

## COMMUTATORS OF SINGULAR INTEGRALS AND FRACTIONAL INTEGRALS ON HOMOGENEOUS SPACES

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ABSTRACT.  $L^p - L^q$  estimates are proved for commutators of singular and fractional integral operators with multiplication with a  $BMO$  function on homogeneous spaces. For both types of operators the exponents  $p$  and  $q$  have the same values as in the corresponding theorems in Euclidean space. A characterization of  $BMO$  functions is also proved.

### 1. INTRODUCTION

This paper deals with commutators of integral operators with multiplication by a  $BMO$  function on a homogeneous space. The results we obtain confirm and sometimes even improve the results that have been proved to hold in  $R^n$ . More precisely, let  $(X, d, \mu)$  be a homogeneous space ( $X$  a set,  $d$  a quasidistance,  $\mu$  a Borel measure: see beginning of sect. 2 for the precise definition) and let  $T$  be either a singular integral operator or a fractional integral operator and  $a \in BMO(X)$ , (definition in sect. 2). In this paper we prove that the commutator  $[a, T]$  defined as  $[a, T]f = [a \cdot Tf - T(af)]$  is a bounded continuous operator  $[a, T] : L^p(X) \rightarrow L^q(X)$  satisfying

$$(1.1) \quad \|[a, T]f\|_q \leq C \|a\|_* \|f\|_p$$

where  $\|a\|_*$  is the  $BMO$  semi-norm of  $a$ ,  $p = q$  with  $1 < p < \infty$  when  $T$  is a singular integral operator, and  $\frac{1}{q} = \frac{1}{p} - \alpha$  with  $1 < p < \frac{1}{\alpha}$  when  $T$  is a fractional integral operator ( $\alpha \in (0, 1)$  is the number appearing in the fractional integral kernel, see sect.2). For the precise statement of these results see theorems 2.9, 2.10

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and 2.12. These results were known to hold for Euclidean spaces (see [6] for the result for singular integrals and [4] for the results for fractional integrals) and were first proved for homogeneous spaces in [1] and [2]. In the case of fractional integrals we actually prove a result which is stronger than (1.1) and therefore stronger than the result proved by Chanillo in [4] for Euclidean spaces. Our result is contained in theorem 2.11 and in particular implies that the estimate (1.1) holds when we replace the fractional integral kernel with an equivalent kernel. Here we also prove a "converse theorem" which is completely new, i.e. we prove that the validity of theorem 2.11 for any triple  $p, q, \alpha$  characterizes functions in  $BMO(X)$ . Chanillo in [4] has a result in this direction, where he proves, in the Euclidean case, that if an estimate of the kind (1.1) holds for a function  $a$  and a triple  $p, q, \alpha$  with  $n \cdot \alpha$  an even number, then  $a \in BMO(R^n)$ .

The proofs we present here of estimate (1.1) in the two cases, follow the main lines of the proofs in [1] and [2], though they are somewhat improved and simplified. The methods used in the two cases are very different. Here we would also like to discuss the applicability of those methods and to point out why, we believe, it is in the nature of each of these problems to be treated differently.

The method used to prove the result for singular integrals follows an idea of Torchinsky (see [13]) and makes use of powerful harmonic analysis tools and results that have been proved to hold in the context of homogeneous spaces, such as maximal function and sharp function estimates, as well as a John-Nirenberg lemma. A main point is the assumption in the definition of a singular integral operator that the kernel satisfies a pointwise Hörmander condition (see (2.4)). We point out that we prove the above estimates for singular integrals also in the case  $X$  is a finite measure space, under a few extra natural geometric conditions on  $X$  and the kernel of  $T$ .

In the case of fractional integral operators we could not use the above proof because the fractional integral kernel does not, in general, satisfy a Hörmander condition. It seemed instead more natural to follow an idea contained in [6] as a suggestion for an alternative proof for their  $L^p$  continuity of commutators of singular integrals on Euclidean spaces. As mentioned above, we prove a stronger result in this case. We point out that if we replace a kernel with an equivalent one in commutators of singular integral operators, in general,  $L^p$  estimates are no longer true, since in this case the cancellation properties of the kernels play a major role. The idea of the proof is to show that it is possible to reduce the problem of getting estimates (1.1) to that of obtaining certain *weighted* estimates for the operator  $T$ . The reduction is done by showing that a certain operator is a holomorphic function in a neighborhood of the origin of the complex plane and then using classic complex analysis tools. We need to assume a natural geometric condition on the space  $X$  (see property **(P)**, (2.8) below) in order to assure that the proof of weighted estimates by Sawyer and Wheeden (see [12]) carries through to our case. Condition (2.8) is quite general, since it allows for  $X$  being a finite measure space as well as for existence of atoms in  $X$ .

The reason why we did not apply this proof in the singular integral case is that, to our knowledge, there are no general weighted estimates of the kind we would need for singular integrals. Moreover, while it seemed that these estimates may follow quite straightforwardly from the proofs in  $R^n$  in the case  $\mu(X) = \infty$ , it seemed that in the finite measure case it would have been harder to prove those

estimates than to use Torchinsky's approach to obtain the commutator estimates directly.

2. DEFINITIONS, PRELIMINARY RESULTS AND STATEMENT OF MAIN RESULTS

Let  $X$  be a set. A *quasi-distance* on  $X$  is a function  $d : X \times X \rightarrow [0, +\infty)$  such that

- (i)  $d(x, y) = 0$  if and only if  $x = y$ ;
- (ii)  $d(x, y) = d(y, x)$  for every  $x, y \in X$ ;
- (iii)  $d(x, y) \leq c_d (d(x, z) + d(z, y))$  for some  $c_d \geq 1$  and for every  $x, y, z \in X$ .

The balls  $B_r(x) = \{y \in X : d(x, y) < r\}$ ,  $r > 0$ , form a base for a complete system of neighborhoods of  $x \in X$ . The space  $X$  endowed with this topology (which we will refer to as the  $d$ -topology of  $X$ ) is a Hausdorff space. We'll say that two quasi-distances  $d(x, y)$  and  $d'(x, y)$  on  $X$  are equivalent if there exists two positive constants  $a_1$  and  $a_2$  such that  $a_1 d(x, y) \leq d'(x, y) \leq a_2 d(x, y)$  for every  $x$  and  $y$  in  $X$ . In particular equivalent quasi-distances induce the same topology on  $X$ .

Observe that the balls are not in general open sets in the  $d$ -topology. Let  $X$  be a set endowed with a quasi-distance  $d(x, y)$  such that the balls are open sets in the  $d$ -topology and let  $\mu$  be a positive Borel measure defined on a  $\sigma$ -algebra of subsets of  $X$  which contains the  $d$ -balls, for which there exists a constant,  $c_\mu \geq 1$  such that the following *doubling condition* is satisfied

$$(2.1) \quad 0 < \mu(B_{2r}(x)) \leq c_\mu \cdot \mu(B_r(x)) < \infty$$

for every  $x \in X$  and  $r > 0$ . We will call  $(X, d, \mu)$  a *homogeneous space*.

We'll say that a homogeneous space  $(X, d, \mu)$  is *normal* if there exist two positive constants  $c_1$  and  $c_2$ , such that

$$(2.2) \quad c_1 r \leq \mu(B_r(x)) \leq c_2 r$$

for every  $x \in X$  and every  $r \in (\mu\{x\}, \mu(X))$ .

The following three theorems, due to Macías and Segovia (see [10]), contain very useful results about quasi-distances and homogeneous spaces.

**Theorem 2.1.** *Let  $X$  be a homogeneous space. Then the set  $M$  of atoms of  $X$  (i. e. of points  $x \in X$  with  $\mu\{x\} > 0$ ) is countable and for each  $x \in M$  there exists  $r > 0$  such that  $B_r(x) = \{x\}$ .*

**Theorem 2.2.** *Let  $d(x, y)$  be a quasi-distance on a set  $X$ . Then there exists a quasi-distance  $d'(x, y)$  on  $X$  equivalent to  $d$  and a number  $0 < \alpha < 1$ , such that for every  $x, y$  and  $z$  in  $X$  and  $r > 0$*

$$|d'(x, z) - d'(y, z)| \leq C [d'(x, z) + d'(y, z)]^{1-\alpha} (d'(x, y))^\alpha$$

whenever are both smaller than  $r$ . Observe that balls corresponding to the quasi-distance  $d'$  are open in the  $d'$ -topology.

**Theorem 2.3.** *Let  $(X, d, \mu)$  be a homogeneous space. Then the function  $\delta(x, y)$  defined as follows*

$$(2.3) \quad \delta(x, y) = \begin{cases} \inf\{\mu(B) \mid B \text{ ball containing } x, y\} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is a quasi-distance on  $X$ . Moreover  $(X, \delta, \mu)$  is a normal space and the topologies induced by  $d$  and by  $\delta$  coincide.

Another interesting known result, about homogeneous spaces is the following:

**Theorem 2.4.** *Let  $X$  be a homogeneous space. Then  $\mu(X) < \infty$  if and only if  $X$  is bounded, i.e. if there exists a ball  $B = B_r(x)$  such that  $X = B$ .*

We will adapt the definitions of some spaces of functions on  $R^n$  to the context of homogeneous spaces.

The space  $C_0^\alpha(X)$  of Hölder continuous functions on  $X$ , is the space of functions  $f$  with bounded support in  $X$  and such that

$$\|f\|_\alpha = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} < \infty.$$

The spaces  $L^p(X)$ ,  $1 \leq p \leq \infty$  are defined as usual. The space  $\mathcal{D}(X)$  of test functions, is the space of functions  $f \in L^\infty(X)$  with bounded support.

Let  $\int_\Omega f = \frac{1}{|\Omega|} \int_\Omega f d\mu = f_\Omega$ . In what follows  $B$  will denote a ball in  $X$ . Given  $f : X \rightarrow R$ , we will call *maximal function* of  $f$ , the function

$$Mf(x) \equiv \sup_{B \ni x} \int_B |f(y)| d\mu(y)$$

and *sharp function* of  $f$ , the function

$$f^\sharp(x) \equiv \sup_{B \ni x} \int_B |f(y) - f_B| d\mu(y)$$

We will say that  $f \in BMO(X)$ , if  $f \in L_{loc}^1(X)$  and  $f^\sharp \in L^\infty$ . If  $f \in BMO(X)$  we will call *BMO seminorm* of  $f$  the quantity

$$\|f\|_* \equiv \sup_x f^\sharp(x) = \sup_B \int_B |f(y) - f_B| d\mu(y)$$

The following two lemmas contain important properties of *BMO* functions (proofs may be found e.g. in [13]).

**Lemma 2.5.** *Let  $f \in BMO(X)$  and let*

$$f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| < n \\ n & \text{if } f(x) \geq n \\ -n & \text{if } f(x) \leq -n \end{cases}$$

Then  $f_n \in BMO(X)$  and  $\|f_n\|_* \leq c \|f\|_*$ .

**Lemma 2.6.** *Let  $f \in BMO$  and  $M > 1$ . Then for every ball  $B_r(x)$*

$$|f_{B_{M^j r}} - f_{B_r}| \leq c \cdot j \|f\|$$

*for every positive integer  $j$ , where  $c$  depends only on  $M$  and  $c_\mu$ .*

Let's now define singular integral and fractional integral operators on homogeneous spaces.

**Definition 2.7.** We say that  $K$  is a Calderón-Zygmund operator (CZ operator) on the homogeneous space  $(X, d, \mu)$  if:

- (i)  $K : L^p(X) \rightarrow L^p(X)$  is linear continuous for every  $p \in (1, \infty)$ ;
- (ii) there exists a measurable function  $k : X \times X \rightarrow R$  such that for every  $f \in \mathcal{D}$  and a.e.  $x \notin \text{supp} f$ ,  $Kf(x) = \int_X k(y, x)f(y)d\mu(y)$ ;
- (iii)  $k$  satisfies a *pointwise Hörmander condition* i.e. there exist  $c > 0$ ,  $\beta > 0$  and  $h > 1$  such that for every  $x_0 \in X$ ,  $r > 0$ ,  $x \in B_r(x_0)$ ,  $y \notin B_{hr}(x_0)$

$$(2.4) \quad |k(x_0, y) - k(x, y)| \leq \frac{c}{\mu(B(x_0; y))} \cdot \frac{d(x_0, x)^\beta}{d(x_0, y)^\beta}$$

(where  $B(x; y) = B_{d(x,y)}(x)$ ).

Incidentally observe that condition (2.4) in definition 2.7 implies an integral Hörmander condition (for a proof see [1]); this condition is assumed in [5] and [7] to prove  $L^p$  continuity of a CZ operator which is  $L^2$  continuous.

**Definition 2.8.** For  $\alpha \in (0, 1)$  we call *fractional integral kernel* the function  $k_\alpha(x, y) = \mu(B(x; y))^{\alpha-1}$  and *fractional integral operator* the operator  $I_\alpha$  defined by

$$(2.5) \quad I_\alpha f(x) = \int_{X \setminus \{x\}} k_\alpha(x, y)f(y)d\mu(y)$$

We are now ready to state the main results in this paper. Theorems 2.9 and 2.10 contain the results for singular integrals; theorems 2.11 and 2.12 contain the results for fractional integrals and theorem 2.13 the "converse fractional integral" result.

**Theorem 2.9.** *Let  $(X, d, \mu)$  be an infinite measure homogeneous space,  $K$  a CZ operator on  $X$  and  $a \in BMO(X)$ . Then the commutator  $[a, K]$  satisfies*

$$(2.6) \quad \|[a, K]f\|_p \leq C \|a\|_* \|f\|_p$$

*for every  $p \in (1, \infty)$*

**Theorem 2.10.** *Let  $(X, d, \mu)$  be a finite measure homogeneous space,  $K$  a CZ operator on  $X$  and  $a \in BMO(X)$ . Moreover let the following conditions hold:*

- (i)  $X$  is separable and  $\mu$  is regular;
- (ii) the adjoint kernel  $k^*$  (defined as  $k^*(x, y) = k(y, x)$ ) satisfies the pointwise Hörmander condition (2.4);

(iii)  $k$  satisfies the growth condition:

$$(2.7) \quad |k(x, y)| \leq \frac{c}{\mu(B(x; y))}, \text{ for every } x, y \in X.$$

Then (2.6) holds.

We need a few more definitions in order to state the precise result for fractional integrals. For  $x \in X$  let

$$R_x = \begin{cases} \inf \{R | B_R(x) = X\} & \text{if } \mu(X) < \infty \\ \infty & \text{otherwise} \end{cases}$$

and

$$r_x = \begin{cases} \sup \{r | B_r(x) = x\} & \text{if } \mu\{x\} > 0 \\ 0 & \text{otherwise} \end{cases}.$$

We'll say the space  $X$  satisfies property **(P)** if:

$$(2.8) \quad \text{there exists } m \in (0, 1) \text{ such that } \forall r \in (r_x, R_x), B_r(x) \setminus B_{mr}(x) \neq \emptyset.$$

Finally let the operator  $C_\alpha^a$  be defined as  $C_\alpha^a f(x) = \int_X |a(x) - a(y)| k_\alpha(x, y) f(y) dy$ .

**Theorem 2.11.** *Let  $(X, d, \mu)$  be a homogeneous space, satisfying property **(P)** and  $a \in BMO(X)$ . Then the operator  $C_\alpha^a$  satisfies*

$$(2.9) \quad \|C_\alpha^a f\|_q \leq C \|a\|_* \|f\|_p$$

for every  $\alpha \in (0, 1)$ ,  $p \in (1, \frac{1}{\alpha})$  and  $\frac{1}{q} = \frac{1}{p} - \alpha$ .

**Theorem 2.12.** *Let  $(X, d, \mu)$  be a homogeneous space, satisfying property **(P)** and  $a \in BMO(X)$ . Then the commutator  $[a, I_\alpha]$  satisfies*

$$(2.10) \quad \|[a, I_\alpha]f\|_q \leq C \|a\|_* \|f\|_p$$

for every  $\alpha \in (0, 1)$ ,  $p \in (1, \frac{1}{\alpha})$  and  $\frac{1}{q} = \frac{1}{p} - \alpha$ .

Moreover inequality (2.10) holds if the kernel  $k_\alpha$  is replaced with any equivalent kernel  $\tilde{k}_\alpha$ .

**Theorem 2.13.** *Assume that for  $a \in L_{loc}^1(X)$  the operator  $C_\alpha^a$  satisfies inequality (2.9) for some  $\alpha \in (0, 1)$ , some  $p \in (1, \frac{1}{\alpha})$  and  $\frac{1}{q} = \frac{1}{p} - \alpha$ . Then  $a \in BMO(X)$ .*

## 3. COMMUTATORS OF SINGULAR INTEGRALS: PROOFS OF THMS. 2.9 AND 2.10

In the next theorem we'll recall known results about the  $L^p$ -continuity of maximal function and the sharp function as well as a John-Nirenberg type lemma that have been proved to hold in homogeneous spaces. For the proofs see [7] and [3]

**Theorem 3.1.** (i) (Maximal Inequality) *The operator  $M : f \mapsto Mf$  maps  $L^p$  in  $L^p$  for every  $p \in (1, \infty]$  and*

$$(3.1) \quad \|Mf\|_p \leq c \|f\|_p.$$

(ii) (Sharp Inequality) *For every  $f \in L^p$ ,  $p \in (1, \infty]$  if  $\mu(X) = \infty$ :*

$$(3.2) \quad \|f\|_p \leq c \|f^\sharp\|_p ;$$

*if  $\mu(X) < \infty$ :*

$$(3.3) \quad \left\| f - \int_X f \right\|_p \leq c \|f^\sharp\|_p .$$

(iii) (John-Nirenberg Lemma) *Let  $p \in [1, \infty)$  and  $B \subset X$  a ball; then  $f \in BMO(X)$  with norm  $\|f\|_*$  if and only if*

$$(3.4) \quad \left( \int_B |f(x) - f_B|^p d\mu(x) \right)^{\frac{1}{p}} \leq \|f\|_* .$$

*Inequality (3.4) also implies that there exists  $\eta > 0$  such that  $f^\eta \in A_2$  ( $A_2$  is the Muckenhoupt class).*

In view of John-Nirenberg lemma (theorem 3.1, equation (3.4)), lemma 2.5 and the density of  $\mathcal{D}$  in  $L^p$ , it is enough to prove theorems 2.9 and 2.10 for  $a \in L^\infty$  and  $f \in \mathcal{D}$ . The first main result in the proof of estimates for commutators of singular integrals is the following.

**Theorem 3.2.** *Let  $K$  be a CZ operator and  $p \in (1, \infty)$ ; then there exists  $c = c(p, K, X)$  such that for every  $a \in L^\infty$  and every  $f \in \mathcal{D}$*

$$(3.5) \quad \|[a, K]^\sharp\| \leq c \|a\|_* \|f\|_p$$

*Proof.* In view of the  $L^p$ -continuity of  $K$  and of the maximal inequality (theorem 3.1 (i)) in order to prove (3.5) it is enough to prove that for every  $r \in (1, \infty)$  there exists  $c = c(r, K, X)$  such that the following inequality

$$(3.6) \quad |[a, K]f^\sharp(x)| \leq c \cdot \|a\|_* \left\{ (M|Kf|^r(x))^{\frac{1}{r}} + (M|f|^r(x))^{\frac{1}{r}} \right\}$$

holds for every  $x \in X$ .

The proof follows Torchinsky ([13], p.418). Let's fix  $B_\delta = B_\delta(x_0)$  containing  $x$ .

Then for  $y \in B_\delta$

$$\begin{aligned} & [a, K]f(y) = \\ & = K \left( (a - a_{B_\delta}) f \cdot \chi_{B_{h\delta}} \right) (y) + K \left( (a - a_{B_\delta}) f \cdot \chi_{B \setminus X_{h\delta}} \right) (y) - (a(y) - a_{B_\delta}) \cdot Kf(y) = \\ & = A(y) + B(y) + C(y). \end{aligned}$$

Now by the John-Nirenberg lemma (theorem 3.1 (iii)), lemma 2.6 and the  $L^p$  continuity of  $K$ , we immediately get

$$\int_{B_\delta} |A(y) - A_{B_\delta}| d\mu(y) \leq c \|a\|_* [M(|f|^r)(x)]^{\frac{1}{r}}$$

and

$$\int_{B_\delta} |C(y) - C_{B_\delta}| d\mu(y) \leq c \|a\|_* [M(|Kf|^r)(x)]^{\frac{1}{r}}.$$

Next observe that from Hörmander, we obtain

$$\begin{aligned} & |B(y) - B(x_0)| \leq \\ (3.7) \quad & \leq C \cdot \delta^\beta \left( \int_{X \setminus B_{h\delta}} \frac{|a(z) - a_{B_\delta}|^{r'}}{\mu(B(x_0; z)) \cdot d(x_0, z)^\beta} d\mu(z) \right)^{\frac{1}{r'}} \cdot \\ & \cdot \left( \int_{X \setminus B_{h\delta}} \frac{|f(z)|^r}{\mu(B(x_0; z)) \cdot d(x_0, z)^\beta} d\mu(z) \right)^{\frac{1}{r}}. \end{aligned}$$

The last term can be written as

$$\sum_{j=1}^{\infty} \int_{h^j \delta \leq d(x_0, z) < h^{j+1} \delta} \frac{|f(z)|^r}{\mu(B(x_0; z)) \cdot d(x_0, z)^\beta} d\mu(z)$$

and by the doubling condition (2.1) it is  $\leq C \cdot \frac{1}{\delta^\beta} \cdot M(|f|^r)(x)$ . Using the analogous expansion together with lemma 2.6 for the first term in (3.7) we get that it is  $\leq \frac{1}{\delta^\beta} \sum_{j=1}^{\infty} c \cdot \|a\|_*^{r'} \left( \frac{1 + j^{r'}}{M^{\beta j}} \right) = c \cdot \frac{1}{\delta^\beta} \|a\|_*^{r'}$ .

Observing finally that  $\int_{B_\delta} |B(y) - B_{B_\delta}| d\mu(y) \leq 2 \int_{B_\delta} |B(y) - B(x_0)| d\mu(y)$ , the theorem is proved.  $\square$

From the above theorem, combined with the sharp inequality (theorem 3.1 (ii)) we obtain, respectively, theorem 2.9 when  $\mu(X) = \infty$  and



$$(3.8) \quad \left\| [a, K]f - \int_X [a, K]f \right\|_p \leq c \|a\|_* \|f\|_p,$$

when  $\mu(X) < \infty$ .

*Proof (of theorem 2.10).* From now on we'll assume  $\mu(X) < \infty$  together with hypothesis (i)-(iii) of theorem 2.10. We may also assume  $d$  to be Hölder continuous, since if we replace  $d$  with  $d'$  as defined in theorem 2.2 all the conditions in theorem 2.10 still hold. Accordingly the number  $\alpha$  will be the Hölder exponent of  $d$ .

Let's now define the truncating function

$$\psi_\varepsilon(t) = \begin{cases} 0 & t \leq \varepsilon \\ \frac{t}{\varepsilon} - 1 & \varepsilon < t < 2\varepsilon \\ 1 & t \geq 2\varepsilon \end{cases}$$

and the truncated kernels

$$k_\varepsilon(x, y) = k(x, y) \cdot \psi_\varepsilon(\delta(x, y)),$$

for  $\varepsilon \in (0, 1)$ .

The first result we need is

**Lemma 3.3.** *The truncated kernels  $k_\varepsilon$  satisfy Hörmander condition (2.4) with constants independent of  $\varepsilon$  and in particular with exponent  $\gamma = \min(\alpha, \beta)$ .*

(For a proof see [2].)

The following  $L^p$  continuity result may now be obtained adapting to our context the proof of the well known result by M. Cotlar's (see e. g. [11]).

**Theorem 3.4.** *Let  $K$  be a CZ operator satisfying the hypothesis of theorem 2.10 and let the operators  $K_\varepsilon$  be defined as  $K_\varepsilon f(x) = \int_X k_\varepsilon(x, y) f(y) d\mu(y)$ . Then, for every  $p \in (1, \infty)$ , there exists a constant  $c$  independent of  $\varepsilon$  such that*

$$(3.9) \quad |K_\varepsilon f(x)| \leq c \left\{ M(Kf)(x) + (M(|f|^p)(x))^{\frac{1}{p}} + Mf(x) \right\},$$

for every  $f \in L^p$ . Moreover the operator  $K^*$  defined by  $K^* f(x) = \sup_{\varepsilon > 0} |K_\varepsilon f(x)|$  is continuous from  $L^p$  into  $L^p$  for every  $p \in (1, \infty)$ .

The next step is now to prove that the commutators of  $K_\varepsilon$  are  $L^p$  continuous. This will be a consequence of theorem 3.4, lemma 3.3 and the following theorem.

**Theorem 3.5.** *Let  $K$  be an operator as in the hypothesis of theorem 2.10. Assume moreover that for  $f \in \mathcal{D}$  the kernels  $k$  and  $k^*$  are such that  $\int_X |k(x, y)| f(y) d\mu(y)$  and  $\int_X |k^*(x, y)| f(y) d\mu(y)$  are bounded. Then the commutator  $[a, K]$  is a continuous operator on  $L^p$  and equation (2.6) holds with constant  $C$  depending on  $K$  only through the norms of  $K$  and  $K^*$  as operators on  $L^p$  and the constants in (2.4) and (2.7).*

*Proof.* Recalling equation (3.8) we have that

$$(3.10) \quad \|[a, K]f\|_p \leq c \|a\|_* \|f\|_p + \mu(X)^{\frac{1}{p}-1} \left| \int_X \left( \int_X k(x, y)[a(x) - a(y)] d\mu(y) \right) d\mu(x) \right|.$$

By Fubini's theorem the term in absolute value equals

$$(3.11) \quad \left| \int_X f(y) \left( \int_X k(x, y)[a(y) - a(x)] d\mu(x) \right) d\mu(y) \right| \leq \|f\|_p \|[K^*, a]1\|_{p'}.$$

Recalling that  $K^*$  satisfies the same hypothesis as  $K$ , and applying (3.10) and (3.11) with  $K^*$  in place of  $K$  we obtain

$$(3.12) \quad \|[K^*, a]1\|_{p'} \leq c \|a\|_* \mu(X)^{\frac{1}{p'}} + \mu(X)^{-\frac{1}{p}} \left| \int_X K a(y) d\mu(y) - \int_X K^* a(x) d\mu(x) \right|.$$

Finally, using Hölder's inequality, the  $L^p$  estimates and the John-Nirenberg lemma (see theorem 3.1 (iii)) we get

$$(3.13) \quad \left| \int_X K a(y) d\mu(y) \right| \leq c \|a\|_p \cdot \mu(X)^{\frac{1}{p}} \leq c \|a\|_* \cdot \mu(X),$$

and the analogous inequality for  $K^*$ . The thesis of the theorem can now be obtained by combining inequalities (3.10) to (3.13).  $\square$

As a final step we need to show that  $[a, K_\varepsilon] \rightarrow [a, K]$  in some sense. In general it is not true that the truncated operators  $K_\varepsilon$  converge to  $K$ , but we'll show in the following theorem that they converge to an operator  $T$  (in a suitable weak sense) and that  $T$  and  $K$  have the same commutator. The analogous result has been proved for Calderón-Zygmund operators on  $R^n$ ; in this last case the proof uses tools, such as Fourier transforms, which are characteristic of  $R^n$ ; our proof instead relies entirely on topological and measure theoretic arguments.

**Theorem 3.6.** *In the hypothesis of theorem 2.10 there exists a sequence  $\varepsilon_n \rightarrow 0$  and a function  $b \in L^\infty$  such that*

$$(3.14) \quad \int_X f K_{\varepsilon_n} g d\mu \rightarrow \int_X f K g d\mu + \int_X f b g d\mu$$

for every  $f, g \in \mathcal{D}$ .

*Proof.* Because of the hypothesis we have made on  $X$  and on the measure  $\mu$ , the space  $L^p(X)$  is separable for every  $p \in [1, \infty)$ . Therefore theorem 3.4 implies that there exists a sequence  $\varepsilon_n \rightarrow 0$  such that  $K_{\varepsilon_n} f$  converges weakly in  $L^p$  for  $f$  in a dense subset of  $L^p$ . Again theorem 3.4 implies that

$$(3.15) \quad \int_X g K_{\varepsilon_n} f d\mu \rightarrow \int_X g \tilde{K} f d\mu$$

for every  $f \in L^p$  and  $g \in L^{p'}$ . Our goal is to show that  $K_1 = K - \tilde{K}$  is multiplication times an  $L^\infty$  function. Observe first of all that if  $f, g \in \mathcal{D}$  and  $d(\text{sprt } f, \text{sprt } g) \geq \varepsilon_0 > 0$  then

$$\int_X g K f d\mu \rightarrow \int_X g \tilde{K} f d\mu.$$

Taking  $f = \chi_B$  with  $E$  a Borel set whose closure is contained in  $X \setminus \text{sprt } f$  this implies that  $Kg = \tilde{K}g$  a. e. in  $X \setminus \text{sprt } f$ . Now for  $E$  a Borel set let  $C_n$  be an increasing sequence of closed sets and  $A_n$  be an decreasing sequence of open sets such that  $\mu(A_n \setminus C_n) \rightarrow 0$  and  $C_n \subset E \subset A_n$ . Let now

$$b(x) = (K_1 1)(x) = \left( K_1 \chi_{C_n} \right) (x) + \left( K_1 \chi_{A_n \setminus C_n} \right) (x) + \left( K_1 \chi_{A_n^c} \right) (x).$$

Taking limits in  $n$  we get that  $K_1 \chi_E(x) = b(x)$  for a.e.  $x \in E$ . Analogously  $K_1 \chi_E(x) = 0$  for a.e.  $x \notin E$ , i. e.  $K_1 \chi_E(x) = b(x) \cdot \chi_E(x)$ . By linearity this proves the assertion. To see that  $b \in L^\infty$  observe that for  $f \in \mathcal{D}$  from the linearity of  $K_1$  we have that

$$\left| \int_X b h d\mu \right| = \left| \int_X h^{\frac{1}{p}} K_1 h^{\frac{1}{p'}} d\mu \right| \leq c \|h\|_1. \quad \square$$

The proof of theorem 2.10 is now complete.  $\square$

#### 4. COMMUTATORS OF FRACTIONAL INTEGRALS: PROOF OF THM. 2.11

First of all let's recall the following  $L^p - L^q$  continuity result that holds for the operator  $I_\alpha$ .

**Theorem 4.1.** *Let  $(X, d, \mu)$  be a homogeneous space; then the fractional integral operator  $I_\alpha$  satisfies*

$$(4.1) \quad \|I_\alpha f\|_q \leq C \|f\|_p$$

for every  $\alpha \in (0, 1)$ ,  $p \in (1, \frac{1}{\alpha})$  and  $\frac{1}{q} = \frac{1}{p} - \alpha$ .

Observe that no extra hypothesis are needed on  $X$ . The proof of this result can be obtained by the results in [8] combined with theorem 2.1.

Let's now make a few observations and remarks on property **(P)** (see (2.8)).

(i) By the definitions of  $R_x$  and  $r_x$  (2.8) is equivalent to the following

$$(4.2) \quad \exists m \in (0, 1) \text{ such that } \forall r \in \left( r_x, \frac{R_x}{m} \right), B_r(x) \setminus B_{mr}(x) \neq \emptyset.$$

(ii) (2.8) is quite general since it allows for the existence of atoms in  $X$  as well as for  $X$  to be bounded (and therefore finite measure). What it actually requires is that the ratio between outer and inner radii of empty annuli is bounded. In [2], though, we construct an example of a (very pathological) homogeneous space for which condition (2.8) does not hold.

(iii) It is trivial that if (2.8) holds for a certain  $p$  than it holds for  $p' < p$ . Because of (i) the same is true for (4.2).

(iv) (2.8) implies the following *reverse doubling condition* (RDC) there exists  $\delta > 0$ ,  $K > 1$  and  $k_1, k_2 \in (0, 1)$  such that

$$(4.3) \quad \mu(B_{Kr}(x)) \geq (1 + \delta) \cdot \mu(B_r(x)), \quad \forall x \in X, \quad r \in (k_1 r_x, k_2 R_x).$$

In order to prove theorem 2.11 observe first of all that, because of lemma 2.5 and Fatou's theorem, it is enough to prove the theorem for  $f \geq 0$  and  $a \in L^\infty$ . The first main result is the following theorem.

**Theorem 4.2.** *For  $a \in BMO$  and  $z \in \mathbf{C}$  (i.e.  $z$  complex), let  $b(x) = e^{(\operatorname{Re} z \cdot a(x))}$ . Assume there exists  $\varepsilon > 0$  such that the following weighted estimate*

$$(4.4) \quad \|I_\alpha f\|_{L^q(b^q d\mu)} \leq c \cdot \|f\|_{L^p(b^p d\mu)}$$

holds for every  $z$  with  $|z| < \varepsilon$ . Then theorem 2.11 holds.

*Proof.* We follow an idea contained in [6] as a suggestion for an alternative proof to their result on the commutator of classical Calderón-Zygmund operators on euclidean space. For  $z \in \mathbf{C}$  define

$$T_z f(x) = \int_{X \setminus \{x\}} e^{(a(x) - a(y)) \cdot z} k_\alpha(x, y) f(y) \operatorname{sgn}(a(x) - a(y)) d\mu(y);$$

then formally

$$\frac{d}{dz} T_z \Big|_{z=0} = 0 f(x) = \int_X |a(x) - a(y)| k_\alpha(x, y) f(y) dy = C_\alpha^a f(x).$$

Therefore our first goal is to show that the map  $z \rightarrow T_z$  is analytic in a neighborhood of the origin of  $\mathbf{C}$ . From the definition of  $T$  we get

$$|T_z f(x)| \leq b(x) \int_{X \setminus \{x\}} k_\alpha(x, y) |f(y)| b^{-1}(y) d\mu(y)$$

and from the weighted estimates

$$\|T_z\|_q \leq \|I_\alpha(b^{-1} f)\|_{L^q(b^q d\mu)} \leq c \|b^{-1} f\|_{L^p(b^p d\mu)} = c \|f\|_p.$$

Therefore  $T$  maps the disk  $|z| < \varepsilon$  into the linear continuous operators from  $L^p$  to  $L^q$ . By taking difference quotients and using the John-Nirenberg lemma together with the above reasoning it is easy to see that  $T_z$  is analytic in the same disk. We can now use Cauchy's integral formula and get  $C_\alpha^a f(x) = \frac{1}{2\pi i} \int_{|w|=\frac{\varepsilon}{2}} \frac{T_w f(x)}{w^2} dw$ .

Finally using the continuity of  $T_z$  the above gives  $\|C_\alpha^a f\|_q \leq c \|f\|_p$ . The linear dependence of the constant  $c$  on  $\|a\|_*$  follows by the homogeneity of the estimate. The proof of theorem 4.2 is complete.  $\square$

**Theorem 4.3.** *Let  $(X, d, \mu)$  be a homogeneous space satisfying property (2.8) and let  $K$  the operator acting on functions on  $X$  defined by*

$$Kf(x) = \int_X k(x, y) f(y) d\mu(y)$$

where the kernel  $k : X \times X \rightarrow \mathbf{R}^+$  is a measurable function. Let  $p, q, \alpha$  be as in theorem 2.11 and assume further:

- (i) the kernel  $k$  is such that the quantity defined by  $\Phi(B) = \sup\{k(x, y) \mid x, y \in B, d(x, y) \geq hr\}$  is equivalent to  $\mu(B)^{1-\alpha}$  for every ball  $B = B_r(x)$  with  $r \in (r_x, R_x)$ , i.e.  $c_1 \mu(B)^{1-\alpha} \leq \Phi(B) \leq c_2 \mu(B)^{1-\alpha}$ .
- (ii)  $v$  and  $w$  are two nonnegative measurable function such that for some  $s > 1$   $v^{(1-p)s} \in A_{t_1}$  and  $w^s \in A_{t_2}$  with  $t_1 = 1 + \frac{p'}{q}$  and  $t_2 = 1 + \frac{q}{p'}$ .

Then

$$(4.5) \quad \|I_\alpha f\|_{L^q(w d\mu)} \leq c \cdot \|f\|_{L^p(v d\mu)}$$

The above theorem is a consequence of the weighted estimates proved by Sawyer and Wheeden in [12]. Actually the hypothesis on  $X$  in [12] are more restrictive since they require  $X$  to have no empty annuli. To see how to adapt their proof to our case as well as how theorem 4.3 follows from Sawyer and Wheeden's estimates see [2].

*Proof.* (of Theorem 2.11) To conclude the proof of theorem 2.11 we only need to show that our kernel  $k_\alpha$  satisfies hypothesis (i) and our weights  $v = b^p$  and  $w = b^q$  satisfy hypothesis (ii) in the above theorem. Observe that the RDC (4.3) implies that for every  $\eta \in (0, 1)$  there exists two positive constants  $c, \gamma$  depending on  $X$  and  $\eta$  such that for every  $x \in X$  we have  $\mu(B_R(x)) \geq c \left(\frac{R}{r}\right)^\gamma \mu(B_r(x))$ , for  $\eta r_x \leq r \leq R \leq \frac{1}{\eta} R_x$ . Now (i) follows from the above property, triangle inequality and the doubling condition. As for condition (ii), since  $a \in L^\infty$ , by the John-Nirenberg Lemma for every  $t \in (1, \infty)$  there exists  $\varepsilon$  such that  $e^{(\varepsilon \cdot a(x))} \in A_t$  with  $A_t$  constant bounded by  $c(t, \|a\|_*)$ . Therefore, for every  $t \in (1, \infty)$ ,  $\beta > 0$  there exists  $\varepsilon$  such that for any  $z \in \mathbf{C}$  with  $|z| < \varepsilon$  the function  $b^\beta \in A_p$ . The proof is now complete.  $\square$

## 5. A CHARACTERIZATION OF BMO FUNCTIONS: PROOF OF THM. 2.13

We will now prove theorem 2.13. Let  $B = B_r(z)$  be a fixed ball in  $X$  and  $x \in B$ ; then

$$(5.1) \quad \left| a(x) - \int_B a(y) dy \right| \leq \int_B |a(x) - a(y)| dy.$$

Moreover, observing that if  $y \in B$  then  $B_{d(x,y)}(x) \subset B_{3c_d r}(z)$ , we have

$$(5.2) \quad \begin{aligned} C_\alpha^a(\chi_B)(x) &= \int_B |a(x) - a(y)| \frac{1}{\mu(B_{d(x,y)}(x))^{1-\alpha}} dy \geq \\ &\geq \frac{1}{\mu(B_{3c_d r}(z))^{1-\alpha}} \int_B |a(x) - a(y)| dy \geq \frac{C}{\mu(B)^{1-\alpha}} \int_B |a(x) - a(y)| dy \end{aligned}$$

where we used the doubling condition and where  $C = C(c_d, c_\mu)$ . From (5.1) and (5.2) we get

$$|a(x) - a_B| \leq C \mu(B)^{-\alpha} C_\alpha^a(\chi_B)(x).$$

From the continuity of  $C_\alpha$  we obtain

$$(5.3) \quad \left\{ \int_B |a(x) - a_B|^q dx \right\}^{\frac{1}{q}} \leq C \mu(B)^{\frac{1}{p} - \alpha} = C \mu(B)^{\frac{1}{q}}$$

In view of the John-Nirenberg Lemma (see (3.4) (i)) (5.3) proves the theorem.  $\square$

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