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Commutators of Fractional Integrals on Homogeneous Spaces

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0. INTRODUCTION

Let (X,d,μ) be a homogeneous space, I_α the fractional integral operator of exponent $\alpha \in (0,1)$ defined by

$$I_\alpha f(x) \;=\; \int_{X\setminus\{x\}} k_\alpha(x,y) f(y)\, d\mu(y)$$

with $k_\alpha(x,y) = \mu(B(x,d(x,y)))^{\alpha-1}$. In this paper we study the commutator

$$C[I_\alpha,a](f) \;=\; a\cdot I_\alpha f \;-\; I_\alpha(af),$$

where $a \in BMO(X)$, and we prove that, under a suitable geometric assumption on (X,d), it satisfies the $\mathcal{L}^p$-$\mathcal{L}^q$ estimate

$$\| \, C[I_\alpha,a](f) \, \|_q \;\le\; c \, \| \, a \, \|_* \, \| \, f \, \|_p$$

for $p \in (1, \frac{1}{\alpha})$, $\frac{1}{q} = \frac{1}{p} - \alpha$, where $\| \cdot \|_*$ is the BMO seminorm and $c = c(\alpha,p,X)$. We actually prove a stronger result which improves known results even in the euclidean case; namely we prove that the above estimate holds for the operator C_α^a defined by

$$C_\alpha^a(f)(x) \;= I_\alpha\Big(\,|\, a(x) - a(\cdot) \,|\, f(\cdot) \Big)(x).$$

This in particular implies that the estimate holds if in the definition of I_α we replace the fractional integral kernel k_α by any equivalent one.

1. DEFINITIONS AND STATEMENT OF RESULTS

Let X be a nonempty set. A function $d{:}X\times X \to [0,\infty)$ is called *quasidistance* if:

i) $d(x,y) \;=\; 0 \;\Leftrightarrow\; x = y$;

ii) $d(x,y) \;=\; d(y,x)$;

iii) there exists a constant $c_d \ge 1$ such that for every $x,\, y,\, z \in X$

$$d(x,y) \;\le\; c_d[d(x,z) + d(z,y)]. \tag{1.1}$$

If (X,d) is a set endowed with a quasidistance, the "balls" $B_r(x) \equiv B(x,r) = \Big\{ y \in X{:}\, d(x,y) < r \Big\}$ (for $x \in X$ and $r > 0$) always induce a topology in X.

Among the different definitions of homogeneous space that appear in the literature, here we use the one in [CW1], [MS], [Bu]. We say that (X,d,μ) is a *homogeneous space* if:

i) X is a set endowed with a quasidistance d, such that the balls are open sets in the topology induced by d (in particular, they form a base);

ii) μ is a positive Borel measure on X, satisfying the *doubling condition*:

$$0 \;<\; \mu(B_{2r}(x)) \;\le\; c_\mu \cdot \mu(B_r(x)) \;<\infty \tag{1.2}$$

for every $x \in X$, $r > 0$, some constant $c_\mu > 1$.

For any $\alpha \in (0,1)$, set

$$k_\alpha(x,y) \;=\; \mu(B(x;y))^{\alpha-1} \quad \text{where} \quad B(x;y) = B(x,\, d(x,y)) \tag{1.3}$$

and define the fractional integral operator

$$I_\alpha f(x) \;=\; \int_{X\setminus\{x\}} k_\alpha(x,y) f(y)\, d\mu(y) \tag{1.4}$$

for any measurable function f defined on X.

Fractional integrals on homogeneous spaces have been studied first by Gatto and Vagi (see [GV1], [GV2]), who proved in this context results analogous to those which hold in the euclidean case ($X = \mathbb{R}^n$,

$k_\alpha(x,y) = |x-y|^{n(\alpha-1)}$). The result which is most important to us, although stated in [GV2] with more restrictive hypotheses on X, can be restated as follows (in view of [MS]):

Theorem 1.1. (Gatto and Vagi).
Let $\alpha \in (0,1)$, $p \in (1, \frac{1}{\alpha})$, $\frac{1}{q} = \frac{1}{p} - \alpha$. (So $1 < p < q < +\infty$). Then there exists a constant $c = c(c_d, c_\mu, \alpha, p)$ such that for every $f \in \mathcal{L}^p(X)$

$$\| I_\alpha f \|_q \leq c \| f \|_p. \tag{1.5}$$

This result has been recently extended by Sawyer and Wheeden (see [SW]), who proved some weighted estimates for fractional integrals on (euclidean and) homogeneous spaces. Here we study the commutator of the fractional integral operator I_α with a BMO function. In another paper (see [BC]) we study the commutator of singular integral operators of Calderon-Zygmund type, on homogeneous spaces. We assume the reader is familiar with the definition and basic properties of BMO in the context of homogeneous spaces (see [Bu]). For $a \in BMO(X)$, define the commutator

$$C[I_\alpha, a](f)(x) = a(x) \cdot I_\alpha f(x) - I_\alpha(af)(x) = \int_{X \setminus \{x\}} k_\alpha(x,y)\, [a(x) - a(y)] f(y)\, d\mu(y).$$

Also, set

$$C_a^\alpha f(x) = \int_{X \setminus \{x\}} k_\alpha(x,y)\, |a(x) - a(y)|\, f(y)\, d\mu(y).$$

In order to state our main results, we need two more definitions.

Definition 1.2. For every $x \in X$, set:

$$R_x = \inf\left\{R \colon B_R(x) = X\right\}; \quad r_x = \sup\left\{r \colon B_r(x) = \{x\}\right\}$$

(with the conventions $\sup \emptyset = 0$, $\inf \emptyset = +\infty$).
Note that $R_x < +\infty$ if and only if X is bounded; $r_x > 0$ if and only if $\mu\{x\} > 0$, that is x is an "atom". (This follows from results in [MS]).

Definition 1.3. We say that (X,d) satisfies the Property (P) if there exists $m \in (0,1)$ such that for every $x \in X$, every number $r \in (r_x, \frac{R_x}{m})$,

$$B_r(x) \setminus B_{mr}(x) \neq \emptyset. \tag{P}$$

Throughout the paper, we will write $c(X)$ for a constant depending on the the numbers c_d, c_μ appearing in (1.1)-(1.2) and m in (P).

Property (P) requires that certain annuli be nonempty. Actually, it requires that for any empty annulus the ratio between the outer and inner radii is bounded. Note that if X is bounded or has atoms, there are always *some* empty annuli. Nevertheless, condition (P) can still hold (see example A.1 in the *Appendix*). But condition (P) can fail to be true in some pathological cases (see example A.2).

The following two theorems contain our main results.

Theorem 1.4. Let (X,d,μ) be a homogeneous space satisfying property (P), α, p, q as in Thm.1.1, i.e. $\alpha \in (0,1)$, $p \in (1, \frac{1}{\alpha})$, $\frac{1}{q} = \frac{1}{p} - \alpha$, $a \in BMO(X)$. Then there exists a constant $c = c(X, \alpha, p)$ such that for every $f \in \mathcal{L}^p(X)$

$$\| C_a^\alpha f \|_q \leq c \| a \|_* \| f \|_p. \tag{1.6}$$

($\| \cdot \|_*$ denotes the BMO seminorm).

Theorem 1.5. Under the same assumptions of Thm.1.4, we also have

$$\| C[I_\alpha, a](f) \|_q \leq c \| a \|_* \| f \|_p. \tag{1.7}$$

Moreover, the same kind of estimate holds if the kernel k_α is replaced with any equivalent kernel $\widetilde{k}_\alpha$ (that is, such that $c_1 \cdot k_\alpha(x,y) \leq \widetilde{k}_\alpha(x,y) \leq c_2 \cdot k_\alpha(x,y)$ for some positive constants c_1, c_2).

Theorem 1.5 obviously follows from Theorem 1.4 and it generalizes to homogeneous spaces an analogous result proved by Chanillo for fractional integrals in $\mathbb{R}^n$ (see [Cha]). Thm. 1.4 is a stronger result, which is new even in the euclidean case. Finally, we note that the second part of the statement of Thm. 1.5

does not follow from the first part, without assuming Thm. 1.4. The interest of this fact is that fractional integrals on homogeneous spaces may be actually defined by different (equivalent) kernels.

Before closing this section we point out two results concerning the above definitions.

Lemma 1.6. If X is bounded, then for every $x, y \in X$ the numbers R_x, R_y are equivalent:

$$\frac{1}{4c_d} R_y \leq R_x \leq 4c_d R_y.$$

Proof. Observe that for every $y \in X$, $B_{2R_y}(y) \equiv X$. For fixed x, let x_k be a sequence of points such that $d(x,x_k) \to R_x$; then

$$R_x = \lim_{k \to \infty} d(x,x_k) \leq \limsup_{k \to \infty} c_d \left(d(x,y) + d(y,x_k) \right) \leq 4c_d R_y.$$

Interchanging the roles of x, y we obtain the reverse inequality. □

Lemma 1.7. If condition (P) holds for some $m \in (0,1)$, then it holds for any positive $m' < m$.

Proof. It is enough to show that (P) holds for $m^2 < m' < m$ (then by iteration it holds for any $m' < m$). For $x \in X$, let $r \in (r_x, \frac{R_x}{m'})$. If $r < \frac{R_x}{m}$, then $B_r \backslash B_{m'r} \supseteq B_r \backslash B_{mr} \neq \emptyset$ by (P). If $\frac{R_x}{m} \leq r < \frac{R_x}{m'}$, let $r' = \frac{m'}{m} r$, then $r' < \frac{R_x}{m}$. Also $r' \geq \frac{m' R_x}{m^2} > R_x \geq r_x$. Therefore $r' \in (r_x, \frac{R_x}{m})$ and hence $B_{r'} \backslash B_{mr'} \neq \emptyset$. But $B_{mr'} = B_{m'r}$ and $B_{r'} \subseteq B_r$ so that $B_r \backslash B_{m'r} \supseteq B_{r'} \backslash B_{mr'} \neq \emptyset$. □

The proof of Thm. 1.4 consists in two steps. The first one (§ 2) consists in showing that the estimate on $C_a^\alpha f$ follows from a suitable *weighted* estimate on $I_\alpha f$. This idea was suggested by Coifman-Rochberg-Weiss in [CRW] as a possible "alternative proof" of their result about commutators of singular integrals in $\mathbb{R}^n$. The second step (§ 3), consists then in proving this weighted estimate. This will follow from the weighted estimates of Sawyer-Wheeden (see Thm 3.1 and [SW]), modified in order to include our more general geometric assumptions on the space X.

2. OUTLINE OF THE PROOF OF THM. 1.4.

Observe, first of all, that it is enough to prove the theorem for $f \geq 0$ and $a \in \mathcal{L}^\infty$, since for $a \in BMO$ there exists a sequence $\{a_k\} \subset \mathcal{L}^\infty$ such that $a_k \to a$ pointwise a.e. and $\| a_k \|_* \leq c \| a \|_*$ for some absolute constant c; therefore by Fatou's theorem:

$$\| C_a^\alpha f \|_q \leq \| C_a^\alpha (|f|) \|_q \leq \liminf_{k \to \infty} \| C_{a_k}^\alpha (|f|) \|_q \leq$$

$$\leq c \cdot \liminf_{k \to \infty} \| a_k \|_* \| f \|_p \leq c \| a \|_* \| f \|_p.$$

Claim 2.1. For $a \in \mathcal{L}^\infty$, $z \in \mathbb{C}$, let

$$b(x) = exp\,(Re\, z \cdot a(x)) \tag{2.1}$$

and suppose we know that the operator I_α satisfies for some $\epsilon > 0$, every $z \in \mathbb{C}$ with $|z| < \epsilon$, the weighted estimate:

$$\| I_\alpha f \|_{\mathcal{L}^q(b^q d\mu)} \leq c \cdot \| f \|_{\mathcal{L}^p(b^p d\mu)} \tag{2.2}$$

with $c = c(X,p,q,\epsilon, \| a \|_*)$, p and q as in Thm. 1.1. Then

$$\| C_a^\alpha f \|_q \leq c \| a \|_* \| f \|_p \tag{2.3}$$

with $c = c(X,p,q,\epsilon)$.

Proof. For $f \geq 0$, define:

$$T_z f(x) = \int_{X \backslash \{x\}} e^{z(a(x)-a(y))} k_\alpha(x,y) f(y)\, sgn\,(a(x) - a(y))\, d\mu(y). \tag{2.4}$$

Then

$$| T_z f(x) | \leq b(x) \int_{X \backslash \{x\}} k_\alpha(x,y) f(y)\, b^{-1}(y)\, d\mu(y) \tag{2.5}$$

so that

$$\| T_z f \|_q \leq \| I_\alpha(b^{-1} f) \|_{\mathcal{L}^q(b^q d\mu)} \leq \text{(by (2.2))}$$

$$\leq c \cdot \| b^{-1} f \|_{\mathcal{L}^p(b^p d\mu)} = c \cdot \| f \|_p \tag{2.6}$$

for $|z| < \epsilon$, with $c = c(X,p,q,\epsilon, \| a \|_*)$.

Therefore $F: z \mapsto T_z$ is a mapping from the disk $D_\epsilon = \{z \in \mathbb{C}: |z| < \epsilon\}$ into the space of linear continuous operators $\mathcal{L}(\mathcal{L}^p, \mathcal{L}^q)$. Let us show that F is analytic in D_ϵ. Writing

$$\frac{T_{z+h} f(x) - T_z f(x)}{h} =$$

$$= \int_{X \setminus \{x\}} \left(\frac{e^{h(a(x)-a(y))} - 1}{h} \right) e^{z(a(x)-a(y))} k_\alpha(x,y) f(y)\, sgn\,(a(x) - a(y))\, d\mu(y),$$

since

$$\left| \frac{e^{h(a(x)-a(y))} - 1}{h} \right| \leq 2 \| a \|_\infty + o(1) \quad \text{as } h \to 0,$$

and since the integral defining $T_z f$, by (2.6), converges absolutely for a.e. x, if $|z| < \epsilon$, we get:

$$\frac{d}{dz} T_z f(x) = \int_{X \setminus \{x\}} e^{z(a(x)-a(y))} \, | a(x) - a(y) | \, k_\alpha(x,y) f(y)\, d\mu(y) \tag{2.7}$$

for $|z| < \epsilon$.

Then by the Cauchy's integral formula we can write

$$\frac{d}{dz} T_z f(x) \Big|_{z=0} = \frac{1}{2\pi i} \int_{|w| = \frac{\epsilon}{2}} \frac{T_w f(x)}{w^2}\, dw. \tag{2.8}$$

But the left hand side of (2.8) is exactly $C_a^\alpha f(x)$, by (2.7), so we have:

$$\| C_a^\alpha f \|_q \leq c(\epsilon) \cdot \sup_{|w| = \frac{\epsilon}{2}} \| T_w f \|_q \leq \text{(by (2.6))} \leq c \, \| f \|_p \tag{2.9}$$

with $c = c(X,p,q,\epsilon, \| a \|_*)$.

Finally, note that the constant c in (2.9) must actually be of the form: $c(X,p,q,\epsilon) \cdot \| a \|_*$. To see this, observe that from (2.9) we have

$$\| C_a^\alpha f \|_q \leq c(X,p,q,\epsilon) \, \| f \|_p \tag{2.10}$$

for every a with $\| a \|_* = 1$. But then, for every $a \in \mathcal{L}^\infty$, applying (2.10) to $\frac{a}{\|a\|_*}$ we get

$$\| C_a^\alpha f \|_q \leq c(X,p,q,\epsilon) \, \| a \|_* \, \| f \|_p.$$

This completes the proof of the claim. □

3. WEIGHTED INEQUALITIES FOR FRACTIONAL INTEGRALS

In order to prove Theorem 1.4 we now need to prove the weighted inequality (2.2). This result is actually contained as a particular case in the following theorem by Sawyer-Wheeden (see [SW]):

Theorem 3.1. Suppose $1 < p \leq q < \infty$, $k: X \times X \to \mathbb{R}^+$ is a measurable function,

$$Kf(x) = \int_{X \setminus \{x\}} k(x,y) f(y)\, d\mu(y).$$

For a suitable positive number $h \leq m$, define

$$\phi(B) = \sup \left\{ k(z,y) : z, y \in B,\ d(z,y) \geq h\, r \right\}, \tag{3.1}$$

for every ball B of the kind

$$B_r(x) \quad \text{with} \quad r \in (r_x, \tfrac{R_x}{h}). \tag{3.2}$$

Assume further:

(i) v and w are two nonnegative measurable functions, such that for some $s > 1$, $w^s\, d\mu$ and $v^{(1-p')s}\, d\mu$ are doubling measures and satisfy:

$$\phi(B)\,\mu(B)^{\frac{1}{q}+\frac{1}{p'}}\left(⨍_B w^s\,d\mu\right)^{\frac{1}{qs}}\left(⨍_B v^{(1-p')s}\,d\mu\right)^{\frac{1}{p's}}\;\leq\;C_s \tag{3.3}$$

for all balls satisfying (3.2), with C_s independent of B.

(ii) There exists $\beta > 0$ such that

$$\frac{\mu(B')}{\mu(B)}\;\leq\;c\left(\frac{\phi(B)}{\phi(B')}\right)\left(\frac{r(B')}{r(B)}\right)^{\beta} \tag{3.4}$$

for all pairs of balls $B' \subseteq B$ and B', B satisfying (3.2).

Then:

$$\|\,Kf\,\|_{\mathcal{L}^q(w\,d\mu)}\;\leq\;c\;\|\,f\,\|_{\mathcal{L}^p(v\,d\mu)}$$

where c depends on all the constants appearing in the assumptions.

Observe that, by property (P) and Lemma 1.7, if $h \leq m$, the set $\{(y,z) \in B\times B:\ d(z,y) \geq h\,r\}$ is nonempty, so $\phi(B)$ is well defined. The above statement is not exactly the one proved in [SW]. More precisely, they define $\phi(B)$ for *any* ball (not necessarily satisfying (3.2)) and require (3.3)-(3.4) to hold for any ball. On the other hand, to make sense of $\phi(B)$ they require X to have no empty annuli, condition that implies in particular that X is unbounded and has no atoms. We shall show, at the end of this section, that it is still possible to prove their theorem if the assumptions on X are relaxed requiring only certain annuli to be nonempty (property (P)), if (3.3) and (3.4) hold only for balls satisfying (3.2).

In order to get the estimate (2.2), we will apply Thm. 3.1 to $k = k_\alpha$, with $p,\ q,\ \alpha$ as in Thm. 1.1, $w = b^q$, $v = b^p$ and b as in (2.1). Therefore we need to check assumptions (3.3)-(3.4), and show that the constants depend on a only through its BMO seminorm.

In all the following lemmas, X will be a homogeneous space satisfying (P).

Lemma 3.2. (Reverse Doubling Condition). There exist two constants $\delta(X) > 0,\ K(X) > 1$ such that

$$\mu(B_{Kr}(x)) \geq (1+\delta)\cdot\mu(B_r(x)) \tag{3.5}$$

for every $x \in X$ and $r \in \left(\frac{m}{2c_d}\,r_x\,,\ \frac{m}{3c_d^2}\,R_x\right)$. (Recall m is the constant in property (P)).

Proof.

Let $r \in \left(\frac{m}{k}\,r_x\,,\ \frac{R_x}{K}\right)$ with $1 < k < K$ to be determined. Condition (P) implies that

$$\exists\, y \in X \text{ such that } kr < d(x,y) < \frac{kr}{m}.$$

We will show that $\qquad \exists\,\bar r > 0$ such that $B_{\bar r}(y) \subseteq B_{Kr}(x)\backslash B_r(x)$, *i.e.*: (3.6)

$a)$ we want $B_{\bar r}(y) \subseteq B_{Kr}(x)$. Let $w \in B_{\bar r}(y)$, then

$$d(x,w) \leq c_d\left(d(x,y)+d(y,w)\right)\;<\;c_d\left(\frac{k}{m}r+\bar r\right)$$

then $w \in B_{Kr}(x)$ if $c_d\left(\frac{k}{m}r+\bar r\right) \leq Kr$, i.e. $\bar r \leq \left(\frac{K}{c_d}-\frac{k}{m}\right)r$, which implies $K > \frac{c_d}{m}\,k$;

$b)$ we also want $B_{\bar r}(y)\cap B_r(x) = \emptyset$. Let again $w \in B_{\bar r}(y)$, then

$$kr < d(x,y) \leq c_d\left(d(x,w)+d(w,y)\right)\;<\;c_d\left(d(x,w)+\bar r\right).$$

Therefore $d(x,w) > \frac{kr}{c_d}-\bar r$, and $w \notin B_r(x)$ if $\frac{kr}{c_d}-\bar r \geq r$, that is $\bar r \leq \left(\frac{k}{c_d}-1\right)r$, which implies $\frac{k}{c_d}-1 > 0.$

By (a) and (b), if we choose $k = 2c_d$ and $K = \frac{3c_d^2}{m}$, (3.6) holds for $\bar r = r$. Moreover, reasoning as in (a), we find that $B_{Kr}(y) \supseteq B_r(x)$. Therefore:

$$\mu(B_{Kr}(x))\;\geq\;\mu(B_r(x))\;+\;\mu(B_r(y))\;\geq\;\text{(by doubling)}$$

$$\geq\mu(B_r(x))\;+\;c(c_d,c_\mu,m)\cdot\mu(B_{Kr}(y))\;\geq\;(1+\delta)\cdot(B_r(x)).\qquad\square$$

Lemma 3.3. For every $\eta \in (0,1)$ there exist two positive constants $c,\ \gamma$ depending on X and η such that for every $x \in X$

$$\frac{\mu(B_R(x))}{\mu(B_r(x))} \geq c\left(\frac{R}{r}\right)^{\gamma} \tag{3.7}$$

if $\eta r_x \leq r \leq R \leq \frac{1}{\eta} R_x$.

Proof. By Lemma 3.2, (3.5) holds in particular *for* $r \in (r_x, \frac{R_x}{c})$, where c is a suitable constant > 1. Let K be as in (3.5), fix R, r as above and choose an integer n such that $K^n r < R \leq K^{n+1} r$. Then from (3.5) we obtain that, for $r \in \left(r_x, \frac{R_x}{cK^{n-1}}\right)$

$$\mu(B_R(x)) \geq \mu(B_{K^n r}(x)) \geq (1+\delta)^n \mu(B_r(x)).$$

Since $log_K \left(\frac{R}{rK}\right) \leq n$,

$$(1+\delta)^n \geq (1+\delta)^{log_K\left(\frac{R}{rK}\right)} = \left(\frac{R}{rK}\right)^{log_K(1+\delta)} = c\left(\frac{R}{r}\right)^{\gamma}.$$

So (3.7) is proved for $r > r_x$ and $R \leq K^{n+1} r \leq \frac{K^2 R_x}{c}$, that is for $r_x \leq r \leq R \leq c_1 R_x$, where c_1 can be assumed < 1.

Now take $r_x \leq r \leq R$ with $c_1 R_x \leq R \leq \frac{R_x}{\eta}$. Then if $r < c_1 \eta R$, by the above proof:

$$\frac{\mu(B_R(x))}{\mu(B_r(x))} \geq \frac{\mu(B_{c_1\eta R}(x))}{\mu(B_r(x))} \geq c \cdot \left(\frac{c_1 \eta R}{r}\right)^{\beta} = c(\eta) \cdot \left(\frac{R}{r}\right)^{\beta}.$$

If $c_1 \eta R \leq r \leq R$, then $\mu(B_R(x))$ and $\mu(B_r(x))$ are equivalent, by the doubling condition; therefore (3.5) is proved for $r_x \leq r \leq R \leq \frac{R_x}{\eta}$. Analogously we can prove the same for $\eta r_x \leq r \leq R \leq \frac{1}{\eta} R_x$. □

Lemma 3.4. There exist two positive constants c_1, c_2 depending on X such that if ϕ is as in (3.1) and $B = B_r(x)$ is any ball as in (3.2)

$$c_1\, \mu(B)^{\alpha-1} \leq \phi(B) \leq c_2\, \mu(B)^{\alpha-1}.$$

Proof. Let $(y,z) \in B \times B$ such that $d(z,y) \geq h\, r$. By triangle inequalities we can write

$$B(y, d(y,z)) \subseteq B(y, 2c_d r) \subseteq B(x, c_d(1+2c_d)\, r).$$

Then by doubling

$$\mu\Big(B(y, d(y,z))\Big) \leq c(c_d, c_\mu) \cdot \mu(B(x,r))$$

and

$$k_\alpha(y,z) \geq c \cdot \mu(B)^{\alpha-1}$$

so that

$$\phi(B) \geq c_1 \cdot \mu(B)^{\alpha-1}.$$

Conversely, since $d(z,y) \geq h\, r$ and, again by triangle inequalities and doubling,

$$\inf_{y \in B_r(x)} \mu(B(y,hr)) \geq c(c_d, c_\mu) \cdot \mu(B(x,r)),$$

we have

$$\mu\Big(B(y, d(y,z))\Big) \geq c\, \mu(B)$$

and therefore

$$\phi(B) \leq c_2 \cdot \mu(B)^{\alpha-1}.$$ □

Theorem 3.5. Conditions (3.3)-(3.4) in Thm. 3.1 hold under the assumptions of Thm. 1.4.

Proof. First, we need to bound the quantity at the left hand side of (3.3). Note that for $\frac{1}{q} = \frac{1}{p} - \alpha$, by Lemma 3.4:

$$\phi(B)\, \mu(B)^{\frac{1}{q}+\frac{1}{p'}} \leq c. \tag{3.8}$$

Recall that $w = b^q$, $v = b^p$, $b(x) = exp\,(Re\, z \cdot a(x))$ with $a \in \mathcal{L}^\infty$, $z \in \mathbb{C}$.
By the John-Nirenberg Lemma, for every $p \in (1,\infty)$ there exists $\epsilon = \epsilon(p, \| a \|_*)$ such that $exp(\epsilon\, a(x)) \in A_p$, with A_p constant bounded by $c(p, \| a \|_*)$. Therefore for every $p \in (1,\infty)$, $\beta > 0$, there exists

$\epsilon = \epsilon(p,\beta, \parallel a \parallel_*)$ such that for any $z \in \mathbb{C}$ with $|z| < \epsilon$, $b^\beta \in A_p$ with A_p constant bounded by $c = c(p,\beta, \parallel a \parallel_*)$.

Let $u \equiv w^s = b^{sq}$. Then we can write

$$v^{(1-p')s} = b^{p(1-p')s} = u^{\frac{p(1-p')}{q}} = u^{-\frac{1}{t-1}}$$

with $t = \frac{q}{p'} + 1$. Therefore

$$\left(⨍_B w^s \, d\mu\right)^{\frac{1}{qs}} \left(⨍_B v^{(1-p')s} \, d\mu\right)^{\frac{1}{p's}} = \left(\left(⨍_B u \, d\mu\right)\left(⨍_B u^{-\frac{1}{t-1}} \, d\mu\right)^{t-1}\right)^{\frac{1}{qs}} \leq c \tag{3.9}$$

with c depending on q, s and the A_t constant of u, that is of b^{sq}. By the above remark, there exists $\epsilon = \epsilon(p,q, \parallel a \parallel_*)$ such that for $|z| < \epsilon$ (3.9) holds, with $c = c(p,q, \parallel a \parallel_*)$. Then (3.3) follows from this fact and (3.8).

As to (3.4), in view of Lemma 3.4 it is enough to prove

$$\frac{\mu(B')}{\mu(B)} \leq c \cdot \left(\frac{r(B')}{r(B)}\right)^\beta$$

for some $\beta > 0$, and $B = B(x_0,r_0)$, $B' = B(x_1,r_1) \subseteq B$. By the triangle inequality and doubling:

$$\mu(B(x_1,r_0)) \leq c(c_d,c_\mu)\, \mu(B(x_0,r_0))$$

Then by Lemma 3.3, for $r_1 > r_{x_1}$, $r_0 < \frac{R_{x_1}}{\eta}$

$$\frac{\mu(B')}{\mu(B)} \leq c \cdot \frac{\mu(B')}{\mu(B(x_1,r_0))} \leq c \cdot \left(\frac{r_1}{r_0}\right)^\beta. \tag{3.10}$$

Note that by (3.2) and Lemma 1.6, $r_0 < \frac{R_{x_0}}{h} < \frac{R_{x_1}}{\eta}$, for η small enough (independent of x_0, x_1), and this proves (3.4) under our assumptions. □

Finally we need to show that Sawyer-Wheeden's proof can be adapted to hold in the hypotheses of Theorem 3.1. In [SW], they construct a family of "dyadic balls", taking for each $k \in \mathbb{Z}$ $\left\{\hat{B}_j^k\right\}_j$ to be a sequence of balls of radius λ^{k-1}, maximal with respect to the property that $\hat{B}_j^k \cap \hat{B}_i^k = \emptyset$ for $i \neq j$, where $\lambda = c_d + 2c_d^2$. If X is bounded (case implicitely excluded in [SW]) we need to modify slightly this construction. Recalling that, by Lemma 1.6, if X is bounded there exists a constant K such that $R_x \leq K$ for every $x \in X$, let $k_0 \in \mathbb{Z}$ be such that $\lambda^{k_0-1} \leq K < \lambda^{k_0}$. Then for every $k \geq k_0$ and every j, $B_j^k = B(x_j^k, \lambda^k) = X$. Define dyadic balls setting $B_j^k = B_j^{k_0}$ for $k \geq k_0$, every j; these balls have the same properties as those of [SW], therefore we can say that for, every dyadic ball $B(x,r)$,

$$r < M \cdot R_x \tag{3.11}$$

with M independent of x and r. We now choose the number h appearing in Thm. 3.1 (see (3.2)) as

$$h = min\left(m,\, c_d^2/4,\, 1/M\right). \tag{3.12}$$

By Lemma 1.7, Lemmas 3.2-3.4 and Theorem 3.5 still hold if m is replaced by h. The condition $h \leq c_d^2/4$ is required by the proof in [SW], and the condition $h \leq m$ is required to make sense of the function ϕ in (3.2). Moreover in [SW]'s proof the dyadic balls that need to satisfy (3.3)-(3.4) are only those with $r < \frac{R_x}{h}$.

With this remark, the proof of Theorem 1.4 is completed.

APPENDIX. Some examples

Example A.1. ***A bounded space, with atoms, satisfying condition (P).***

Let $X \subset \mathbb{R}^n$, $X = \{0\} \cup \{x: |x| = 1\}$; put in X the euclidean distance and the following measure μ: μ is the usual surface measure on $\{|x| = 1\}$ and $\mu\{0\} = 1$. Then μ is doubling, so that (X,d,μ) is a homogeneous space. Note that for $|x| = 1$, $r_x = 0$ and $R_x = 2$; for $x = 0$, $r_0 = R_0 = 1$. In this case (P) holds for any $m \in (0,1)$.

Example A.2. ***A homogeneous space such that (P) and (RD) do not hold.***
In other words, we show that the *doubling* condition does not imply the *reverse doubling* condition.

In $\mathbb{R}^n$, let C_k $(k = 1,2,\dots)$ be the point $(k^k + \frac{1}{2},0,\dots,0)$, B_k the ball $B_{1/2}(C_k)$ for $k \geq 2$, $B_1 = B_{1/2}(0)$. Let $X = \bigcup_{k=1}^{\infty} B_k$ with the euclidean distance d and the measure μ such that $\mu(B_k) = 2^k$ and, on each ball B_k, μ is uniformly distributed.

Claim 1. μ does not satisfy the (RD) condition. To see this, observe that the annuli $\left\{b_k < |x| < a_{k+1}\right\}$, with $a_k = k^k$, $b_k = k^k + 1$, are empty, and $\frac{a_{k+1}}{b_k} = \frac{(k+1)^{k+1}}{k^k+1} \to \infty$. Therefore for every $M > 0$ there exists k_0 such that $a_{k_0+1} \geq M b_{k_0}$ and therefore $B_{Mb_{k_0}}(0) = B_{b_{k_0}}(0)$.

Claim 2. μ satisfies the doubling condition. Let $B_r = B_r(P)$, with $P = (p_1,\dots,p_n)$.

Case 1. Assume for some k, $B_k \subseteq B_r$ and let $k_0 = max\left\{k \mid B_k \subseteq B_r\right\}$. Then certainly

$$p_1 + r \leq b_{k_0+1} = (k_0+1)^{k_0+1} + 1 \quad \text{and} \quad \mu(B_r) \geq 2^{k_0}.$$

But then

$$p_1 + 2r \;\leq\; 2\left((k_0+1)^{k_0+1} + 1\right) \;\leq\; (k_0+2)^{k_0+2} \;=\; a_{k_0+2}\,.$$

Therefore $B_{2r} \subseteq B_{a_{k_0+2}}(0) \equiv B_0$. But $\mu(B_0) = \sum_{k=0}^{k_0+1} 2^k \;\leq\; 2^{k_0+2} \;\leq\; 4\mu(B_r)$, therefore the doubling condition holds with $c_\mu = 4$.

Case 2. If for all k, $B_k \nsubseteq B_r$, then $r < 1$ so that B_r and B_{2r} intersect only one ball B_k. Then the doubling condition holds.

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Detailed version of the changes in Sawyer-Wheeden's theorem:

Remark 3.2. Observe that, by property (P) and Lemma 1.7, if $h \leq m$, the set $\{(y,z) \in B \times B\colon d(z,y) \geq h\,r\}$ is nonempty, so $\phi(B)$ is well defined. The above statement is not exactly the one proved in [SW]. More precisely, they define $\phi(B)$ for *any* ball (not necessarily satisfying (3.2)) and require (3.3)-(3.4) to hold for any ball. On the other hand, to make sense of $\phi(B)$ they require X to have no empty annuli, condition that implies in particular that X is unbounded and has no atoms. We shall show, at the end of this section, that it is still possible to prove their theorem if the assumptions on X are relaxed requiring only certain annuli to be nonempty (property (P)), if (3.3) and (3.4) hold only for balls satisfying (3.2).

We have now to justify the statement of Theorem 3.1 (see Remark 3.2). To do this, we recall some facts from the proof of this result in [SW]. They construct a family of "dyadic balls", at the following way. Set $\lambda = c_d + 2c_d^2$. For each $k \in \mathbb{Z}$, let $\left\{\hat{B}_j^k\right\}_j$ be a sequence of balls of radius λ^{k-1}, maximal with respect to the property that $\hat{B}_j^k \cap \hat{B}_i^k = \emptyset$ for $i \neq j$. Set $B_j^k = B(x_j^k, \lambda^k)$, where x_j^k is the centre of $\hat{B}_j^k$. The balls B_j^k will be called "dyadic balls". They prove that:

i) every ball of radius λ^{k-1} is contained in at least one of the balls B_j^k.

ii) $\sum_j \chi_{B_j^k} \leq M$ for all $k \in \mathbb{Z}$, where M is a constant depending only on c_d, c_μ.

iii) $\hat{B}_j^k \cap \hat{B}_i^k = \emptyset$ for $i \neq j$, $k \in \mathbb{Z}$.

If X is bounded (this case is implicitely excluded in [SW], as we noted in Remark 3.2) we need to modify slightly the previous construction. First we need the following

By the above Remark, if X is bounded there exists a constant K such that $R_x \leq K$ for every $x \in X$. Let $k_0 \in \mathbb{Z}$ such that $\lambda^{k_0-1} \leq K < \lambda^{k_0}$. Then for every $k \geq k_0$ and every j, $B_j^k = B(x_j^k, \lambda^k) = X$. Note that properties (*i*)-(*ii*)-(*iii*) still hold if we change the previous definition of dyadic balls setting $B_j^k = B_j^{k_0}$ for $k \geq k_0$, every j. So we can say that for, every dyadic ball $B(x,r)$,

$$r < M \cdot R_x \tag{3.11}$$

with M independent of x and r. We now fix the number h appearing in Thm. 3.1 (see (3.2)) at the following way:

$$h = min\left(m,\ c_d^2/4,\ 1/M\right). \tag{3.12}$$

By Lemma 1.7, Lemmas 3.2-3.4 and Theorem 3.5 still hold if m is replaced by h. The condition $h \leq c_d^2/4$ is required by the proof in [SW]; recall also that the condition $h \leq m$ is required to make sense of the function ϕ in (3.2). Reading carefully the proof in [SW], one can see that they only apply assumptions (3.3)-(3.4) to *particular* balls, namely dyadic balls of the kind $B_r(x)$, with $r > r_x$. Moreover, by (3.11)-(3.12), $r < \frac{R_x}{h}$. So, in the statement of Thm. 3.1, we only need to assume (3.3)-(3.4) for balls satisfying (3.1). Therefore the theorem holds exactly in the form in which we have stated it.

With this remark, the proof of Theorem 1.4 is completed.