \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi_0)S^0_{ij}(\xi_0)f(\xi)\end{aligned}$$

where $P_k(\xi_0)$ $(k = 0, 1, \ldots, q)$ is a frozen operator of type 2, while $S^k_{ij}(\xi_0)$ $(k = 0, 1, \ldots, q)$ is a frozen operator of type 1. We now differentiate with respect to $\tilde{X}_m\tilde{X}_l$ both sides of the last identity. By Lemma 2.9, $\tilde{X}_m\tilde{X}_l P_k(\xi_0)$ $(k = 0, 1, \ldots, q)$ is a frozen operator of type 0. To compute $\tilde{X}_m\tilde{X}_l S^k_{ij}(\xi_0)$, we apply once Theorem 2.10, and then Lemma 2.9. After some computation, we obtain

$$\begin{aligned}\tilde{X}_m\tilde{X}_l\tilde{X}_h\,(af)\,(\xi) &= \sum_{k=1}^{q} T_k(\xi_0)\tilde{X}_k\tilde{\mathcal{L}}_0 f(\xi)\\ &+ \sum_{i,j,k,s=1}^{q} \tilde{a}_{ij}(\xi_0)T^{sk}_{ij}(\xi_0)\tilde{X}_s\tilde{X}_k f(\xi) + \sum_{i,j,k=1}^{q} \tilde{a}_{ij}(\xi_0)T^k_{ij}(\xi_0)\tilde{X}_k f(\xi)\\ &+ \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi_0)T_{ij}(\xi_0)f(\xi),\end{aligned} \tag{3.9}$$

where all the $T_k(\xi_0), T^{sk}_{ij}(\xi_0), T^k_{ij}(\xi_0), T_{ij}(\xi_0)$ are frozen operators of type 0.

Let us rewrite

$$\begin{aligned}\tilde{X}_k\tilde{\mathcal{L}}_0 f(\xi) &= \left(\tilde{X}_k\left(\tilde{\mathcal{L}}_0 - \tilde{\mathcal{L}}\right)f + \tilde{X}_k\tilde{\mathcal{L}}f\right)(\xi)\\ &= -\sum_{i,j=1}^{q} \tilde{X}_k\tilde{a}_{ij}(\xi)\,\tilde{X}_i\tilde{X}_j f(\xi) + \sum_{i,j=1}^{q} \left[\tilde{a}_{ij}(\xi_0) - \tilde{a}_{ij}(\xi)\right]\tilde{X}_k\tilde{X}_i\tilde{X}_j f(\xi) + \tilde{X}_k\tilde{\mathcal{L}}f(\xi).\end{aligned} \tag{3.10}$$

Substituting (3.10) in (3.9) and finally letting $\xi = \xi_0$, we get

$$\begin{aligned}\tilde{X}_m\tilde{X}_l\tilde{X}_h\,(af)\,(\xi) &= \sum_{k=1}^{q} T_k(\xi)\tilde{X}_k\tilde{\mathcal{L}}f(\xi) - \sum_{i,j,k=1}^{q} T_k(\xi)\left(\tilde{X}_k\tilde{a}_{ij}(\cdot)\,\tilde{X}_i\tilde{X}_j f\right)(\xi)\\ &+ \sum_{i,j=1}^{q} \left[\tilde{a}_{ij}, T_k\right]\tilde{X}_k\tilde{X}_i\tilde{X}_j f(\xi) + \sum_{i,j,k,s=1}^{q} \tilde{a}_{ij}(\xi)T^{sk}_{ij}\tilde{X}_s\tilde{X}_k f(\xi)\\ &+ \sum_{i,j,k=1}^{q} \tilde{a}_{ij}(\xi)T^k_{ij}\tilde{X}_k f(\xi) + \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi)T_{ij}f(\xi).\end{aligned} \tag{3.11}$$

Formula (3.11) holds for every test function f. Assuming the support of f small enough, we can take $\mathcal{L}^p$-norms of both sides and apply Theorem 2.11, obtaining

$$\left\|\tilde{X}_m \tilde{X}_l \tilde{X}_h f\right\|_p \leq c\left\{\left\|D\tilde{\mathcal{L}}f\right\|_p + \|D\,\tilde{a}_{ij}\|_\infty \left\|\tilde{X}_i \tilde{X}_j f\right\|_p + \varepsilon\left\|D^3 f\right\|_p + \|f\|_{S^{2,p}}\right\}$$

where

$$\left\|D^h f\right\|_p = \sum_{i_j=1}^{q} \left\|\tilde{X}_{i_1}\tilde{X}_{i_2}\ldots\tilde{X}_{i_h} f\right\|_p .$$

Therefore, applying also Theorem 3.2,

$$\|f\|_{S^{3,p}} \leq c\left\{\left\|\tilde{\mathcal{L}}f\right\|_{S^{1,p}} + \|f\|_p\right\}.$$

Second step. We now need suitable extensions of Lemma 3.3, Proposition 3.5 and Theorem 3.8. The extensions of Lemma 3.3 and Proposition 3.5 are straightforward computations. The extension of Theorem 3.8 is a consequence of an analogous extension of Theorem 3.6, which can be proved applying Theorem 3.6 to $\tilde{X}_i f$ and then iterating. Once we have these tools, the result follows as in the proof of Theorem 0.1 (for details, see [4]). □

3.2. **Local estimates in Hölder spaces.** We now deal with the problem of local Hölder continuity for the solutions to the equation $\mathcal{L}f = g$. The precise statements of our results are the following:

Theorem 0.3. *Under the same assumptions of Theorem 0.1, if $f \in S_X^{2,p}(\Omega)$ for some $p \in (1,\infty)$ and $\mathcal{L}f \in \mathcal{L}^s(\Omega)$ for some $s > Q$, then*

$$\|f\|_{\Lambda_X^\alpha(\Omega')} \leq c\left\{\|\mathcal{L}f\|_{\mathcal{L}^r(\Omega)} + \|f\|_{\mathcal{L}^p(\Omega)}\right\}$$

for $r = \max(p,s), \alpha = \alpha(Q,p,s) \in (0,1), c = c(X,n,q,p,r,\mu,\Omega,\Omega')$.

Theorem 0.4. *Under the same assumptions of Theorem 0.1, if $a_{ij} \in S_X^{k,\infty}(\Omega)$, $f \in S_X^{2,p}(\Omega)$ for some $p \in (1,\infty)$ and $\mathcal{L}f \in S^{k,s}(\Omega)$ for some positive integer k, some $s > Q$, then*

$$\|f\|_{\Lambda_X^{k,\alpha}(\Omega')} \leq c\left\{\|\mathcal{L}f\|_{S^{k,r}(\Omega)} + \|f\|_{\mathcal{L}^p(\Omega)}\right\}$$

for $r = \max(p,s), \alpha = \alpha(Q,p,s) \in (0,1), c = c(X,n,q,p,r,\mu,\Omega,\Omega')$.

The basic tool for the proof of the above results is a suitable version of Sobolev embedding theorems, in the context of "lifted variables". We proceed through several steps.

Lemma 3.10. *For $i = 1,2,\ldots,Q$, let*

$$K_i(\xi,\eta) = a_i(\xi)W_i(\Theta(\eta,\xi))b_i(\eta),$$

where a_i, b_i are fixed test functions and W_i is a function defined on G, smooth outside the origin and homogeneous of degree $i - Q$. Then the "fractional integral operator" T_{K_i} with kernel K_i satisfies the following estimates:

(a) for $i < Q$, $1 < p < Q/i$, $1/q_i = 1/p - i/Q$,

$$\|T_{K_i}f\|_{q_i} \leq c\,\|f\|_p\,;$$

(b) for $i = Q$, $1 < p < \infty$,

$$\|T_{K_i}f\|_\infty \leq c\,\|f\|_p\,;$$

(c) there exists $\beta \in (0,1)$ such that for $Q/i < p < Q/(i-\beta)$, $\alpha_i = i - Q/p$,

$$\|T_{K_i} f\|_{\Lambda^{\alpha_i}_{\tilde{X}}} \leq c \|f\|_p .$$

Actually, β is the number appearing in Proposition 2.17.

Proof. By our assumptions, the kernel K_i satisfies properties (i), (ii) stated in Proposition 2.17. By (i), we can apply the standard result on fractional integrals in homogeneous spaces (see Theorem 2.13), and get the $\mathcal{L}^p - \mathcal{L}^q$ estimate stated in (a). (b) is trivial, because in that case the kernel is bounded. By (i) and (ii), we can prove the Hölder estimate stated in (c), reasoning as in the proof of Theorem 3.1 in [4]. □

Lemma 3.11. *For $1 \leq \ell \leq Q$, let*

$$K(\xi,\eta) = \sum_{i=\ell}^{Q} K_i(\xi,\eta)$$

where $K_i(\xi,\eta)$ are the same as in previous lemma. Then the integral operator T_K with kernel K satisfies the following estimates:

(a) for $1 < p < Q/\ell$, $1/q = 1/p - \ell/Q$,

$$\|T_K f\|_q \leq c \|f\|_p ;$$

(b) for $Q/\ell < p < Q/(\ell-\beta)$, $\alpha = \ell - Q/p$,

$$\|T_K f\|_{\Lambda^{\alpha}_{\tilde{X}}} \leq c \|f\|_p$$

where β is the same as in Proposition 2.17.

Proof. (a) The estimate follows from Theorem 2.13 observing that

$$|K(\xi,\eta)| \leq \frac{c}{|B(\xi,\rho(\xi,\eta))|^{(\ell-Q)/Q}}.$$

(b) Let $Q/\ell < p < Q/(\ell-\beta)$. Then $Q/i < p$ for every $i = \ell, \dots, Q$.

If $p < Q/(i-\beta)$, by Lemma 3.10 we can write

$$\|T_{K_i} f\|_{\Lambda^{\ell-Q/p}_{\tilde{X}}} \leq c \|T_{K_i} f\|_{\Lambda^{i-Q/p}_{\tilde{X}}} \leq c \|f\|_p .$$

If $p \geq Q/(i-\beta)$ (and therefore $i > \ell$), let $p_i' \leq p$ be chosen later, such that

$$Q/i < p_i' < Q/(i-\beta).$$

Then we can write the embeddings

$$\|T_{K_i} f\|_{\Lambda^{\ell-Q/p}_{\tilde{X}}} \leq c \|T_{K_i} f\|_{\Lambda^{i-Q/p_i'}_{\tilde{X}}} \leq c \|f\|_{p_i'} \leq c \|f\|_p$$

provided that $\ell - Q/p \leq i - Q/p_i'$. Therefore we have to choose p_i' such that the conditions

$$\begin{cases} p_i' \leq p, \\ Q/i < p_i' < Q/(i-\beta), \\ \ell - Q/p \leq i - Q/p_i' \end{cases} \tag{3.12}$$

hold, under the assumptions

$$\begin{cases} Q/\ell < p < Q/(\ell-\beta), \\ p \geq Q/(i-\beta). \end{cases} \tag{3.13}$$

By (3.13), the system (3.12) is equivalent to

$$\begin{cases} \frac{i-\beta}{Q} < \frac{1}{p_i'} \le \frac{1}{p} + \frac{i-\ell}{Q}, \\ p_i' \le p, \end{cases}$$

which is possible whenever

$$\frac{i-\beta}{Q} < \frac{1}{p} + \frac{i-\ell}{Q}.$$

This condition is contained in (3.13), so we are done. □

Theorem 3.12 (Sobolev embeddings)**.** *Let $f \in S^{2,p}_{\tilde{X},0}(B_r)$ for some ball $B_r \subset \mathbb{R}^N$. Then:*

(a) if $1 < p < Q/2$ and $1/p^* = 1/p - 2/Q$, then

$$\|f\|_{p^*} \le c \|f\|_{S^{2,p}};$$

(b) if $Q/2 < p < Q/(2-\beta)$, and $\alpha = 2 - Q/p$, then

$$\|f\|_{\Lambda^{\alpha}_{\tilde{X}}(B_r)} \le c \|f\|_{S^{2,p}}$$

where β is the same as in Proposition 2.17.

Proof. For $f \in C_0^\infty(B_r)$ and a a cutoff function equal to 1 on B_r, we can write again equation (3.8):

$$f(\xi) = P_2 \Delta f(\xi) + P_1 f(\xi)$$

where P_2, P_1 are, respectively, constant operators of type 2, 1. Explicitly, the kernels of P_ℓ $(\ell = 1, 2)$ can be written as

$$k_\ell(\xi, \eta) = \sum_{i=\ell}^{Q} K_i(\xi, \eta) + K_0(\xi, \eta),$$

where the K_i's are as in Lemma 3.10, while

$$K_0(\xi, \eta) = a(\xi) W(\Theta(\eta, \xi))\, b(\eta)$$

with $W(u)$ a Lipschitz function on G, that is,

$$|W(u) - W(v)| \le c \left\| v^{-1} \circ u \right\|.$$

Claim 3.13. The operator T_{K_0} with kernel $K_0(\xi, \eta)$ satisfies the estimates contained in Lemma 3.11.

Proof of the claim. Since $K_0(\xi, \eta)$ is bounded, the operator maps every $\mathcal{L}^p$ in $\mathcal{L}^\infty$. Moreover, since W is Lipschitz, by Theorem 1.7, the operator T_{K_0} satisfies

$$|T_{K_0} f(\xi_1) - T_{K_0} f(\xi_2)| \le c \rho(\xi_1, \xi_2)^{1/s} \|f\|_1$$

and therefore maps every $\mathcal{L}^p$ in Λ^β, with $\beta = 1/s$. Now, fix p, ℓ with $Q/\ell < p < Q/(\ell - \beta)$, and let $\alpha = \ell - Q/p$. Then $\alpha < \beta$, so that

$$\|T_{K_0} f\|_{\Lambda^\alpha} \le \|T_{K_0} f\|_{\Lambda^\beta} \le c \|f\|_p.$$

□

Once we know that the operator with kernel $k_\ell(\xi,\eta)$ satisfies the estimates of Lemma 3.11, we can prove the theorem as follows:

a) If $1<p<Q/2$, let $1/q_1=1/p-1/Q$; $1/q_2=1/p-2/Q$. Then

$$\|f\|_{q_2}\le c\,\|\Delta f\|_p+\|P_1 f\|_{q_2}\le c\left\{\|\Delta f\|_p+\|P_1 f\|_{q_1}\right\}$$

$$\le c\left\{\|\Delta f\|_p+\|f\|_p\right\}\le c\,\|f\|_{S^{2,p}_{\tilde X}}.$$

b) If $Q/2<p<Q/(2-\beta)$, let us rewrite equation (3.8) in the form

$$f(\xi)=P_2\Delta f(\xi)+P_1P_2\Delta f(\xi)+P_1P_1 f(\xi).$$

Then, by Lemma 3.11 (b),

$$\|f\|_{\Lambda^{2-Q/p}_{\tilde X}}\le c\,\|f\|_{S^{2,p}_{\tilde X}}+\|P_1P_2\Delta f+P_1P_1 f\|_{\Lambda^{2-Q/p}_{\tilde X}}.$$

To bound the last term, we note that:

(i)

$$\|P_2\Delta f\|_{\Lambda^{2-Q/p}_{\tilde X}}\le c\,\|\Delta f\|_p\,;$$

but the Hölder norm of $P_2\Delta f$ controls the $\mathcal{L}^\infty$-norm, which, in turn, controls the $\mathcal{L}^{p'}$-norm, for every p'. Then, for $Q<p'<Q/(1-\beta)$ we can write

$$\|P_1P_2\Delta f\|_{\Lambda^{1-Q/p'}_{\tilde X}}\le c\,\|P_2\Delta f\|_{p'}\le c\,\|\Delta f\|_p.$$

Choosing $1/p'=1/p-1/Q$, the conditions $Q<p'<Q/(1-\beta)$ are fulfilled, and

$$\|P_1P_2\Delta f\|_{\Lambda^{2-Q/p}_{\tilde X}}=\|P_1P_2\Delta f\|_{\Lambda^{1-Q/p'}_{\tilde X}}\le c\,\|f\|_{S^{2,p}_{\tilde X}}.$$

(ii) Let $1/q=1/p-1/Q$. Then $2-Q/p=1-Q/q$. Moreover, the conditions $Q/2<p<Q/(2-\beta)$ imply that $Q<q<Q/(1-\beta)$ and $1<p<Q$. Therefore by Lemma 3.11, we can write

$$\|P_1P_1 f\|_{\Lambda^{2-Q/p}_{\tilde X}}=\|P_1P_1 f\|_{\Lambda^{1-Q/q}_{\tilde X}}\le c\,\|P_1 f\|_q\le c\,\|f\|_p.$$

So we can conclude that

$$\|f\|_{\Lambda^{2-Q/p}_{\tilde X}}\le c\,\|f\|_{S^{2,p}_{\tilde X}}.$$

□

Proof of Theorem 0.3. Theorem 3.12 (Sobolev embeddings), Theorem 3.9 (local $\mathcal{L}^p$-estimates in the space of lifted variables) and Proposition 3.5 (approximation with test functions) allow us to repeat the argument contained in the proof of Theorem 1.6 in [4]. This leads to the following result: if $f\in S^{2,p}_{\tilde X}(\tilde\Omega)$ for some $p\in(1,\infty)$ and $\tilde{\mathcal{L}}f\in\mathcal{L}^s(\tilde\Omega)$ for some $s>Q$, then

$$\|f\|_{\Lambda^\alpha_{\tilde X}(\tilde\Omega')}\le c\left\{\left\|\tilde{\mathcal{L}}f\right\|_{\mathcal{L}^r(\tilde\Omega)}+\|f\|_{\mathcal{L}^p(\tilde\Omega)}\right\}$$

for $r=\max(p,s),\alpha=\alpha(Q,p,s)\in(0,1),c=c(X,n,q,p,r,\mu,\Omega,\Omega')$.

With the same reasoning as in the proof of Theorem 0.1 we come back from the lifted space to the original one. We only need to note that, in view of Theorem 1.13,

$$\sup\frac{|f(x_1)-f(x_2)|}{d_{\tilde X}((x_1,t_1),(x_2,t_2))^\alpha}=\sup\frac{|f(x_1)-f(x_2)|}{d_X(x_1,x_2)^\alpha}.$$

This concludes the proof. □

Proof of Theorem 0.4. We start noting that, if $f \in S^{k+2,p}_{\tilde{X},0}(B_r)$ with $Q/2 < p < Q/(2-\beta)$, and $\alpha = 2 - Q/p$, then

$$\|f\|_{\Lambda^{k,\alpha}_{\tilde{X}}} \leq c\,\|f\|_{S^{k+2,p}_{\tilde{X}}}.$$

To get this, it's enough to apply Theorem 3.12 to the k-th derivative of f. Using this estimate, Theorem 0.4 follows from Theorem 0.2 as Theorem 0.3 follows from Theorem 0.1. We omit the details. □

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