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$\mathscr{L}^p$ Estimates for Some Ultraparabolic Operators with Discontinuous Coefficients

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We consider a class of ultraparabolic operators of the kind

$$L \equiv \sum_{i,j=1}^{q} a_{ij}(z)\partial_{x_i x_j} + \langle x, BD\rangle - \partial_t, \qquad D = \left(\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_N}\right)$$

$(z = (x,t) \in \mathbb{R}^{N+1})$, where the principal part is uniformly elliptic on $\mathbb{R}^q$, $q \le N$, and the constant matrix B is upper triangular and such that the operator obtained by freezing the coefficients a_{ij} at any point $z_0 \in \mathbb{R}^{N+1}$ is hypoelliptic. We prove local $\mathscr{L}^p$-estimates for the derivatives $\partial_{x_i x_j} u$ $(i, j = 1, \dots, q)$ of a solution to the equation $Lu = f$, under the assumption that the coefficients a_{ij} belong to the space VMO (''vanishing mean oscillation'') with respect to a suitable metric related to B. © 1996 Academic Press, Inc.

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0. INTRODUCTION

In this paper we will consider a class of of Kolmogorov−Fokker−Planck type evolution operators on $\mathbb{R}^{N+1}$, of the form

$$\begin{aligned} L &\equiv \sum_{i,j=1}^{q} a_{ij}(z)\,\partial_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t \\ &\equiv \sum_{i,j=1}^{q} a_{ij}(z)\,\partial_{x_i x_j} + \langle x, BD\rangle - \partial_t, \qquad D = \left(\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_N}\right), \end{aligned} \tag{0.1}$$

where $z = (x,t) \in \mathbb{R}^{N+1}, 1 \le q \le N,$ and $b_{ij} \in \mathbb{R}$ for every $i, j = 1, \dots, N$. These operators have been widely studied by Lanconelli and Polidoro in [11, 12]. We shall make the following assumptions on the coefficients of L:

(H_1) $a_{ij} = a_{ji} \in \mathscr{L}^\infty(\mathbb{R}^{N+1})$ and there exists $\mu > 0$ such that

$$\frac{1}{\mu}\sum_{i=1}^{q} \xi_i^2 \le \sum_{i,j=1}^{q} a_{ij}(z)\,\xi_i \xi_j \le \mu \sum_{i=1}^{q} \xi_i^2$$

for every $z \in \mathbb{R}^{N+1}$ and $(\xi_1, \xi_2, \dots, \xi_q) \in \mathbb{R}^q$.

(H_2) B has the form

$$B \equiv \begin{bmatrix} 0 & B_1 & 0 & \dots & 0 \\ 0 & 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_r \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

where for every $k = 1, \dots, r, B_k$ is a matrix $p_{k-1} \times p_k$ with rank p_k and $q = p_0 \ge p_1 \ge \dots \ge p_r,\ p_0 + p_1 + \dots + p_r = N$.

It is known that under conditions H_1−H_2, for any fixed $z_0 \in \mathbb{R}^{N+1}$, the ‘‘frozen’’ operator

$$L_{z_0} = \sum_{i,j=1}^{q} a_{ij}(z_0)\,\partial_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t \tag{0.2}$$

is hypoelliptic (see [11]).

In the following section we will define a *quasidistance* associated to the operator L. With respect to this quasidistance, we will define a space VMO (‘‘vanishing mean oscillation’’), which we will denote by

$\mathrm{VMO}(\mathbb{R}^{N+1}, L)$. Then the last assumption about the coefficients of L is:

(H_3) $a_{ij} \in \mathrm{VMO}(\mathbb{R}^{N+1}, L)$.

Now, for Ω an open set in $\mathbb{R}^{N+1}$, $p \in (1, \infty)$, define the space

$$S^p(L, \Omega) = \left\{ u \in \mathscr{L}^p(\Omega) : u_{x_i}, u_{x_i x_j}; Yu \in \mathscr{L}^p(\Omega), i, j = 1, \dots, q \right\},$$

where $Yu = (\sum_{i,j=1}^N b_{ij} x_i \partial_{x_j} - \partial_t)u$. Set

$$\|u\|^p_{S^p(L,\Omega)} = \|u\|^p_{\mathscr{L}^p(\Omega)} + \sum_{i=1}^q \|u_{x_i}\|^p_{\mathscr{L}^p(\Omega)} + \sum_{i,j=1}^q \|u_{x_i x_j}\|^p_{\mathscr{L}^p(\Omega)} + \|Yu\|^p_{\mathscr{L}^p(\Omega)}.$$

We will prove local a priori estimates in $S^p(L, \Omega)$ for solutions to the equation $Lu = f$, when conditions (H_1)–(H_3) are fulfilled. More precisely:

THEOREM 0.1. *Assume* (H_1)–(H_3) *hold. If* $\Omega' \subset\subset \Omega \subset \mathbb{R}^{N+1}$ (Ω, Ω' *bounded open sets*) *then there exists a positive constant c such that*

$$\|u\|_{S^p(L,\Omega')} \le c\left(\|Lu\|_{\mathscr{L}^p(\Omega)} + \|u\|_{\mathscr{L}^p(\Omega)}\right) \qquad \text{for every } u \in S^p(L, \Omega). \quad (0.3)$$

The constant c depends only on p, μ, Ω', Ω, the matrix B (both through the entries b_{ij} and through the numbers p_i), and the "VMO moduli" η of the coefficients a_{ij} (see Definition 1.6). Throughout the paper the dependence of a constant on B will be understood in the above sense.

The paper is divided into three sections. Section 1 contains some geometric preliminaries (most of them are known results); the "geometry" is related to the structure of the matrix B, while the variable coefficients a_{ij} do not play any role. In Section 2 we establish a representation formula for the second derivatives of a solution to $Lu = f$ and prove some properties of the fundamental solution of L; here the coefficients a_{ij} enter only through their boundedness and ellipticity (on $\mathbb{R}^q$). In Section 3 we show how to obtain the suitable singular integral estimates, in order to prove Theorem 0.1 from the representation formula. The proof follows the analogous technique used in [3] for elliptic equations and in [1] for parabolic equations, including the classical tool of expansion in spherical harmonics (see also [5, 7]). To apply the above technique to our case we use results on singular integral operators in homogeneous spaces developed in [2]. It is in these singular integral estimates that the "VMO assumption" on the a_{ij}'s is used.

1. SOME GEOMETRIC PRELIMINARIES

We start by recalling two geometric structures in the space $\mathbb{R}^{N+1}$, induced by the constant matrix B. Proofs and further references about the properties listed below can be found in [11].

The Group of Translations

For every $z = (x, t), \zeta = (y, \tau) \in \mathbb{R}^{N+1}$, set

$$(x, t) \circ (y, \tau) = (y + E(\tau)x, t + \tau), \qquad \text{where } E(\tau) = \exp(-\tau B^T).$$

(Note that, since B is nilpotent, $E(\tau)$ is a polynomial of degree r in τ, with coefficients $N \times N$ matrices). Then $(\mathbb{R}^{N+1}, \circ)$ is a (noncommutative) group with neutral element $(0, 0)$; the inverse of an element $(x, t) \in \mathbb{R}^{N+1}$ is

$$(x, t)^{-1} = (-E(-t)x, -t).$$

We will call left translation by z the mapping $\zeta \mapsto z \circ \zeta$. The operator L_{z_0} is invariant for left translations.

The Group of Dilations

There exists a group of dilations on $\mathbb{R}^{N+1}$, which we denote by $(D(\lambda))_{\lambda > 0}$, with respect to which L_{z_0} is homogeneous of degree 2:

$$L_{z_0}(u(D(\lambda)z)) = \lambda^2 \cdot D(\lambda)\big((L_{z_0}u)(z)\big).$$

More precisely, $D(\lambda)$ acts as

$$D(\lambda)(x, t) = \big(\lambda^{\alpha_1}x_1, \dots, \lambda^{\alpha_N}x_N, \lambda^2 t\big), \tag{1.1}$$

where

$$\alpha_1 = \dots = \alpha_{p_0} = 1, \alpha_{p_0+1} = \dots = \alpha_{p_0+p_1} = 3, \dots,$$
$$\alpha_{p_0+ \dots +p_{r-1}+1} = \dots = \alpha_N = 2r + 1.$$

Therefore we can write

$$D(\lambda) = \operatorname{diag}\big(\lambda I_{p_0}, \lambda^3 I_{p_1}, \lambda^5 I_{p_2}, \dots, \lambda^{2r+1} I_{p_r}, \lambda^2\big),$$

where I_k is the identity matrix $k \times k$.

We will denote by $Q + 2$ the *homogeneous dimension* of $\mathbb{R}^{N+1}$ with respect to $(D(\lambda))_{\lambda > 0}$

$$Q + 2 = p_0 + 3p_1 + 5p_2 + \dots + (2r + 1)p_r + 2.$$

Note also that

$$\det D(\lambda) = \lambda^{Q+2}. \tag{1.2}$$

We will also call Q the *spatial homogeneous dimension* of $\mathbb{R}^{N+1}$ with respect to $(D(\lambda))_{\lambda > 0}$, and denote by $D_0(\lambda)$ the restriction of $D(\lambda)$ to $\mathbb{R}^N$;

note that

$$\det D_0(\lambda) = \lambda^Q. \tag{1.3}$$

Remark 1.1. An important property linking the two structures (translations and dilations) is

$$E(\lambda^2 t) = D_0(\lambda) E(t) D_0\left(\frac{1}{\lambda}\right) \qquad \text{for every } \lambda > 0, t \in \mathbb{R}. \tag{1.4}$$

This implies that

$$\text{Det } E(\lambda^2 t) = \text{Det } E(t) \qquad \text{for every } \lambda > 0, t \in \mathbb{R},$$

and, for $\lambda \to 0$,

$$\text{Det } E(t) = \text{Det } E(0) = 1 \qquad \text{for every } t \in \mathbb{R}. \tag{1.5}$$

Then a straightforward computation shows that the mappings

$$z \mapsto z \circ \zeta \qquad (\zeta \text{ fixed});$$
$$z \mapsto \zeta \circ z \qquad (\zeta \text{ fixed});$$
$$z \mapsto z^{-1}$$

have jacobian determinant equal to 1, and therefore *preserve the Lebesgue measure*.

We introduce now a *norm* and a *quasidistance* in $\mathbb{R}^{N+1}$, related to the groups of translations and dilations defined above.

DEFINITION 1.2. (see [7]). For any $z \in \mathbb{R}^{N+1} \setminus \{0\}$, define $\|z\| = \rho$ if ρ is the unique positive solution to the equation

$$\frac{x_1^2}{\rho^{2\alpha_1}} + \frac{x_2^2}{\rho^{2\alpha_2}} + \ldots + \frac{x_N^2}{\rho^{2\alpha_N}} + \frac{t^2}{\rho^4} = 1, \tag{1.6}$$

and $\|0\| = 0$. Moreover, define the following "polar-type" coordinate system,

$$\begin{cases} x_1 = \rho^{\alpha_1} \cos\psi_1 \ldots \cos\psi_{N-1} \cos\psi_N \\ x_2 = \rho^{\alpha_2} \cos\psi_1 \ldots \cos\psi_{N-1} \sin\psi_N \\ \qquad \qquad \ldots \\ x_N = \rho^{\alpha_N} \cos\psi_1 \sin\psi_2 \\ t = \rho^2 \sin\psi_1 \end{cases} \tag{1.7}$$

and note that

$$dx\,dt = \rho^{Q+1} J(\psi_1, \dots, \psi_N)\, d\rho\, d\psi_1 \dots d\psi_N = \rho^{Q+1} d\rho\, d\sigma, \quad (1.8)$$

where $d\sigma$ is the area element on $\Sigma_{N+1} = \{(x,t) \in \mathbb{R}^{N+1} : |(x,t)| = 1\}$ ($|\cdot|$ denotes the euclidean norm in $\mathbb{R}^{N+1}$).

PROPOSITION 1.3. *The functional* $z \mapsto \|z\|$ *has the following properties*:

(N_1) $\|D(\lambda)z\| = \lambda\|z\|$ *for every* $z \in \mathbb{R}^{N+1}$, $\lambda > 0$;

(N_2) *The set* $\{z \in \mathbb{R}^{N+1} : \|z\| = 1\}$ *is the euclidean sphere* Σ_{N+1};

(N_3) *For every* $z, \zeta \in \mathbb{R}^{N+1}$

(i) $\|z + \zeta\| \le \|z\| + \|\zeta\|$;

(ii) $\|z \circ \zeta\| \le c(\|z\| + \|\zeta\|)$ *for some constant* $c = c(B) \ge 1$;

(N_4) *There exists a constant* $c = c(B) \ge 1$ *such that for every* $z \in \mathbb{R}^{N+1}$

$$\frac{1}{c}\|z\| \le \|z^{-1}\| \le c\|z\|;$$

(N_5) *For every compact set* K *of* $\mathbb{R}^{N+1}$ *there exists* $c = c(K, B) > 0$ *such that if* $z \in K$ *and* $\|\zeta\| \le 1$,

$$|z \circ \zeta - z| \le c\|\zeta\|.$$

(N_6) *There exists* $\beta = \beta(r) \in (0, 1]$, $c = c(B) > 0$, *and* $M = M(B) \ge 1$ *such that for every* $z, \eta, \zeta \in \mathbb{R}^{N+1}$ *with* $\|\eta^{-1} \circ z\| \ge M\|\zeta^{-1} \circ z\|$,

(i) $\|\eta^{-1} \circ z - \eta^{-1} \circ \zeta\| \le c\|\zeta^{-1} \circ z\|^{\beta}\|\eta^{-1} \circ z\|^{1-\beta}$

(ii) $\|z^{-1} \circ \eta - \zeta^{-1} \circ \eta\| \le c\|\zeta^{-1} \circ z\|^{\beta}\|\eta^{-1} \circ z\|^{1-\beta}$.

Proof. Properties (N_1) and (N_2) follow immediately from (1.1) and (1.6). Property (N_3)(i) is proved in [7]. To complete the proof of the proposition, let us define the "norm"

$$(x,t) \mapsto |||(x,t)||| = |||x|||' + |t|^{1/2} = \sum_{i=1}^{N} |x_i|^{1/\alpha_i} + |t|^{1/2}.$$

Observe that $|||z_1 + z_2||| \le |||z_1||| + |||z_2|||$ and $|||\cdot|||$ satisfies property (N_1); also,

$$\frac{1}{N+1}\|z\| \le |||z||| \le (N+1)\|z\|.$$

Therefore, it is enough to prove the inequalities of Proposition 1.3 for this "norm" $|||\cdot|||$.

To prove (N_3)(ii), observe that

$$\|| z \circ \zeta \|| = \|| y + E(\tau) x \||' + |t + \tau|^{1/2}$$
$$\leq \|| y \||' + \|| E(\tau) x \||' + |t|^{1/2} + |\tau|^{1/2}.$$

Then it is enough to show that

$$\|| E(\tau) x \||' \leq c\big(\|| x \||' + |\tau|^{1/2} \big). \tag{1.9}$$

Let first $\tau > 0$. Then, by (1.4),

$$\|| E(\tau) x \||' = \left\| D_0(\sqrt{\tau}) E(1) D_0\left(\frac{1}{\sqrt{\tau}}\right) x \right\|' = \tau^{1/2} \left\| E(1) D_0\left(\frac{1}{\sqrt{\tau}}\right) x \right\|'. \tag{1.10}$$

Moreover, by the definition of B and $E(1)$, the matrix $E(1)$ is lower triangular; therefore

$$\|| E(1) y \||' = \sum_{i=1}^{N} \left(\left| \sum_{j=1}^{i-1} c_{ij} y_j \right| \right)^{1/\alpha_i} \qquad \left(\text{if } c = \max_{i,j} |c_{ij}|\right)$$
$$\leq \sum_{i=1}^{N} \left(c \cdot \sum_{j=1}^{i-1} |y_j| \right)^{1/\alpha_i} \qquad \left(\text{for } j < i, \frac{1}{\alpha_j} \geq \frac{1}{\alpha_i};\right.$$
$$\left. \text{then } |y_j|^{1/\alpha_i} \leq \left(1 + |y_j|^{1/\alpha_j}\right)\right)$$
$$\leq c(B)(1 + \|| y \||'). \tag{1.11}$$

From (1.10) and (1.11), (1.9) follows, if $\tau > 0$. If $\tau < 0$, an analogous proof can be done, using the matrix $E(-1)$, since, in this case,

$$E(\tau) x = D_0(\sqrt{-\tau}) E(-1) D_0\left(\frac{1}{\sqrt{-\tau}}\right) x.$$

Proof of (N_4).

$$\|| z^{-1} \|| = \|| E(-t) x \||' + |t|^{1/2} \qquad (\text{by } (1.9))$$
$$\leq c\big(\|| x \||' + |\tau|^{1/2} \big) = c \|| z \||.$$

Proof of (N_5). Let $z = (x, t), \zeta = (y, \tau)$.

$$z \circ \zeta - z = \big(y + (E(\tau) - I) x, \tau \big) = \zeta + \big((E(\tau) - I) x, 0 \big). \tag{1.12}$$

Moreover

$$|(E(\tau) - I)x| = \left| \sum_{k=1}^{r} \frac{(-\tau B^T)^k}{k!} x \right| \qquad (\text{since } |\tau| \leq 1)$$
$$\leq |\tau| \sum_{k=1}^{r} \frac{|(B^T)^k x|}{k!} \qquad (\text{since } z \in K)$$
$$\leq c(K, B)|\tau| \tag{1.13}$$

and since $\| \zeta \| \leq 1$, also

$$(|y| + |\tau|) \leq c\| \zeta \|. \tag{1.14}$$

Property (N_5) follows from (1.12)–(1.14).

Proof of (N_6). (i) Setting

$$\zeta^{-1} \circ z = u$$
$$\eta^{-1} \circ z = v,$$

we have to prove that

$$\text{if } \|v\| \geq M\|u\| \quad \text{then} \quad \|v \circ u^{-1} - v\| \leq c\|u\|^{\beta}\|v\|^{1-\beta}.$$

By (N_4) and a suitable choice of M, it is enough to prove

$$\text{if } \|v\| \geq \|u\| \quad \text{then} \quad \|v - v \circ u\| \leq c\|u\|^{\beta}\|v\|^{1-\beta}.$$

Let $\lambda = \|v\|$. Writing

$$v = D(\lambda) D\left(\frac{1}{\lambda}\right) v \equiv D(\lambda) v',$$
$$u = D(\lambda) u',$$

we can assume

$$\|v'\| = 1 \qquad \text{and} \qquad \|u'\| \leq 1.$$

Set $v' = (x, t), u' = (y, \tau)$. From (1.13)

$$\||(E(\tau) - I)x\||' \leq c \sum_{i=1}^{N} |\tau|^{1/\alpha_i} \leq c|\tau|^{1/(2r+1)} \leq c\,\|| u' \||^{1/(2r+1)},$$

so that (1.12) implies

$$\|v' \circ u' - v'\| \leq c\big(\|u'\| + \|u'\|^{\beta}\big) \leq c\|u'\|^{\beta}$$

with $\beta = 1/(2r+1)$.

(ii) Analogously to (i), it is enough to prove that

$$\text{if } \|v'\| = 1 \text{ and } \|u'\| \le 1, \quad \text{then} \quad \|u' \circ v' - v'\| \le c\|u'\|^{\beta}.$$

With the same notations as in (i), we have

$$\|u' \circ v' - v'\| = \|(E(t)y, \tau)\| \le c\left(|||\, E(t)y\, |||' + |\tau|^{1/2}\right).$$

As in (1.11), since $|t| \le 1$, we have

$$|||\, E(t)y\, |||' \le c \sum_{i=1}^{N} \left(\sum_{j=1}^{i-1} |y_j| \right)^{1/\alpha_i} \qquad (\text{since } \|u'\| \le 1)$$

$$\le c \sum_{i=1}^{N} \left(\sum_{j=1}^{i-1} |y_j|^{1/\alpha_j} \right)^{1/\alpha_i} \le c \sum_{i=1}^{N} |||\, y\, |||'^{\,1/\alpha_i} \le c\, |||\, y\, |||'^{\,1/(2r+1)},$$

and (ii) is proved with $\beta = 1/(2r+1)$. ∎

DEFINITION 1.4. For every $z, \zeta \in \mathbb{R}^{N+1}$ define

$$d(z, \zeta) = \|\zeta^{-1} \circ z\|.$$

From properties (N_3)–(N_4) it follows that d is a *quasidistance*, that is;

$$d(z,\zeta) \ge 0, \qquad d(z,\zeta) = 0 \text{ if and only if } z = \zeta;$$

$$\frac{1}{c} d(\zeta, z) \le d(z,\zeta) \le c\, d(\zeta, z) \qquad \text{and}$$

$$d(z,\zeta) \le c\left(d(z,z') + d(z',\zeta)\right)$$

for every $z, \zeta, z' \in \mathbb{R}^{N+1}$, some positive constant $c = c(B)$.

We define the balls with respect to d:

$$\mathscr{B}(z,r) \equiv \mathscr{B}_r(z) \equiv \left\{\zeta \in \mathbb{R}^{N+1} : d(z,\zeta) < r\right\}.$$

Note that $\mathscr{B}(0,r) = D(r)\mathscr{B}(0,1)$.

Remark 1.5. $|\mathscr{B}(z,r)| = |\mathscr{B}(0,r)| = |\mathscr{B}(0,1)| r^{Q+2}$, for every $z \in \mathbb{R}^{N+1}$ and $r > 0$. This fact follows from the invariance of Lebesgue measure (see Remark 1.1), $D(\lambda)$-homogeneity of degree 1 of $\|\cdot\|$ (see (N_1)), and the definition of homogeneous dimension of $\mathbb{R}^{N+1}$ (see (1.1)):

$$|\mathscr{B}(z,r)| = \int_{\|\zeta^{-1}\circ z\|<r} d\zeta = \left(d\zeta = d\zeta^{-1}\right) = \int_{\|\zeta\circ z\|<r} d\zeta$$

$$= (d\zeta = d(\zeta\circ z)) = \int_{\|\zeta\|<r} d\zeta = \int_{\|D(1/r)\zeta\|<1} d\zeta$$

$$= \left(d\left(D\left(\frac{1}{r}\right)\zeta\right) = r^{-Q-2}d\zeta\right)$$

$$= r^{Q+2}\int_{\|\zeta\|<1} d\zeta = r^{Q+2}|\mathscr{B}(0,1)|.$$

This implies that dz is a *doubling measure* with respect to d, since

$$|\mathscr{B}(z,2r)| = 2^{Q+2}|\mathscr{B}(z,r)| \qquad \text{for every } z\in\mathbb{R}^{N+1} \text{ and } r>0$$

and therefore the space $(\mathbb{R}^{N+1}, dz, d)$ is a *homogeneous space*. To be more precise, the standard definition of homogeneous space (see for instance [4]) requires the distance d to be symmetric, while here it is only *quasisymmetric*. However, for the results on homogeneous spaces that we will use (in the proof of Theorem 3.6), this difficulty can be avoided. (See Remark 3.7.)

The quasidistance d allows us to define the spaces BMO (‘‘bounded mean oscillation’’) and VMO (‘‘vanishing mean oscillation’’) in a natural way:

DEFINITION 1.6. For $f\in\mathscr{L}^1_{loc}(\mathbb{R}^{N+1})$, define

$$\|f\|_* = \sup_{\mathscr{B}} ⨍_{\mathscr{B}} |f(z) - f_{\mathscr{B}}|\,dz,$$

where the sup is taken over all the d-balls, $⨍$ denotes average, and $f_{\mathscr{B}} = ⨍_{\mathscr{B}} f(z)\,dz$. Then, by definition,

$$\mathrm{BMO}(\mathbb{R}^{N+1}, L) = \left\{f\in\mathscr{L}^1_{loc}(\mathbb{R}^{N+1}) : \|f\|_* < \infty\right\}.$$

Note that the distance (and therefore the definition of BMO) depends on the operator L, or, more precisely, on the matrix B. Moreover, for $f\in\mathrm{BMO}(\mathbb{R}^{N+1}, L)$, define

$$\eta_f(r) = \sup_{\rho<r} ⨍_{\mathscr{B}_\rho} |f(z) - f_{B_\rho}|\,dz$$

and

$$\mathrm{VMO}(\mathbb{R}^{N+1}, L) = \left\{f\in\mathrm{BMO}(\mathbb{R}^{N+1}, L) : \eta_f(r)\to 0 \text{ for } r\to 0\right\}.$$

2. FUNDAMENTAL SOLUTION FOR THE "FROZEN" OPERATOR AND REPRESENTATION FORMULAS

Let us fix a point $z_0 \in \mathbb{R}^{N+1}$ and call L_{z_0} the operator L "frozen at z_0":

$$L_{z_0} = \sum_{i,j=1}^{q} a_{ij}(z_0)\,\partial_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij}\,x_i\,\partial_{x_j} - \partial_t. \tag{0.2}$$

Recall that, under assumptions (H_1)–(H_2), this operator is hypoelliptic. The fundamental solution for L_{z_0} with pole at zero is (see [9–11])

$$\Gamma^0(x,t) = \frac{1}{(4\pi)^{N/2}\left(\det C(t,z_0)\right)^{1/2}} \exp\left(-\frac{1}{4}\langle C^{-1}(t,z_0)x, x\rangle\right) \tag{2.1}$$

if $t > 0, \Gamma^0(x,t) = 0$ if $t \le 0$; the fundamental solution of L_{z_0} with pole at (y,τ) is the translated of Γ^0 with respect to the group $(\mathbb{R}^{N+1}, \circ)$:

$$\Gamma^0(x,t;y,\tau) = \Gamma^0\left((y,\tau)^{-1}\circ(x,t);0,0\right) = \Gamma^0\left((y,\tau)^{-1}\circ(x,t)\right). \tag{2.2}$$

In (2.1), $C(t,z_0)$ is the matrix

$$C(t,z_0) = \int_0^t E(x)\,A(z_0)\,E^T(s)\,ds,$$

where $A(z_0)$ is the $N \times N$ constant matrix

$$A(z_0) = \begin{bmatrix} A_0(z_0) & 0 \\ 0 & 0 \end{bmatrix}$$

and $A_0(z_0)$ is the $q \times q$ matrix $A_0(z_0) = (a_{ij}(z_0))_{i,j=1,\dots,q}$.

Γ^0 can be rewritten at the following way, using the dilations $D(\lambda)$:

$$\Gamma^0(x,t) = \frac{t^{-Q/2}}{(4\pi)^{N/2}\left(\det C(1,z_0)\right)^{1/2}} \times \exp\left(-\frac{1}{4}\left\langle C^{-1}(1,z_0)\,D_0\left(\frac{1}{\sqrt{t}}\right)x, D_0\left(\frac{1}{\sqrt{t}}\right)x\right\rangle\right) \tag{2.3}$$

for $t > 0$ (see [11]).

Remark 2.1. The matrix $C(t,z_0)$ is symmetric and, since B is nilpotent, is a polynomial in t with coefficients $N \times N$ matrices. Moreover, by conditions (H_1)–(H_2), $C(1,z_0)$ is positive, *uniformly in* z_0. (See [11]). More precisely, set

$$A^- = \begin{bmatrix} \frac{1}{\mu} I_q & 0 \\ 0 & 0 \end{bmatrix}; \qquad A^+ = \begin{bmatrix} \mu I_q & 0 \\ 0 & 0 \end{bmatrix},$$

where μ is the ellipticity constant in (H_1), and

$$C^- = \int_0^1 E(s) A^- E^T(s)\, ds, \qquad C^+ = \int_0^1 E(s) A^+ E^T(s)\, ds.$$

Then

$$C^- \le C(1, z_0) \le C^+ \tag{2.4}$$

for every $z_0 \in \mathbb{R}^{N+1}$, and therefore

$$\det C^- \le \det C(1, z_0) \le \det C^+ \tag{2.5}$$

for every $z_0 \in \mathbb{R}^{N+1}$. Inequality (2.5) can be read as a condition of *uniform subellipticity* for the operator L; the quantity $\det C^-$ is the right substitute for $1/\mu$, in the study of these "subelliptic operators with variable coefficients"; it depends on the number μ and the matrix B.

The next theorem summarizes the properties of the function Γ^0.

THEOREM 2.2. *Let $z_0 \in \mathbb{R}^{N+1}$ and define $\Gamma(z_0, \cdot) \equiv \Gamma^0(\cdot)$. Then*:

(i) $\Gamma^0 \in C^\infty(\mathbb{R}^{N+1} \setminus \{0\})$;

(ii) *Γ^0 is $D(\lambda)$-homogeneous of degree $-Q$. Moreover $\Gamma_i^0 = \partial_{x_i}\Gamma^0$ and $\Gamma_{ij}^0 = \partial_{x_i x_j}\Gamma^0$ are $D(\lambda)$-homogeneous of degree $-Q-\alpha_i$ and $-Q-\alpha_i-\alpha_j$, respectively, for every $i, j = 1, \dots, N+1$. In particular Γ_{ij}^0 is homogeneous of degree $-Q-2$ for $i, j = 1, \dots, q$. (Here $x_{N+1} = t$ and $\alpha_{N+1} = 2$.)*

(iii) *The following estimates hold*

$$|\Gamma_i^0(z)| \le \frac{c}{\|z\|^{Q+\alpha_i}} \qquad \text{and}$$

$$|\Gamma_{ij}^0(z)| \le \frac{c}{\|z\|^{Q+\alpha_i+\alpha_j}} \text{ for every } z \in \mathbb{R}^{N+1} \setminus \{0\},$$

$$i, j = 1, \dots, N+1,$$

$$\text{with } c = \max\left\{ \sup_{\Sigma_{N+1}} |\Gamma_i^0(z)|, \sup_{\Sigma_{N+1}} |\Gamma_{ij}^0(z)|; i, j = 1, \dots, N+1 \right\}.$$

(iv) *For every* $m \in \mathbb{N}$, $z \in \mathbb{R}^{N+1}$, $\beta = (\beta_1, \beta_2, \ldots, \beta_{N+1})$ *a multi-index of height* $|\beta| = \beta_1 + \beta_2 + \ldots + \beta_{N+1}$, *we have*

$$\sup_{\|\zeta\|=1, |\beta|=2m} \left| \left(\frac{\partial}{\partial \zeta} \right)^{\beta} \Gamma_{ij}(z; \zeta) \right| \le c(m, \mu, B) \qquad \text{for every } z \in \mathbb{R}^{N+1}.$$

In particular, the constant c in the previous point (iii) *depends on* μ, B.

(v) *Vanishing property of* Γ^0:

$$\int_{a<\|\zeta\|<b} \Gamma_{ij}^0(\zeta)\, d\zeta = 0 = \int_{\|\zeta\|=1} \Gamma_{ij}^0(\zeta)\, d\sigma(\zeta),$$

for $i, j = 1, \ldots, q, 0 < a < b$.

Proof. Parts (i) and (ii) are obvious from (2.1)–(2.3); (iii) follows from (ii); (iv) follows from (2.3)–(2.5).

Let us prove (v). Using the "polar" change of variables (1.6) and recalling that, by (ii), Γ_{ij}^0 is homogeneous of degree $-Q-2$ for $i, j = 1, \ldots, q$, we get

$$\int_{a<\|\zeta\|<b} \Gamma_{ij}^0(\zeta)\, d\zeta = \log \frac{b}{a} \cdot \int_{\|\zeta\|=1} \Gamma_{ij}^0(\zeta)\, d\sigma(\zeta).$$

Therefore, it is enough to prove that the first integral is zero. Let us write

$$\int_{a<\|\zeta\|<b} \Gamma_{ij}^0(\zeta)\, d\zeta = \int_{\|\zeta\|=b} \Gamma_i^0(\zeta)\nu_j\, d\sigma(\zeta) - \int_{\|\zeta\|=a} \Gamma_i^0(\zeta)\nu_j\, d\sigma(\zeta). \tag{2.6}$$

Moreover

$$\int_{\|\zeta\|=a} \Gamma_i^0(\zeta)\nu_j\, d\sigma(\zeta) = \int_{\|\zeta\|=a, t>0} \ldots + \int_{\|\zeta\|=a, t<0} \ldots = I + II.$$

The surface $\{\|\zeta\| = a, t > 0\}$ can be represented as

$$t \equiv u(x) = a^2 \sqrt{1 - \left(\frac{x_1^2}{a^{2\alpha_1}} + \ldots + \frac{x_n^2}{a^{2\alpha_N}} \right)}$$

so that, for $i, j = 1, \dots, q$,

$$I = \int_{x_1^2/a^{2\alpha_1} + \dots + x_n^2/a^{2\alpha_N} \le 1} \frac{\Gamma_i^0\Big(x_1, \dots, x_N, a^2\sqrt{1 - \big(x_1^2/a^{2\alpha_1} + \dots + x_n^2/a^{2\alpha_N}\big)}\Big)x_j}{a^2\sqrt{1 - \big(x_1^2/a^{2\alpha_1} + \dots + x_N^2/a^{2\alpha_N}\big)}}\, dx$$

(using the change of variable $x = D_0(a)x'$ and the homogeneity of Γ_i^0)

$$= \int_{|x'| \le 1} \Gamma_i^0\Big(x', \sqrt{1 - |x'|^2}\Big)x'_j\, dx' = \int_{\|\zeta\| = a,\, t > 0} \Gamma_i^0(\zeta)\nu_j\, d\sigma(\zeta).$$

II can be handled analogously, so that

$$\int_{\|\zeta\| = a} \Gamma_i^0(\zeta)\nu_j\, d\sigma(\zeta) = \int_{\|\zeta\| = 1} \Gamma_i^0(\zeta)\nu_j\, d\sigma(\zeta).$$

Since this is true for every $a > 0$, the right hand side of (2.6) vanishes, and (v) is proved. ∎

Remark 2.3. By (i), (ii), (v) of Theorem 2.2, the kernel $\Gamma_{ij}^0(z)$ defines a *distribution of kind zero* in the sense of Rothschild and Stein [13, p. 263].

THEOREM 2.4. (Representation Formula for the Second Derivatives). *Let* $u \in C_0^\infty(\mathbb{R}^{N+1})$, $u = 0$ *for* $t \le 0$, $z \in \operatorname{sprt} u$. *Then, for* $i, j = 1, \dots, q$,

$$\begin{aligned} u_{x_i x_j}(z) = &- \lim_{\epsilon \to 0} \int_{\|\zeta^{-1} \circ z\| \ge \epsilon} \Gamma_{ij}(z; \zeta^{-1} \circ z) \\ &\times \left(\sum_{h,k=1}^{q} [a_{hk}(z) - a_{hk}(\zeta)] u_{x_h x_k}(\zeta) + Lu(\zeta) \right) d\zeta \\ &- Lu(z) \cdot \int_{\|\zeta\| = 1} \Gamma_j(z; \zeta)\nu_i\, d\sigma(\zeta), \end{aligned} \tag{2.7}$$

where ν_i *is the ith component of the outer normal to the surface* Σ_{n+1}.

Proof. For $u \in C_0^\infty(\mathbb{R}^{N+1})$, $u = 0$ for $t \le 0$, fix $z_0 \in \operatorname{sprt} u$ and set $g(z) = L_{z_0}u(z)$. Then we can write, for $z \in \operatorname{sprt} u$;

$$u(z) = -\int_{\mathbb{R}^{N+1}} \Gamma^0(\zeta^{-1} \circ z) g(\zeta)\, d\zeta.$$

By (iii) of Theorem 2.2, Γ_i^0 is locally integrable, for $i = 1, \dots, q$. Then

$$u_{x_i}(z) = -\int_{\mathbb{R}^{N+1}} \Gamma_i^0(\zeta^{-1} \circ z) g(\zeta)\, d\zeta.$$

Now, let $\eta \in C_0^\infty(\mathbb{R}^{N+1}), 0 \le \eta \le 1$, such that $\eta(z) = 1$ if $\|z\| \ge 1$ and $\eta(z) = 0$ is some neighborhood of the origin. Set $\eta_\epsilon(z) = \eta(D(1/\epsilon)z)$ and

$$v_\epsilon(z) = -\int_{\mathbb{R}^{N+1}} \eta_\epsilon(\zeta^{-1} \circ z)\Gamma_i^0(\zeta^{-1} \circ z) g(\zeta)\, d\zeta.$$

Clearly, $v_\epsilon(z) \to u_{x_i}(z)$ for $\epsilon \to 0$, and

$$\begin{aligned} D_j v_\epsilon(z) &= -\int_{\mathbb{R}^{N+1}} D_j\big(\eta_\epsilon(\zeta^{-1} \circ z)\Gamma_i^0(\zeta^{-1} \circ z)\big) g(\zeta)\, d\zeta \\ &= -\int_{\|\zeta^{-1} \circ z\| \ge h} \dots - \int_{\|\zeta^{-1} \circ z\| \le h} \dots = -I_1(h,\epsilon) - I_2(h,\epsilon) \end{aligned}$$

for every fixed $h > 0$. Now

$$\begin{aligned} I_1(h,\epsilon) = &\int_{\|\zeta^{-1} \circ z\| \ge h} D_j\big(\eta_\epsilon(\zeta^{-1} \circ z)\big)\Gamma_i^0(\zeta^{-1} \circ z) g(\zeta)\, d\zeta \\ &+ \int_{\|\zeta^{-1} \circ z\| \ge h} \eta_\epsilon(\zeta^{-1} \circ z) D_j\big(\Gamma_i^0(\zeta^{-1} \circ z)\big) g(\zeta)\, d\zeta \qquad (\text{for } \epsilon < h) \\ = &\int_{\|\zeta^{-1} \circ z\| \ge h} \Gamma_{ij}^0(\zeta^{-1} \circ z) g(\zeta)\, d\zeta. \end{aligned}$$

$$\begin{aligned} I_2(h,\epsilon) = &\int_{\|\zeta^{-1} \circ z\| \le h} D_j\big(\eta_\epsilon(\zeta^{-1} \circ z)\Gamma_i^0(\zeta^{-1} \circ z)\big) \cdot (g(\zeta) - g(z))\, d\zeta \\ &+ g(z) \int_{\|\zeta^{-1} \circ z\| \le h} D_j\big(\eta_\epsilon(\zeta^{-1} \circ z)\Gamma_i^0(\zeta^{-1} \circ z)\big)\, d\zeta \\ = &\, I_2'(h,\epsilon) + I_2''(h,\epsilon). \end{aligned}$$

Let us rewrite $I_2'(h,\epsilon)$ as

$$I_2'(h,\epsilon) = \int_{\|w\| \le h} D_j\big(\eta_\epsilon(w)\Gamma_i^0(w)\big) \cdot \big(g(z \circ w^{-1}) - g(z)\big)\, dw.$$

By (N_4)–(N_5) of Proposition 1.3, for $z \in \operatorname{sprt} g$, $\|w\| \le h$ and h small enough so that $\|w^{-1}\| \le 1$,

$$|g(z \circ w^{-1}) - g(z)| \le c|z \circ w^{-1} - z| \le c\|w\|.$$

Moreover

$$|D_j(\eta_\epsilon(w))| \le \frac{c}{\epsilon}.$$

Then,

$$|I_2'(h,\epsilon)| \leq \frac{c}{\epsilon}\int_{\|w\|\leq\epsilon} |\Gamma_i^0(w)|\,\|w\|\,d\zeta + \int_{\|w\|\leq h} |\Gamma_{ij}^0(w)|\,\|w\|\,d\zeta \qquad \text{(by (iii) of Theorem 2.2)}$$

$$\leq c\cdot\epsilon + c\cdot h.$$

$$I_2''(h,\epsilon) = g(z)\int_{\|\zeta^{-1}\circ z\|=h} \eta_\epsilon(\zeta^{-1}\circ z)\Gamma_i^0(\zeta^{-1}\circ z)\nu_j\,d\sigma(\zeta) \qquad (\text{for } \epsilon < h)$$

$$= g(z)\int_{\|\zeta^{-1}\circ z\|=h} \Gamma_i^0(\zeta^{-1}\circ z)\nu_j\,d\sigma(\zeta)$$

(as in the proof of Theorem 2.2)

$$= g(z)\int_{\|\zeta\|=1} \Gamma_i^0(\zeta)\nu_j\,d\sigma(\zeta).$$

Therefore

$$\lim_{\epsilon\to 0} D_j v_\epsilon(z) \qquad (\text{for every fixed } h > 0)$$

$$= -\int_{\|\zeta^{-1}\circ z\|\geq h} \Gamma_{ij}^0(\zeta^{-1}\circ z) g(\zeta)\,d\zeta + O(h) - g(z)\int_{\|\zeta\|=1} \Gamma_i^0(\zeta)\nu_j\,d\sigma(\zeta).$$

Taking limits for $h \to 0$, we can write

$$\lim_{\epsilon\to 0} D_j v_\epsilon(z) = -\lim_{\epsilon\to 0}\int_{\|\zeta^{-1}\circ z\|\geq\epsilon} \Gamma_{ij}^0(\zeta^{-1}\circ z) g(\zeta)\,d\zeta - g(z)\int_{\|\zeta\|=1} \Gamma_i^0(\zeta)\nu_j\,d\sigma(\zeta). \tag{2.8}$$

Since the convergence is uniform in ϵ, we can conclude that $u_{x_i x_j}(z)$ equals the right hand side of (2.8). Finally, writing $g(\zeta) = L_{z_0}u(\zeta) = Lu(\zeta) + (L_{z_0} - L)u(\zeta)$ and letting $z = z_0$, we get (2.7). ∎

In order to rewrite (2.7) in a more compact form, set

$$K_{ij}f(z) = \lim_{\epsilon \to 0} \int_{\|\zeta^{-1} \circ z\| \geq \epsilon} \Gamma_{ij}(z; \zeta^{-1} \circ z) f(\zeta)\, d\zeta; \tag{2.9}$$

$$\alpha_{ij}(z) = \int_{\|\zeta\| = 1} \Gamma_j(z; \zeta)\, \nu_i\, d\sigma(\zeta). \tag{2.10}$$

Moreover, for an operator K and a function $a \in \mathscr{L}^\infty(\mathbb{R}^{N+1})$, define the *commutator*

$$C[K, a](f) = K(af) - a \cdot K(f). \tag{2.11}$$

Then (2.7) becomes

$$u_{x_i x_j} = -K_{ij}(Lu) + \sum_{h,k=1}^{q} C\big[K_{ij}, a_{hk}\big](u_{x_h x_k}) + \alpha_{ij} \cdot Lu \qquad \text{for } i, j = 1, \dots, q. \tag{2.12}$$

Now the desired $\mathscr{L}^p$-estimate on $u_{x_i x_j}$ depends on suitable singular integral estimates. These estimates will be the goal of the next section.

3. SINGULAR INTEGRAL ESTIMATES

To get Theorem 0.1 from (2.12), we shall follow the same line of proof as [1, 3], making use of suitable singular integral estimates, proved in [2] in the context of general homogeneous spaces. These abstract results apply to our situation in virtue of the properties of the fundamental solution Γ, with respect to the suitable homogeneous structure in $\mathbb{R}^{N+1}$. The key estimates to be proved are contained in the following:

THEOREM 3.1. *For every $p \in (1, \infty)$ there exists a positive constant $c = c(p, \mu, B)$ such that for every $a \in \mathrm{BMO}(\mathbb{R}^{N+1}, L)$, $f \in \mathscr{L}^p(\mathbb{R}^{N+1})$, $i, j = 1, \dots, q$,*

$$\|K_{ij}(f)\|_{\mathscr{L}^p(\mathbb{R}^{N+1})} \leq c\|f\|_{\mathscr{L}^p(\mathbb{R}^{N+1})} \tag{3.1}$$

$$\|C\big[K_{ij}, a\big](f)\|_{\mathscr{L}^p(\mathbb{R}^{N+1})} \leq c\|a\|_* \|f\|_{\mathscr{L}^p(\mathbb{R}^{N+1})}. \tag{3.2}$$

We briefly discuss how Theorem 0.1 follows from Theorem 3.1 (see [3, 1] for details). Estimate (3.2) can be localized at the following way: if the function a belongs to VMO (and not only to BMO), then for every $\epsilon > 0$ there exists $r_0 > 0$, depending on ϵ and the VMO modulus of a, such that for every $r \in (0, r_0)$, sprt $f \subseteq \mathscr{B}_r$

$$\|C\big[K_{ij}, a\big](f)\|_{\mathscr{L}^p(\mathscr{B}_r)} \leq c(p, \mu, B) \cdot \epsilon \|f\|_{\mathscr{L}^p(\mathscr{B}_r)}. \tag{3.3}$$

Therefore, from (2.12), (3.1)–(3.3) we have:

THEOREM 3.2. *For every* $p \in (1, \infty)$ *there exists* $c = c(p, \mu, B)$ *and* $r_0 = r_0(p, \mu, \eta, B)$ *such that if* $u \in C_0^\infty(\mathbb{R}^{N+1}), u = 0$ *for* $t \leq 0$, $\operatorname{sprt} f \subseteq \mathscr{B}_r$ *with* $0 < r < r_0$, *then, for* $i, j = 1, \ldots, q$

$$\|u_{x_i x_j}\|_p \leq c\left\{\epsilon \cdot \sup_{h,k} \|u_{x_h x_k}\|_p + \|Lu\|_p\right\},$$

that is,

$$\|u_{x_i x_j}\|_p \leq c\|Lu\|_p. \tag{3.4}$$

(*Remember that* η *stands for the VMO moduli of the coefficients* a_{hk}.)

Finally, with standard techniques of cutoff function and interpolation inequalities (se [1]), from the previous result a local a priori estimate for solutions to the equation $Lu = 0$ in a domain of $\mathbb{R}^{N+1}$ can be derived:

THEOREM 3.3. (Interior Estimates). *For every* $p \in (1, \infty)$, *for every open set* $\Omega' \subset\subset \Omega$, *there exists* $c = c(p, \mu, B, \eta, |\Omega|, \operatorname{dist}(\Omega', \partial\Omega))$ *such that for every* $u \in C_0^\infty(\mathbb{R}^{N+1}), u = 0$ *for* $t \leq 0, i, j = 1, \ldots, q$

$$\begin{gathered} \|u_{x_i x_j}\|_{\mathscr{L}^p(\Omega')} \leq c\left\{\|Lu\|_{\mathscr{L}^p(\Omega)} + \|u\|_{\mathscr{L}^p(\Omega)}\right\}; \\ \|Yu\|_{\mathscr{L}^p(\Omega')} \leq c\left\{\|Lu\|_{\mathscr{L}^p(\Omega)} + \|u\|_{\mathscr{L}^p(\Omega)}\right\}. \end{gathered} \tag{3.5}$$

Note that (0.3) *follows from* (3.5).

Therefore everything relies upon estimates (3.1)–(3.2). To get these, the first thing to do is to expand the singular kernel Γ_{ij} in series of *spherical harmonics*. This is a standard technique dating back to [5]. Let us recall some notation and related properties.

Any homogeneous polynomial of degree m which is harmonic in $\mathbb{R}^{N+1}$ is called an $N+1$-dimensional solid harmonic of degree m; its restriction to Σ_{N+1} is called a spherical harmonic of degree m. The space of $N+1$-dimensional spherical harmonics of degree m has dimension

$$g_m = \binom{m+N}{N} - \binom{m+N-2}{N} \leq c(N) \cdot m^{N-1} \tag{3.6}$$

(with the convention $\binom{K}{N} = 0$ if $K < N$). Let

$$\{Y_{km}\}_{\substack{k=1,\ldots,g_m \\ m=0,1,2,\ldots}}$$

be an orthonormal system of spherical harmonics complete in $\mathscr{L}^2(\Sigma_{N+1})$. Then

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta} Y_{km}(x)\right| \le c(N) \cdot m^{((N-1)/2+|\beta|)}$$
$$\text{for } x \in \Sigma_{N+1}, k = 1, \dots, g_m. \quad (3.7)$$

Moreover, if $f \in \mathscr{C}^{\infty}(\Sigma_{N+1})$ and if $f(x) \sim \sum_{k,m} b_{km} Y_{km}(x)$ is the Fourier expansion of $f(x)$ with respect to $\{Y_{km}\}$, that is,

$$b_{km} = \int_{\Sigma_{N+1}} f(x) Y_{km}(x)\, d\sigma$$

then, for every $r > 1$ there exists c_r such that

$$|b_{km}| \le c_r \cdot m^{-2r} \sup_{\substack{|\beta|=2r \\ x \in \Sigma_{N+1}}} \left|\left(\frac{\partial}{\partial x}\right)^{\beta} f(x)\right|. \quad (3.8)$$

Now, for any fixed $z \in \mathbb{R}^{N+1}$, $\zeta \in \Sigma_{N+1}$, we can write the expansion

$$\Gamma_{ij}(z;\zeta) = \sum_{m=0}^{\infty} \sum_{k=1}^{g_m} c_{ij}^{km}(z) Y_{km}(\zeta) \qquad \text{for } i, j = 1, \dots, q. \quad (3.9)$$

If $\zeta \in \mathbb{R}^{N+1}$, let $\zeta' = D(\|\zeta\|^{-1})\zeta$; by (3.9) and homogeneity of Γ_{ij} we have

$$\Gamma_{ij}(z;\zeta) = \sum_{m=0}^{\infty} \sum_{k=1}^{g_m} c_{ij}^{km}(z) \frac{Y_{km}(\zeta')}{\|\zeta\|^{Q+2}} \qquad \text{for } i, j = 1, \dots, q. \quad (3.10)$$

(This kind of expansion in series of spherical harmonics, for kernels with mixed homogeneities, was first used in [7].)

We note explicitly that $c_{ij}^{km} = 0$ for $m = 0$, by the vanishing property of Γ_{ij}, see (v) of Theorem 2.2. Moreover, by (3.8) and (iv) of Theorem 2.2,

$$|c_{ij}^{km}(z)| \le c(s, \mu, B) \cdot m^{-2s} \qquad \text{for any } s > 1, z \in \mathbb{R}^{N+1}. \quad (3.11)$$

Set

$$K_{km}(z) = \frac{Y_{km}(z')}{\|z\|^{Q+2}}.$$

We are going to study singular integrals defined by the kernels $K_{km}(z)$, and their commutators. Let us point out the main properties of $K_{km}(z)$:

(i) regularity: $K_{km}(z) \in C^{\infty}(\mathbb{R}^{N+1} \setminus \{0\})$;

(ii) homogeneity: $K_{km}(z)$ is $D(\lambda)$-homogeneous of degree $-(Q+2)$;

(iii) growth condition (follows from regularity, homogeneity and (3.7)): for any $z \in \mathbb{R}^{N+1} \setminus \{0\}$,

$$|K_{km}(z)| \leq \frac{c_{km}}{\|z\|^{Q+2}} \qquad \text{with } c_{km} \leq c(N) \cdot m^{(N-1)/2}; \tag{3.12}$$

(iv) vanishing property:

$$\int_{\|\zeta\|=1} K_{km}(\zeta)\, d\sigma(\zeta) = 0. \tag{3.13}$$

Property (3.13) follows from the analogous property of the spherical harmonics of degree ≥ 1, and the fact that the spherical harmonic of degree zero does not appear in the expansion of Γ_{ij}, as we noted after (3.10).

The last important property of $K_{km}(z)$ is expressed in the following

Proposition 3.4. (Hörmander Inequality). *There exist* $\beta = \beta(r) \in (0,1]$, $M = M(B) > 1$, $c_{km} = c(N) \cdot m^{(N+1)/2}$ *such that*

$$|K_{km}(\eta^{-1} \circ \zeta) - K_{km}(\eta^{-1} \circ z)| \leq c_{km} \frac{\|\zeta^{-1} \circ z\|^{\beta}}{\|\eta^{-1} \circ z\|^{Q+2+\beta}} \tag{3.14i}$$

and

$$|K_{km}(\zeta^{-1} \circ \eta) - K_{km}(z^{-1} \circ \eta)| \leq c_{km} \frac{\|\zeta^{-1} \circ z\|^{\beta}}{\|\eta^{-1} \circ z\|^{Q+2+\beta}} \tag{3.14ii}$$

for every $z, \zeta, \eta \in \mathbb{R}^{N+1}$, *with* $\|\eta^{-1} \circ z\| \geq M \|\zeta^{-1} \circ z\|$.

To prove Proposition 3.4, we use the following elementary fact:

Lemma 3.5. *If* $f \in C^1(\mathbb{R}^{N+1} \setminus \{0\})$, *$f$ is* $D(\lambda)$-*homogeneous of degree* α, *and* $\|\ \|$ *is a "norm"* $D(\lambda)$-*homogeneous of degree one, then there exists* $c > 0$

such that

$$|f(u) - f(v)| \leq c \cdot \sup_{\Sigma_{N+1}} |Df| \cdot \|u - v\| \, \|u\|^{\alpha - 1}$$

for every $u, v \in \mathbb{R}^{N+1}$ *with* $\|u - v\| \leq \frac{1}{2}\|u\|$.

Proof of Proposition 3.4. If $\|\eta^{-1} \circ z\| \geq M \|\zeta^{-1} \circ z\|$, for M large enough, property (N_6) of Proposition 1.3 says that

$$\|\eta^{-1} \circ z - \eta^{-1} \circ \zeta\| \leq \tfrac{1}{2}\|\eta^{-1} \circ z\|.$$

Then by Lemma 3.5, letting $u = \eta^{-1} \circ z$, $v = \eta^{-1} \circ \zeta$

$$\begin{aligned} &|K_{km}(\eta^{-1} \circ \zeta) - K_{km}(\eta^{-1} \circ z)| \\ &\quad \leq c \cdot \sup_{\Sigma_{N+1}} |DK_{km}| \cdot \frac{\|\eta^{-1} \circ z - \eta^{-1} \circ \zeta\|}{\|\eta^{-1} \circ z\|^{Q+3}} \qquad \big(\text{by } (N_6) \text{ and } (3.7)\big) \\ &\quad \leq c(N) \cdot m^{(N+1)/2} \cdot \frac{\|\zeta^{-1} \circ z\|^{\beta}}{\|\eta^{-1} \circ z\|^{Q+2+\beta}}. \end{aligned}$$

Analogously (3.14ii) follows from the other inequality in (N_6). ∎

Properties (3.12), (3.13), (3.14) allow us to apply the results in [2] and conclude that:

THEOREM 3.6.

(i) *The operator*

$$T_{km} f(z) = \lim_{\epsilon \to 0} \int_{\|\zeta^{-1} \circ z\| > \epsilon} K_{km}(\zeta^{-1} \circ z) f(\zeta)\, d\zeta$$

is well defined and continuous on $\mathscr{L}^p(\mathbb{R}^{N+1})$ *for every* $p \in (1, \infty)$; *moreover*

$$\|T_{km} f\|_p \leq c(p, N) \cdot m^{(N+1)/2} \, \|f\|_p. \tag{3.15}$$

(ii) *If* $a \in \mathrm{BMO}(\mathbb{R}^{N+1}, L)$, *then the commutator*

$$C[T_{km}, a](f) = T_{km}(af) - a \cdot T_{km}(f)$$

is well defined and continuous on $\mathscr{L}^p(\mathbb{R}^{N+1})$ *for every* $p \in (1, \infty)$; *moreover*

$$\|C[T_{km}, a](f)\|_p \leq c(p, N) \cdot m^{(N+1)/2} \, \|a\|_* \, \|f\|_p. \tag{3.16}$$

We are now ready for the

Proof of Theorem 3.1. By the expansion in spherical harmonics,

$$\|K_{ij}(f)\|_{\mathscr{L}^p(\mathbb{R}^{N+1})}$$
$$\leq \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} \|c_{ij}^{km}\|_\infty \|T_{km} f\|_p \qquad (\text{by } (3.6), (3.11), \text{and } (3.15))$$
$$\leq \sum_{m=1}^{\infty} c(N) \cdot m^{N-1} \cdot c(s, \mu, B) \cdot m^{-2s} \cdot c(p, N) \cdot m^{(N+1)/2} \|f\|_p$$

for any $s > 1$. For a suitable choice of s, the series converges and we get (3.1). Analogously (3.2) follows from (3.6), (3.11), and (3.16).

Remark 3.7. We said that properties (3.12), (3.13), (3.14) allow us to apply the results in [2]. We point out, though, that in the standard definition of homogeneous space, the *symmetry* of d is required, whereas our quasidistance is only *quasisymmetric*, i.e.,

$$\frac{1}{c} d(\zeta, z) \leq d(z, \zeta) \leq c\, d(\zeta, z).$$

As observed in [2], in order to overcome this difficulty, define $d'(z, \zeta) = d(z, \zeta) + d(\zeta, z)$. Clearly, d' is a (symmetric) quasidistance, equivalent to d. Moreover, it is not difficult to check that, by (3.12), (3.13), (3.14), all the assumptions of Theorems 2.5, 4.1, 4.5 in [2] are fulfilled by the kernels K_{km}, with respect to the quasidistance d', with constants controlled by the constants appearing in (3.12), (3.14).

This completes the proof of our main result.

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