# Gaussian bounds for heat kernels in the setting of Hörmander vector fields

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**Abstract**

Let $X_1, X_2, \ldots, X_q$ be a family of Hörmander's vector fields in $\mathbb{R}^n$. A systematic study of the properties of nonvariational operators of the kinds

$$L = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j$$

$$H = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x) X_i X_j$$

has begun in recent years. Here the matrix $\{a_{ij}\}$ is symmetric positive definite, and its entries are functions satisfying minimal smoothness assumptions. In this context, we will discuss and announce some recent results regarding Gaussian bounds for the fundamental solution of $H$.

Let us consider a system of smooth real vector fields, defined in a domain $\Omega \subseteq \mathbb{R}^n$

$$X_i = \sum_{j=1}^{n} b_{ij}(x) \partial_{x_j} \qquad i = 1, 2, \ldots, q \quad (q \leq n)$$

and assume they satisfy Hörmander's condition (of step $s$) in $\Omega$: the vector space spanned at every point of $\Omega$ by: the fields $X_i$; their commutators $[X_i, X_j] = X_i X_j - X_j X_i$; the commutators of the $X_k$'s with the commutators $[X_i, X_j]$;...and so on, up to step $s$, is the whole $\mathbb{R}^n$.

Under these assumptions, it is known (Hörmander [11]) that the second order differential operator "Hörmander' sum of squares"

$$L = \sum_{i=1}^{q} X_i^2$$

is hypoelliptic in $\Omega$, that is: if $Lu = f$ in $\Omega$ in distributional sense, and $f \in C^\infty(A)$ with $A \subset \Omega$, then $u \in C^\infty(A)$. Analogously, the evolution operator

$$H = \partial_t - \sum_{i=1}^{q} X_i^2$$

is hypoelliptic in $\mathbb{R} \times \Omega$. Roughly speaking, Hörmander's theorem says that an operator with nonnegative characteristic form, even though degenerate, still shares some good properties of nondegenerate operators (elliptic, parabolic), whenever the "missing directions" in the derivatives involved in the equation are reconstructed by the commutators of the vector fields. The most famous and simple instance of this situation is the following:

**Example 1** *Kohn's Laplacian in* 3 *variables* $(x, y, z)$*:*

$$L = X_1^2 + X_2^2$$

$$X_1 = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z};\ X_2 = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial z};\ [X_1, X_2] = -4\frac{\partial}{\partial z}.$$

*The vector fields* $X_1, X_2, [X_1, X_2]$ *span* $\mathbb{R}^3$ *at any point: Hörmander's condition holds; the operators* $L$ *in* $\mathbb{R}^3$ *and* $\partial_t - L$ *in* $\mathbb{R}^4$ *are hypoelliptic.*

There are two kinds of structures typically associated to a set of Hörmander's vector fields. The first is a metric structure: given a system of Hörmander's vector fields, it is always possible to join any two points of the space by arcs of integral curves of the vector fields (Rashevski-Chow's Theorem, 1938, 1939). Then, the minimal length of these "piecewise integral curves" defines a distance between the two points, called Carnot-Carathéodory distance, or "distance induced by the vector fields". A relevant fact proved by Fefferman-Phong [7] is that Lebesgue's measure turns out to be locally doubling with respect to this distance:

$$|B(x, 2r)| \leq c\,|B(x, r)|$$

at least for $x$ ranging in a compact set and $r \leq r_0$. This fact allows to adapt many typical arguments from real analysis to this context, in the spirit of Coifman-Weiss' theory of spaces of homogeneous type [6].

A second structure is of algebraic nature. In several important instances of systems of Hörmander vector fields in $\mathbb{R}^n$ (but not always!) the space $\mathbb{R}^n$ happens to be endowed with a "Carnot group" structure, that is: a Lie group operation ("translation"):

$$(x, y) \mapsto x \circ y$$

and a family of group automorphism ("dilations"), of the kind:

$$x \mapsto D(\lambda)\,x = (\lambda^{\alpha_1} x_1, ..., \lambda^{\alpha_n} x_n)$$

such that the vector fields $X_i$ are translation left invariant

$$X_i^x\,[f(y \circ x)] = (X^x f)(y \circ x)$$

and homogeneous of degree 1:

$$X_i^x\,[f(D(\lambda)\,x)] = \lambda\,(X_i^x f)(D(\lambda)\,x)$$

Then, Folland [9] has proved that $L$ has a left invariant fundamental solution, homogeneous of degree $2-Q$, where $Q>n$ is the "homogeneous dimension" of the group:

$$\Gamma\left(D\left(\lambda\right)x\right)=\lambda^{2-Q}\Gamma\left(x\right)$$

This is the starting point in order to apply to this context results from the theory of singular and fractional integrals in spaces of homogeneous type. For the "parabolic" operator $H$ one has the analogous behavior, with the vector field $\partial_t$ homogeneous of degree 2.

We now come to our main topic, that is Gaussian bounds for the fundamental solution of "heat-type" operators. For operators of the kind

$$H=\partial_t-\sum_{i=1}^{q}X_i^2$$

with left invariant homogeneous vector fields on a Carnot group in $\mathbb{R}^n$, Varopoulos ([15],[16], see also [17]) has proved the following Gaussian bound for the fundamental solution $h$:

$$\frac{c_1}{t^{Q/2}}e^{-\left\|x^{-1}\circ y\right\|^2/c_2 t}\leq h\left(t,x,y\right)\leq\frac{c_3}{t^{Q/2}}e^{-\left\|x^{-1}\circ y\right\|^2/c_4 t}$$

$\forall x,y\in\mathbb{R}^n,t>0$, where $Q$ is the homogeneous dimension of the group, and $\left\|\cdot\right\|$ the homogeneous norm, so that $\left\|x^{-1}\circ y\right\|$ is the distance in the group, equivalent to the distance induced by the vector fields. This bound is very similar to the classical one which holds for standard parabolic operators.

For the operators of the same form, but without an underlying group structure (that is, assuming that $\{X_1,X_2,...,X_q\}$ is *any* system of Hörmander's vector fields), the analogous result is:

$$c_1\frac{e^{-d(x,y)^2/c_2 t}}{\left|B\left(x,\sqrt{t}\right)\right|}\leq h\left(t,x,y\right)\leq c_3\frac{e^{-d(x,y)^2/c_4 t}}{\left|B\left(x,\sqrt{t}\right)\right|}$$

$\forall x,y\in\mathbb{R}^n$, $t\in(0,\infty)$, where $d\left(x,y\right)$ is the distance induced by the vector fields, and $B\left(x,r\right)$ the metric ball. This result has been accomplished in several steps, by several authors, in two group of papers: Sanchez-Calle [14], Fefferman, Sanchez-Calle [8], and Jerison, Sanchez-Calle [10], with analytical techniques; Kusuoka-Stroock, [12],[13], by means of stochastic techniques and Malliavin calculus.

In recent years, nonvariational operators of the kind

$$H_A=\partial_t-\sum_{i,j=1}^{q}a_{ij}\left(t,x\right)X_iX_j \tag{1}$$

with $X_i$ left invariant homogeneous Hörmander's vector fields on a Carnot group, and

$$\lambda\left|\xi\right|^2\leq\sum_{i,j=1}^{q}a_{ij}(t,x)\;\xi_i\xi_j\leq\lambda^{-1}\left|\xi\right|^2\qquad\forall\xi\in\mathbb{R}^q \tag{2}$$

with coefficients $a_{ij} = a_{ji}$, Hölder continuous with respect to the "parabolic distance" induced by the group structure, have been studied by Bonfiglioli-Lanconelli-Uguzzoni, who have proved the existence of a fundamental solution $h_A$ satisfying the following bound:

$$\frac{c_1}{t^{Q/2}} e^{-\left\|x^{-1}\circ y\right\|^2/c_2 t} \leq h_A\left(t,x,y\right) \leq \frac{c_3}{t^{Q/2}} e^{-\left\|x^{-1}\circ y\right\|^2/c_4 t}$$

$\forall x,y \in \mathbb{R}^n, t>0$. (When $A = I$, this is Varoupulos' estimate). The result has been accomplished in several steps, see [1], [2], [3]. An application of these Gaussian estimates is the proof of an invariant Harnack inequality for the operator $H_A$, which is carried out by Bonfiglioli-Uguzzoni in [4]. Note that operators of kind (1) do not have smooth coefficients; hence they are not hypoelliptic, and the mere existence of a fundamental solution is not trivial.

Finally, we are led to consider the more general case of operators of type (1), when $\{X_1, X_2, ..., X_q\}$ is *any* system of Hörmander's vector fields and the coefficients $a_{ij} = a_{ji}$, are Hölder continuous with respect to the parabolic distance induced by vector fields, and satisfy the ellipticity condition (2). These operators have been recently studied by Bramanti, Brandolini, Lanconelli, Uguzzoni [5]; our main result consists in showing the existence of a fundamental solution $h_A$, satisfying Gaussian bounds of the kind:

$$c_1 \frac{e^{-d(x,y)^2/c_2 t}}{\left|B\left(x,\sqrt{t}\right)\right|} \leq h_A\left(t,x,y\right) \leq c_3 \frac{e^{-d(x,y)^2/c_4 t}}{\left|B\left(x,\sqrt{t}\right)\right|}$$

for any $x,y \in \mathbb{R}^n, t \in (0,T)$. (When $A = I$, this is the result of Jerison-Sanchez-Calle or Kusuoka-Stroock). We assume that $H_A$ coincides with the heat operator outside a large compact set, so in some sense our result is of local nature.

Following the general strategy adopted by Bonfiglioli-Lanconelli-Uguzzoni in the case of Carnot groups, to get our results we first consider the operator $H_A$ with a constant matrix $\{a_{ij}\}$, in a fixed ellipticity class. This is an operator with smooth coefficients which is hypoelliptic, and by known results possesses a global fundamental solution $h_A$; our first, and more difficult, task is to prove a number of uniform estimates on $h_A$, depending on the constant coefficients $a_{ij}$ only through the ellipticity constant $\lambda$ in (2). More precisely, we prove the following uniform bounds:

1. Upper and lower bounds on $h_A$:

$$\frac{c_3}{\left|B\left(x,\sqrt{t}\right)\right|} e^{-d(x,y)^2/c_4 t} \leq h_A\left(t,x,y\right) \leq \frac{c_1}{\left|B\left(x,\sqrt{t}\right)\right|} e^{-d(x,y)^2/c_2 t}$$

2. Upper bounds on the derivatives of $h_A$:

$$\left|X_x^I X_y^J \partial_t^i h_A\left(t,x,y\right)\right| \leq \frac{c_1}{t^{i+\frac{|I|+|j|}{2}}\left|B\left(x,\sqrt{t}\right)\right|} e^{-d(x,y)^2/c_2 t}$$

3. Estimate on the difference of the fundamental solutions of two operators (and their derivatives):

$$\left|X_x^I X_y^J \partial_t^i h_A\left(t,x,y\right) - X_x^I X_y^J \partial_t^i h_B\left(t,x,y\right)\right| \leq$$

$$\leq \|A - B\| \frac{c_1}{t^{i+\frac{|I|+|j|}{2}} \left|B\left(x, \sqrt{t}\right)\right|} e^{-d(x,y)^2/c_2 t}$$

These estimates, by the Levi's parametrix method, enable us to prove the existence of the fundamental solution for the operator $H_A$ with variable (Hölder continuous) coefficients, and to deduce the desired Gaussian bounds for it.

The line of the proofs of these uniform estimates is complex and cannot be summarized here. It will appear in detail in [5].

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