Space regularity for evolution operators modeled on Hörmander vector fields with time dependent measurable coefficients

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Abstract

We consider a heat-type operator $\mathcal{L}$ structured on the left invariant 1-homogeneous vector fields which are generators of a Carnot group, multiplied by a uniformly positive matrix of bounded measurable coefficients depending only on time. We prove that if $\mathcal{L}u$ is smooth with respect to the space variables, the same is true for u, with quantitative regularity estimates in the scale of Sobolev spaces defined by right invariant vector fields. Moreover, the solution and its space derivatives satisfy a 1/2-Hölder continuity estimate with respect to time. The result is proved both for weak solutions and for distributional solutions, in a suitable sense[1].

Let $\mathbb{G} = \left(\mathbb{R}^N, \circ, D_\lambda\right)$ a Carnot group and let $X_1, ..., X_q$ be the generators of its Lie algebra, so that the canonical sublaplacian

$$\sum_{i=1}^{q} X_i^2$$

and the corresponding heat operator

$$\sum_{i=1}^{q} X_i^2 - \partial_t$$

are hypoelliptic in $\mathbb{R}^N$ and $\mathbb{R}^{N+1}$, respectively (precise definitions will be given in Section 1). Let us now consider

$$\mathcal{L} = \sum_{i,j=1}^{q} a_{ij}(t) X_i X_j - \partial_t \tag{0.1}$$

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where $\{a_{ij}(t)\}_{i,j=1}^{q}$ is a real symmetric matrix of bounded measurable coefficients, uniformly positive:

$$\nu |\xi|^2 \leqslant \sum_{i,j=1}^{q} a_{ij}(t)\,\xi_i\xi_j \leqslant \nu^{-1}|\xi|^2 \tag{0.2}$$

for every $\xi \in \mathbb{R}^q$, a.e. $t \in (0,T)$. We want to prove a regularity result for $\mathcal{L}$ in the space variables, that is, roughly speaking: if $u \in W^{1,2}\left((0,T), L^2_{loc}\left(\mathbb{R}^N\right)\right)$ is a weak solution to $\mathcal{L}u = F$, $u(0,\cdot) = 0$ and F is smooth, *with respect to the space variables*, in some domain $(0,T)\times\Omega$, then the same is true for u, with quantitative regularity estimates on u in terms of $\mathcal{L}u$. Also, we will prove that, if F is smooth w.r.t. the space variables, then u and every space derivative $\partial_x^\alpha u$ are $\frac{1}{2}$-Hölder continuous with respect to t. See Theorems 2.14 and 2.15 for the precise statements. This kind of regularity is the best we can hope, even for a uniformly parabolic operator

$$\mathcal{L}u = u_t - a(t)\,u_{xx}$$

as soon as a is only L^∞ (see Example 2.16). The above regularity result can be extended also to distributional solutions belonging to $W^{1,2}\left((0,T), \mathcal{D}'\left(\mathbb{R}^N\right)\right)$ (see Theorem 3.3 for the precise statement). This can be seen as a kind of Hörmander's theorem with respect to the space variables.

Results of this kind have been proved by Krylov [14], who considered operators

$$\mathcal{L} = \partial_t - \sum_{k=1}^{q} L_k^2 + L_0$$

with

$$L_k = \sum_{i=1}^{N} \sigma^{ik}(t,x)\,\partial_{x_i}$$

where the functions $\sigma^{ik}(t,x)$ are assumed to have x-derivatives of every order uniformly bounded for $x \in \mathbb{R}^N$ and $t \in (0,1)$, and the vector fields $L_0, L_1, ..., L_q$ for every fixed t satisfy Hörmander's condition in $\mathbb{R}^N$. Now, every operator (0.1) can be rewritten as

$$-\mathcal{L} = \partial_t - \sum_{k=1}^{q} L_k^2$$

with

$$\sigma^{ik}(t,x) = \sum_{j=1}^{q} m_{jk}(t)\,b_{ji}(x)$$

where

$$X_j = \sum_{i=1}^{N} b_{ji}(x)\,\partial_{x_i}$$

and

$$a_{ij}(t)=\sum_{k=1}^{q}m_{ik}(t)\,m_{jk}(t)$$

so that

$$D_x^{\alpha}\sigma^{ik}(t,x)=\sum_{j=1}^{q}m_{jk}(t)\,D_x^{\alpha}b_{ji}(x)$$

$$\left|D_x^{\alpha}\sigma^{ik}(t,x)\right|\leqslant c_{\nu}\sum_{j=1}^{q}\left|D_x^{\alpha}b_{ji}(x)\right|.$$

Since the coefficients $b_{ji}(x)$ of the generators on a Carnot group are polynomials, the functions $|D_x^{\alpha}b_{ji}(x)|$ are not *globally* bounded on $\mathbb{R}^N$. Therefore, although the class of operators that we consider is strictly contained in the class considered by Krylov as to their structure, the assumption on $\sigma^{ik}(t,x)$ made in [14] is not satisfied in our situation.

Actually, the technique employed in this paper is very different from that in [14]. In [14], following the classical approach introduced by Kohn [13] and Oleĭnik-Radkevič [17], pseudodifferential operators and Sobolev spaces of fractional order are used. Here, instead, we adapt to the evolutionary case the technique introduced in [3] to give a proof of Hörmander's theorem for sublaplacians on Carnot groups. The main idea consists in measuring the regularity of solutions of an equation $\mathcal{L}u=f$, where $\mathcal{L}$ is a *left invariant* operator, in terms of Sobolev spaces induced by *right invariant* vector fields. Since a right invariant and a left invariant operator always commute, this approach greatly simplifies the proof of higher order estimates. We handle Sobolev norms with respect to vector fields by means of equivalent norms defined in terms of finite difference operators, in the directions of the vector fields $X_1,...,X_q$. This feature of our argument is reminiscent of the original proof of Hörmander's theorem given in [12], although in the richer framework of Carnot groups the proof becomes much simpler.

Let us now give some motivation for the present research and describe some related literature. The regularity result proved in [14] has been applied by the same Author in [15] to prove an analogous result for stochastic PDEs, and in [16], in the context of filtering problems. We refer to [15] for motivations to prove this result without any continuity assumption on the coefficients with respect to time.

Hyperbolic operators of the kind

$$Hu=u_{tt}-\sum_{i,j=1}^{n}a_{ij}(t)\,u_{x_ix_j}$$

with merely bounded measurable a_{ij} have been studied by many authors, see for instance [9], [8], [11] and references therein. In particular, [11] gives some physical motivation to study this class of operators under no regularity condition on $a_{ij}(t)$.

Operators of the kind

$$\mathcal{L}=\sum_{i,j=1}^{q}a_{ij}\left(t,x\right)X_{i}X_{j}-\partial_{t}, \tag{0.3}$$

satisfying (0.2) have been studied by several Authors, assuming the coefficients $a_{ij}\left(t,x\right)$ either Hölder continuous or with vanishing mean oscillation, and proving a priori estimates and regularity results in the scale of Hölder or Sobolev spaces induced by the vector fields $\{X_i\}_{i=1}^{q}$ and the distance they induce. See for instance [4], [6], [7] and references therein. In [6], for the operator $\mathcal{L}$ with Hölder continuous coefficients, a heat kernel has been constructed and shown to satisfy sharp Gaussian estimates, which also imply a scale invariant Harnack inequality.

The operators (0.1) studied in the present paper can also be seen as model operators to study the more general class (0.3) with the coefficients satisfying some moderate regularity assumtpion in x, but only L^{∞} with respect to time, an area of research that we plan to attack in the future.

1 Preliminaries about Carnot groups

Let us recall some standard definitions and results that will be useful in the following. For the proofs of these facts the reader is referred to [10], [1, Chap.1]. A *homogeneous group* (in $\mathbb{R}^N$) is a Lie group $\left(\mathbb{R}^N,\circ\right)$ (where the group operation $\circ$ will be thought as a "translation") endowed with a one parameter family $\left\{D_\lambda\right\}_{\lambda>0}$ of group automorphisms ("dilations") which act this way:

$$D_{\lambda}\left(x_{1},x_{2},...,x_{N}\right)=\left(\lambda^{\alpha_{1}}x_{1},\lambda^{\alpha_{2}}x_{2},...,\lambda^{\alpha_{N}}x_{N}\right) \tag{1.1}$$

for suitable integers $1=\alpha_1\leqslant\alpha_2\leqslant...\leqslant\alpha_N$. We will write $\mathbb{G}=\left(\mathbb{R}^N,\circ,D_\lambda\right)$ to denote this structure. The number

$$Q=\sum_{i=1}^{N}\alpha_{i}$$

will be called *homogeneous dimension* of $\mathbb{G}$. A *homogeneous norm* on $\mathbb{G}$ is a continuous function

$$\left\|\cdot\right\|:\mathbb{G}\rightarrow[0,+\infty),$$

such that, for some constant $c>0$ and every $x,y\in\mathbb{G}$,

$$\begin{array}{ll}(i) & \left\|x\right\|=0\Longleftrightarrow x=0\\ (ii) & \left\|D_{\lambda}\left(x\right)\right\|=\lambda\left\|x\right\|\ \ \forall\lambda>0\\ (iii) & \left\|x^{-1}\right\|\leqslant c\left\|x\right\|\\ (iv) & \left\|x\circ y\right\|\leqslant c\left(\left\|x\right\|+\left\|y\right\|\right).\end{array}$$

We will always use the symbol $\left\|\cdot\right\|$, without any subscript, to denote a homogeneous norm in $\mathbb{G}$. Examples of homogeneous norms are the following:

$$\left\|x\right\|=\max_{k=1,2,\ldots,N}\left|x_{k}\right|^{\frac{1}{\alpha_{k}}}$$

or

$$\|x\| = \left(\sum_{k=1}^{N} |x_k|^{\frac{Q}{\alpha_k}} \right)^{1/Q}.$$

It can be proved that any two homogeneous norms on $\mathbb{G}$ are equivalent.

We say that a smooth function f in $\mathbb{G} \setminus \{0\}$ is *D_λ-homogeneous of degree* $\beta \in \mathbb{R}$ (or simply "β-homogeneous") if

$$f(D_\lambda(x)) = \lambda^\beta f(x) \quad \forall \lambda > 0, x \in \mathbb{G} \setminus \{0\}.$$

Given any differential operator P with smooth coefficients on $\mathbb{G}$, we say that P is *left invariant* if for every $x, y \in \mathbb{G}$ and every smooth function f

$$P(L_y f)(x) = L_y(Pf(x)),$$

where

$$L_y f(x) = f(y \circ x).$$

Analogously one defines the notion of *right invariant* differential operator. Also, P is said *β-homogeneous* (for some $\beta \in \mathbb{R}$) if

$$P(f(D_\lambda(x))) = \lambda^\beta (Pf)(D_\lambda(x))$$

for every smooth function f, $\lambda > 0$ and $x \in \mathbb{G}$.

A *vector field* is a first order differential operator

$$X = \sum_{i=1}^{N} c_i(x) \partial_{x_i}.$$

Let $\mathfrak{g}$ be the *Lie algebra of left invariant vector fields* over $\mathbb{G}$, where the Lie bracket of two vector fields is defined as usual by

$$[X, Y] = XY - YX.$$

Let us denote by $X_1, X_2, \ldots, X_N$ the *canonical base* of $\mathfrak{g}$, that is for $i = 1, 2, ..., N$, X_i is the only left invariant vector field that agrees with ∂_{x_i} at the origin. Also, $X_1^R, X_2^R, \ldots, X_N^R$ will denote the right invariant vectors fields that agree with $\partial_{x_1}, \partial_{x_2}, ..., \partial_{x_N}$ (and hence with $X_1, X_2, \ldots, X_N$) at the origin.

We assume that for some integer $q < N$ the vector fields $X_1, X_2, \ldots, X_q$ are 1-homogeneous and the Lie algebra generated by them is $\mathfrak{g}$. If s is the maximum length of commutators

$$\left[X_{i_1}, \left[X_{i_2}, ..., \left[X_{i_{s-1}}, X_{i_s}\right]\right]\right], \quad i_j \in \{1, 2, ..., q\}$$

required to span $\mathfrak{g}$, then we will say that $\mathfrak{g}$ is a *stratified Lie algebra of step s*, $\mathbb{G}$ is a *Carnot group* (or a stratified homogeneous group) and its *generators* $X_1, X_2, \ldots, X_q$ satisfy Hörmander's condition at step s in $\mathbb{G}$. Under these assumptions, by Hörmander's theorem (see [12]), the *canonical sublaplacian*

$$L = \sum_{i=1}^{q} X_i^2$$

is hypoelliptic in $\mathbb{R}^N$, that is: for every domains $\Omega' \subset \Omega \subset \mathbb{R}^N$, whenever $u \in \mathcal{D}'(\Omega)$ solves in distributional sense the equation $Lu = f$ in Ω, then $f \in C^\infty(\Omega') \Rightarrow u \in C^\infty(\Omega')$.

Analogously, the corresponding heat operator

$$H = \sum_{i=1}^{q} X_i^2 - \partial_t$$

is hypoelliptic in $\mathbb{R}^{N+1}$.

We will make use of the *Sobolev spaces* $W_X^{k,p}(\mathbb{G})$, $W_{X^R}^{k,p}(\mathbb{G})$ *induced by the systems of vector fields*

$$X = \{X_1, X_2, \dots, X_q\}, X^R = \{X_1^R, X_2^R, \dots, X_q^R\},$$

respectively. More precisely, given an open subset Ω of $\mathbb{R}^N$, we say that $f \in W_X^{1,2}(\Omega)$ if $f \in L^2(\Omega)$ and there exist, in weak sense, $X_j f \in L^2(\Omega)$ for $j = 1, 2, ..., q$. Inductively, we say that $f \in W_X^{k,2}(\Omega)$ for $k = 2, 3, ...$ if $f \in W_X^{k-1,2}(\Omega)$ and any weak derivative of order $k-1$ of f, $X_{j_1}X_{j_2}...X_{j_{k-1}}f$, belongs to $W_X^{1,2}(\Omega)$. We set

$$\|f\|_{W_X^{k,2}(\Omega)} = \|f\|_{L^2(\Omega)} + \sum_{h=1}^{k} \sum_{j_i=1,2,\dots,q} \|X_{j_1}X_{j_2}...X_{j_h}f\|_{L^2(\Omega)}.$$

The space $W_{X^R}^{k,2}(\Omega)$ has a similar definition. We will also use local Sobolev spaces. For example, we will say that $f \in W_{X,loc}^{k,2}(\Omega)$ if for every $\varphi \in C_0^\infty(\Omega)$, we have $\varphi f \in W_X^{k,2}(\Omega)$.

For homogeneity reasons, the generators $X_1, ..., X_q$ satisfy the simple relation $X_i^* = -X_i$ (where X^* stands for the transposed operator of X). In other words,

$$\int_{\mathbb{G}} f(X_i g) = -\int_{\mathbb{G}} (X_i f) g \tag{1.2}$$

whenever $f \in W_{X,loc}^{1,2}(\mathbb{G})$ and $g \in C_0^1(\mathbb{G})$.

The validity of Hörmander's condition at step s implies the following important:

Proposition 1.1 (See [3, Prop. 2.1]) *Under the above assumptions we have:*

1.

$$\bigcap_{k=1}^{\infty} W_X^{k,2}(\Omega) \subset C^\infty(\Omega).$$

2. For any positive integer k and any $\Omega' \Subset \Omega$ there exists a constant $c > 0$ such that, for every $u \in W_X^{ks,2}(\Omega)$ we have

$$\|u\|_{W^{k,2}(\Omega')} \leqslant c\|u\|_{W_X^{ks,2}(\Omega)},$$

where $W^{k,2}(\Omega')$ denotes the standard Sobolev space.

Let us point out a relation between left and right invariant operators which will be very useful in the following.

Proposition 1.2 (see [3, Prop. 2.2]) *Let $\mathcal{L}, \mathcal{R}$ be any two differential operators on $\mathbb{G}$ with smooth coefficients, left and right invariant, respectively. Then $\mathcal{L}$ and $\mathcal{R}$ commute:*

$$\mathcal{L}\mathcal{R}f = \mathcal{R}\mathcal{L}f$$

for any smooth function f.

For every given couple of measurable functions $\varphi, \psi : \mathbb{G} \to \mathbb{R}$ we define

$$\varphi * \psi (x) = \int_{\mathbb{G}} \varphi (y) \psi \left(y^{-1} \circ x\right) dy$$

whenever the integral makes sense. One can prove the following:

Proposition 1.3 *For every couple of measurable functions f, ψ defined on $\mathbb{G}$ such that the following convolutions are well defined, we have*

i) if $\mathcal{P}$ is a left invariant differential operator then

$$\mathcal{P} (f * \psi) = f * \mathcal{P}\psi, \tag{1.3}$$

ii) if $\mathcal{P}$ is a right invariant differential operator then

$$\mathcal{P} (\psi * f) = \mathcal{P}\psi * f$$

whenever $\mathcal{P}\psi$ exists at least in weak sense.

2 Subelliptic estimates for heat-type operators with t-measurable coefficients

For a domain $\Omega \subseteq \mathbb{G}$, let

$$\Omega_T = (0, T) \times \Omega.$$

We are going to define several function spaces on $\mathbb{G}_T = (0, T) \times \mathbb{G}$ that we will use in the following.

The definitions of the spaces $L^2 ((0, T), X)$, $W^{1,2} ((0, T), X)$, $C^0 ([0, T], X)$ when X is a Banach space are standard. For instance, we will often use the spaces

$$L^2 \left((0, T), W_X^{k,2} (\mathbb{G})\right)$$

(for $k = 1, 2, 3, ...$) normed with

$$\|f\|_{L^2\left((0,T),W_X^{k,2}(\mathbb{G})\right)} = \|f\|_{L^2(\mathbb{G}_T)} + \sum_{j=1}^{k} \sum_{i_1,...,i_j \in\{1,...,q\}} \left\|X_{i_1} X_{i_2} ... X_{i_j} f\right\|_{L^2(\mathbb{G}_T)}$$

and the analogous spaces $L^2 \left((0, T), W_{X^R}^{k,2} (\mathbb{G})\right)$.

We will say that $u \in L^2\left((0,T), W^{k,2}_{X,loc}(\mathbb{G})\right)$ when for every $\zeta \in C_0^\infty(\mathbb{G})$ we have $u\zeta \in L^2\left((0,T), W^{k,2}_X(\mathbb{G})\right)$.

For a function $f \in L^2\left((0,T), W^{1,2}_X(\mathbb{G})\right)$ we will also use the shorthand notation

$$\left\|\nabla_X f\right\|^2_{L^2(\mathbb{G}_T)} = \sum_{i=1}^q \left\|X_i f\right\|^2_{L^2(\mathbb{G}_T)},$$

with the analogous meaning for $\left\|\nabla_{X^R} f\right\|^2_{L^2(\mathbb{G}_T)}$.

Definition 2.1 *We say that a function u belongs to $L^2\left((0,T), C^\infty\left(\overline{\Omega}\right)\right)$ if $u \in L^2\left((0,T), C^k\left(\overline{\Omega}\right)\right)$ for every $k = 0,1,2,\ldots$ Explicitly, this implies that*

$$\int_0^T \left\|u(t,\cdot)\right\|^2_{C^k(\overline{\Omega})} dt < \infty \text{ for every } k = 0,1,2,\ldots$$

We say that a function u belongs to $C^0\left([0,T], C^\infty\left(\overline{\Omega}\right)\right)$ if $u \in C^0\left([0,T], C^k\left(\overline{\Omega}\right)\right)$ for every $k = 0,1,2,\ldots$

Definition 2.2 *We let:*

$$\begin{aligned}\mathcal{H} &= L^2\left((0,T), W^{2,2}_X(\mathbb{G})\right) \cap W^{1,2}\left((0,T), L^2(\mathbb{G})\right) \\ &= \left\{u \in L^2(\mathbb{G}_T) : u_t, X_i u, X_i X_j u \in L^2(\mathbb{G}_T)\right\}.\end{aligned}$$

Note that $\mathcal{H} \subset C^0\left([0,T], L^2(\mathbb{G})\right)$, so that for $u \in \mathcal{H}$ and $t \in [0,T]$, $u(t,\cdot)$ is a well defined element of $L^2(\mathbb{G})$.

We will also use

$$\mathcal{H}_0 = \left\{u \in W^{1,2}\left((0,T), L^2_{loc}(\mathbb{G})\right) : \forall \phi \in C_0^\infty(\mathbb{G}) \;\; u\phi \in \mathcal{H} \text{ and } (u\phi)(0,\cdot) = 0\right\}.$$

Proposition 2.3 *Let $\mathcal{L}$ be as in (0.1) and let (0.2) be in force. Then for every $u \in \mathcal{H}$ such that $u(0,\cdot) = 0$ we have*

$$\left\|\nabla_X u\right\|_{L^2(\mathbb{G}_T)} \leqslant c_\nu \left\{\left\|\mathcal{L}u\right\|_{L^2(\mathbb{G}_T)} + \left\|u\right\|_{L^2(\mathbb{G}_T)}\right\} \tag{2.1}$$

for a constant c_ν only depending on the ellipticity constant ν in (0.2).

Proof. For $u \in \mathcal{H}$ we have, recalling that $X_i^* = -X_i$ (see (1.2)):

$$\begin{aligned}&-\int\int_{\mathbb{G}_T} (u\mathcal{L}u)\, dtdx = \int\int_{\mathbb{G}_T} (u\partial_t u)\, dtdx - \int\int_{\mathbb{G}_T} \left(u \sum_{i,j=1}^q a_{ij}(t) X_i X_j u\right) dtdx \\ &= \frac{1}{2}\int_{\mathbb{G}} \left(\int_0^T \partial_t\left(u^2\right) dt\right) dx - \sum_{i,j=1}^q \int_0^T a_{ij}(t) \left(\int_{\mathbb{G}} (u X_i X_j u)\, dx\right) dt \\ &= \frac{1}{2}\int_{\mathbb{G}} \left(u^2(T,x) - u^2(0,x)\right) dx + \sum_{i,j=1}^q \int_0^T a_{ij}(t) \left(\int_{\mathbb{G}} (X_i u X_j u)\, dx\right) dt.\end{aligned} \tag{2.2}$$

Since

$$\sum_{i,j=1}^{q}\int_0^T a_{ij}(t)\left(\int_{\mathbb{G}}(X_iuX_ju)\,dx\right)dt \geq \nu\sum_{i=1}^{q}\int_0^T\int_{\mathbb{G}}(X_iu)^2\,dxdt$$

we have

$$\|\nabla_X u\|_{L^2(\mathbb{G}_T)}^2 \leqslant \frac{1}{\nu}\|\mathcal{L}u\|_{L^2(\mathbb{G}_T)}\|u\|_{L^2(\mathbb{G}_T)} + \frac{1}{2\nu}\|u(0,\cdot)\|_{L^2(\mathbb{G})}. \tag{2.3}$$

In particular, for u vanishing on $t=0$ we get (2.1). ■

In the following of this section we will recall and adapt several definitions and arguments taken from [3]. The reader is referred to that paper for some details.

Definition 2.4 (Finite difference operators) *For every $h \in \mathbb{G}$ and function f defined in $\mathbb{G}$, let us define the operators:*

$$\Delta_h f(x) = f(x \circ h) - f(x)$$
$$\widetilde{\Delta}_h f(x) = f(h \circ x) - f(x).$$

Whenever the function f also depends on t, we will simply write

$$\Delta_h f(t,x) = \Delta_h[f(t,\cdot)](x)$$

and analogously for $\widetilde{\Delta}_h f(t,x)$.

Definition 2.5 *For $m = 1,2,3,4,...$, let*

$$\Delta_h^m = \underbrace{\Delta_h\Delta_h...\Delta_h}_{m\ times}.$$
$$\widetilde{\Delta}_h^m = \underbrace{\widetilde{\Delta}_h\widetilde{\Delta}_h...\widetilde{\Delta}_h}_{m\ times}.$$

Then, for $\alpha > 0$ and $f \in L^2(\mathbb{G}_T)$ we define the semi-norms

$$|f|_{m,\alpha} = \sup\left\{\frac{\|\Delta_h^m f\|_{L^2(\mathbb{G}_T)}}{\|h\|^\alpha} : h = \operatorname{Exp}(tX_i)\ \ \forall i = 1,...,q, t \in \mathbb{R} : 0 < \|h\| \leqslant 1\right\}$$

$$|f|_{m,\alpha}^R = \sup\left\{\frac{\left\|\widetilde{\Delta}_h^m f\right\|_{L^2(\mathbb{G}_T)}}{\|h\|^\alpha} : h = \operatorname{Exp}(tX_i)\ \ \forall i = 1,...,q, t \in \mathbb{R} : 0 < \|h\| \leqslant 1\right\}.$$

We also set for convenience

$$|f|_0 = |f|_0^R = \|f\|_{L^2(\mathbb{G}_T)}$$
$$|f|_m = |f|_{m,m}$$
$$|f|_m^R = |f|_{m,m}^R.$$

The relations between the above seminorms and Sobolev norms with respect to vector fields are contained in the following two results, which can be derived by [3, Thm. 3.11, Prop.3.13] simply integrating in t.

Proposition 2.6 *For $m = 1, 2, ...$there exists $c = c(m, \mathbb{G})$ such that, for every $f \in L^2(\mathbb{G}_T)$ we have:*

1. If $f \in L^2\left((0,T), W_X^{m,2}(\mathbb{G})\right)$ then

$$\sum_{k=0}^{m} |f|_k \leqslant c \|f\|_{L^2\left((0,T), W_X^{m,2}(\mathbb{G})\right)} \tag{2.4}$$

Analogously,

2. If $f \in L^2\left((0,T), W_{X^R}^{m,2}(\mathbb{G})\right)$ then

$$\sum_{k=0}^{m} |f|_k^R \leqslant c \|f\|_{L^2\left((0,T), W_{X^R}^{m,2}(\mathbb{G})\right)}. \tag{2.5}$$

Proposition 2.7 *There exists $C = C(\mathbb{G})$ such that for every $f \in L^2(\mathbb{G}_T)$ we have:*

1. If $|f|_1 < \infty$ then $f \in L^2\left((0,T), W_X^{1,2}(\mathbb{G})\right)$, with

$$\|\nabla_X f\|_{L^2(\mathbb{G}_T)} \leqslant C |f|_1.$$

2. If $|f|_1^R < \infty$ then $f \in L^2\left((0,T), W_{X^R}^{1,2}(\mathbb{G})\right)$, with

$$\|\nabla_{X^R} f\|_{L^2(\mathbb{G}_T)} \leqslant C |f|_1^R.$$

The following bound instead links the $L^2\left((0,T), W_X^{1,2}(\mathbb{G})\right)$ norm with the operators $\widetilde{\Delta}_h$:

Proposition 2.8 *Let Ω be a bounded domain in $\mathbb{G}$. There exists $c = c(\Omega, \mathbb{G})$ such that for every $u \in L^2\left((0,T), W_X^{1,2}(\mathbb{G})\right)$ with $\operatorname{sprt} u(t,\cdot) \subset \Omega$ for every $t \in (0,T)$ we have*

$$\left\|\widetilde{\Delta}_h u\right\|_{L^2(\mathbb{G}_T)} \leqslant c \|h\|^{1/s} \|\nabla_X u\|_{L^2(\mathbb{G}_T)}.$$

(Recall that s is the step of the Lie algebra).

Proof. It is enough to apply to $u(t,\cdot)$ the computations made in [3, Prop.3.7, Lemma 3.8] for functions in $W_X^{1,2}(\mathbb{G})$ and then integrate on $(0,T)$. ■

If $u \in \mathcal{H}$, $u(t,\cdot)$ is supported in some bounded domain Ω for every $t \in [0,T]$ and $u(0,\cdot) = 0$, then by the previous Proposition and (2.1) we get

$$\left\|\widetilde{\Delta}_h u\right\|_{L^2(\mathbb{G}_T)} \leqslant c_\nu \|h\|^{1/s} \left\{\|\mathcal{L}u\|_{L^2(\mathbb{G}_T)} + \|u\|_{L^2(\mathbb{G}_T)}\right\}$$

that is

$$|u|_{1,1/s}^R \leqslant c_\nu \left\{\|\mathcal{L}u\|_{L^2(\mathbb{G}_T)} + \|u\|_{L^2(\mathbb{G}_T)}\right\}. \tag{2.6}$$

Notation 2.9 *Henceforth, we will write*

$$\zeta_0 \prec \zeta$$

if $\zeta_0, \zeta \in C_0^\infty(\mathbb{G})$ *such that* $0 \leqslant \zeta_0 \leqslant \zeta \leqslant 1$ *and* $\zeta = 1$ *on* $\operatorname{sprt} \zeta_0$.

We have the following analog of Theorem 3.15 in [3]:

Theorem 2.10 *Let* $\zeta_0, \zeta \in C_0^\infty(\mathbb{G})$ *with* $\zeta_0 \prec \zeta$. *For every* $m \in \mathbb{N}$ *the exists* $c = c(\zeta_0, \zeta, m, \mathbb{G}, \nu) > 0$ *such that if* $u \in \mathcal{H}_0$ *then*

$$|\zeta_0 u|_{m,m/s}^R \leqslant c \left(\sum_{j=0}^{m-1} |\zeta \mathcal{L} u|_j^R + \|\zeta u\|_2 \right), \tag{2.7}$$

whenever the right hand side is finite.

Proof. We can repeat the proof of Theorem 3.15 in [3] applying (2.6) to the function $\zeta_0 u \in \mathcal{H}$, since $u \in \mathcal{H}_0$, and exploiting the identity

$$\mathcal{L}(\zeta_0 u) = (\mathcal{L}\zeta_0) u + \zeta_0 (\mathcal{L} u) + 2 \sum_{i,j=1}^{q} a_{ij}(t) X_i \zeta_0 X_j u, \tag{2.8}$$

and the fact that the operators ∂_t and $\widetilde{\Delta}_h$ commute, so that $\mathcal{L}$ and $\widetilde{\Delta}_h$ still commute. ■

Also Proposition 3.16 in [3] (Marchaud inequality on Carnot groups) still holds, with $L^2(\mathbb{G})$ norms replaced with $L^2(\mathbb{G}_T)$ norms, and this implies the following analog of Corollary 3.17 in [3].

Corollary 2.11 *Let* $u \in \mathcal{H}$, $u(0, \cdot) = 0$, *and assume that for* $\varepsilon \in (0,1)$ *and some integer* $m > 1$ *the seminorm* $|u|_{m,1+\varepsilon}^R$ *is finite. Then*

$$|u|_1^R \leqslant c \left\{ |u|_{m,1+\varepsilon}^R + \|u\|_{L^2(\mathbb{G}_T)} \right\},$$

with $c = c(\mathbb{G})$.

We are now in position to state the first step of our regularity estimate:

Proposition 2.12 *Let* $\zeta_0, \zeta \in C_0^\infty(\mathbb{G})$ *with* $\zeta_0 \prec \zeta$. *There exists* $c = c(\zeta_0, \zeta, \mathbb{G}, \nu) > 0$ *such that*

if $u \in \mathcal{H}_0$ *and* $\mathcal{L} u \in L^2\left((0,T), W_{X^R,loc}^{s,2}(\mathbb{G})\right)$ *then* $u \in L^2\left((0,T), W_{X^R,loc}^{1,2}(\mathbb{G})\right)$

and

$$\|\zeta_0 u\|_{L^2\left((0,T), W_{X^R}^{1,2}(\mathbb{G})\right)} \leqslant c \left(\|\zeta \mathcal{L} u\|_{L^2\left((0,T), W_{X^R}^{s,2}(\mathbb{G})\right)} + \|\zeta u\|_{L^2(\mathbb{G}_T)} \right). \tag{2.9}$$

Proof. Applying to $\zeta_0 u$ Corollary 2.11 and Theorem 2.10 with $m = s+1$ and $\varepsilon = 1/s$ we can write:

$$|\zeta_0 u|_1^R \leqslant c\left\{|\zeta_0 u|_{s+1,1+1/s}^R + \|\zeta_0 u\|_{L^2(\mathbb{G}_T)}\right\} \leqslant c\left(\sum_{j=0}^{s} |\zeta \mathcal{L} u|_j^R + \|\zeta u\|_{L^2(\mathbb{G}_T)}\right)$$

From this inequality, by Propositions 2.7 and 2.6 we conclude the desired result. ■

To iterate this result to higher order derivatives, we first need a regularization result allowing to apply (2.9) to functions u satisfying weaker assumptions.

Proposition 2.13 *Let $u \in W^{1,2}\left((0,T), L^2_{loc}(\mathbb{G})\right), u(0,\cdot) = 0,$ be a weak solution to $\mathcal{L}u = F \in L^2\left((0,T), L^2_{loc}(\mathbb{G})\right)$ in the following sense*

$$\begin{aligned}&\int_{\mathbb{G}} \left\{-\partial_t u(t,x)\,\phi(x) + \sum_{i,j=1}^{q} a_{ij}(t)\, X_i X_j \phi(x)\, u(t,x)\right\} dx \\ &= \int_{\mathbb{G}} F(t,x)\,\phi(x)\,dx \text{ for every } \phi \in C_0^\infty(\mathbb{G}) \text{ and a.e. } t \in (0,T). \qquad (2.10)\end{aligned}$$

If $F \in L^2\left((0,T), W^{s^2,2}_{X^R,loc}(\mathbb{G})\right)$ then $u \in L^2\left((0,T), W^{1,2}_{X^R,loc}(\mathbb{G})\right)$ and for every $\zeta, \zeta_1 \in C_0^\infty(\mathbb{G})$ with $\zeta \prec \zeta_1$ the following estimate holds:

$$\|\zeta u\|_{L^2\left((0,T),W^{1,2}_{X^R}(\mathbb{G})\right)} \leqslant c\left\{\|\zeta_1 F\|_{L^2\left((0,T),W^{s,2}_{X^R}(\mathbb{G})\right)} + \|\zeta_1 u\|_{L^2(\mathbb{G}_T)}\right\} \qquad (2.11)$$

with $c = c(\zeta_0, \zeta, \mathbb{G}, \nu)$.

Proof. Let us define the ε-mollified u_ε of u as follows. For $\phi \in C_0^\infty(\mathbb{G})$ such that

$$\phi \geqslant 0,\ \phi(x) = 0 \text{ for } \|x\| \geqslant 1 \text{ and } \int_{\mathbb{G}} \phi(x)\,dx = 1,$$

define, for any $\varepsilon > 0$,

$$\phi_\varepsilon(x) = \varepsilon^{-Q}\phi\left(D_{\varepsilon^{-1}} x\right)$$

and

$$u_\varepsilon(t,x) = (\phi_\varepsilon * u)(t,x) = \int_{\mathbb{G}} \phi_\varepsilon(y)\, u\left(t, y^{-1}\circ x\right) dy = \int_{\mathbb{G}} \phi_\varepsilon\left(x \circ z^{-1}\right) u(t,z)\,dz.$$

Now the function u_ε is smooth with respect to x (as can be seen computing $X_I^R u_\varepsilon$), while

$$\frac{\partial u_\varepsilon}{\partial t} = \phi_\varepsilon * \frac{\partial u}{\partial t}$$

and, for any couple of domains $K \Subset K' \Subset \mathbb{G}$ and ε small enough,

$$\begin{aligned} \left\| \frac{\partial u_\varepsilon}{\partial t}(t,\cdot) \right\|_{L^2(K)} &\leqslant \left\| \frac{\partial u}{\partial t}(t,\cdot) \right\|_{L^2(K')} \\ \left\| \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^2((0,T),L^2(K))} &\leqslant \left\| \frac{\partial u}{\partial t} \right\|_{L^2((0,T),L^2(K'))} . \end{aligned}$$

Here we have used Young's inequality in the form

$$\|f * \phi_\varepsilon\|_{L^2(K)} \leqslant \|f\|_{L^2(K')} \tag{2.12}$$

for $K \Subset K'$, and ε small enough, since ϕ_ε is compactly supported.

Also,

$$u_\varepsilon(0,x) = \int_{\mathbb{G}} \phi_\varepsilon(y)\, u\left(0, y^{-1} \circ x\right) dy = 0,$$

hence $u_\varepsilon \in \mathcal{H}_0$ and we can apply to u_ε the estimate proved in Proposition 2.12:

$$\|\zeta u_\varepsilon\|_{L^2\left((0,T),W^{1,2}_{X^R}(\mathbb{G})\right)} \leqslant c \left\{ \|\zeta_1 \mathcal{L}(u_\varepsilon)\|_{L^2\left((0,T),W^{s,2}_{X^R}(\mathbb{G})\right)} + \|\zeta_1 u_\varepsilon\|_{L^2(\mathbb{G}_T)} \right\}. \tag{2.13}$$

We claim that

$$\mathcal{L}(u_\varepsilon) = F_\varepsilon \tag{2.14}$$

for *a.e.* t and *a.e.* x. This is not trivial since $\mathcal{L}u$ just exists in the above weak sense, hence we cannot simply write $\mathcal{L}(u_\varepsilon) = (\mathcal{L}u)_\varepsilon$. However, for every $\varphi \in C_0^\infty(\mathbb{G})$, letting

$$\mathcal{L} = -\partial_t + \mathcal{A}$$

with

$$\mathcal{A}u(t,x) = \sum_{i,j=1}^{q} a_{ij}(t)\, X_i X_j u(t,x)$$

we can write:

$$\int_{\mathbb{G}} \mathcal{L}(u_\varepsilon)(t,x)\,\varphi(x)\,dx = \int_{\mathbb{G}} -\partial_t(u_\varepsilon)(t,x)\,\varphi(x)\,dx + \int_{\mathbb{G}} u_\varepsilon(t,x)\,\mathcal{A}\varphi(x)\,dx$$

Next,

$$\begin{aligned} &\int_{\mathbb{G}} \mathcal{A}\varphi(x) \left(\int \phi_\varepsilon(y)\, u\left(t, y^{-1} \circ x\right) dy \right) dx \\ &= \int_{\mathbb{G}} \phi_\varepsilon(y) \left(\int \mathcal{A}\varphi(x)\, u\left(t, y^{-1} \circ x\right) dx \right) dy \\ &= \int_{\mathbb{G}} \phi_\varepsilon(y) \left(\int \mathcal{A}\varphi(y \circ z)\, u(t,z)\, dz \right) dy \end{aligned}$$

and

$$
\begin{aligned}
\int_{\mathbb{G}} \partial_t (u_\varepsilon)(t,x)\,\varphi(x)\,dx &= \int_{\mathbb{G}} \left(\int \phi_\varepsilon(y)\,\partial_t u\left(t, y^{-1}\circ x\right) dy \right) \varphi(x)\,dx \\
&= \int_{\mathbb{G}} \phi_\varepsilon(y) \left(\int \partial_t u\left(t, y^{-1}\circ x\right) \varphi(x)\,dx \right) dy \\
&= \int_{\mathbb{G}} \phi_\varepsilon(y) \left(\int \partial_t u(t,z)\,\varphi(y\circ z)\,dz \right) dy
\end{aligned}
$$

letting $\psi_y(z) = \varphi(y\circ z)$

$$
\begin{aligned}
&\int_{\mathbb{G}} \mathcal{L}(u_\varepsilon)(t,x)\,\varphi(x)\,dx \\
&= \int_{\mathbb{G}} \phi_\varepsilon(y) \left(\int_{\mathbb{G}} -\partial_t u(t,z)\,\psi_y(z) + \mathcal{A}\psi_y(z)\,u(z)\,dz \right) dy \\
&= \int_{\mathbb{G}} \phi_\varepsilon(y) \left(\int_{\mathbb{G}} \psi_y(z)\,F(t,z)\,dz \right) dy \\
&= \int_{\mathbb{G}} \phi_\varepsilon(y) \left(\int_{\mathbb{G}} \varphi(x)\,F\left(t, y^{-1}\circ x\right) dx \right) dy \\
&= \int_{\mathbb{G}} \varphi(x) \left(\int_{\mathbb{G}} \phi_\varepsilon(y)\,F\left(t, y^{-1}\circ x\right) dy \right) dx = \int_{\mathbb{G}} \varphi(x)\,F_\varepsilon(t,x)\,dx
\end{aligned}
$$

and (2.14) follows. By known properties of the mollifiers, as $\varepsilon \to 0$ we have $\phi_\varepsilon * u \to u$ in $L^2\left(\mathbb{R}^N\right)$ as soon as $u \in L^2\left(\mathbb{R}^N\right)$. Also, for every left invariant differential operator L we can write $L(\phi_\varepsilon * u) = \phi_\varepsilon * Lu$ as soon as Lu exists in $L^2\left(\mathbb{R}^N\right)$. Therefore

$$
\zeta_1 \mathcal{L}(u_\varepsilon) = \zeta_1 F_\varepsilon \to \zeta_1 F \text{ in } W_X^{k,2}(\mathbb{G}), \text{ for } a.e.\ t \tag{2.15}
$$

as soon as $F \in L^2\left((0,T), W_{X,loc}^{k,2}(\mathbb{G})\right)$.

To prove convergence in $L^2\left((0,T), W_{X^R,loc}^{s,2}(\mathbb{G})\right)$ we make the following rough estimates:

$$
\begin{aligned}
\left\| \zeta_1 \mathcal{L}(u_\varepsilon) - \zeta_1 F \right\|_{L^2\left((0,T), W_{X^R}^{s,2}(\mathbb{G})\right)} &\leqslant c \left\| \zeta_1 (\mathcal{L}u)_\varepsilon - \zeta_1 F \right\|_{L^2((0,T), W^{s,2}(\mathbb{G}))} \\
&\leqslant c \left\| \zeta_1 F_\varepsilon - \zeta_1 F \right\|_{L^2\left((0,T), W_X^{s^2,2}(\mathbb{G})\right)}.
\end{aligned} \tag{2.16}
$$

In the first inequality we have bounded the Sobolev norm $W_{X^R}^{s,2}$ (on a compact set containing the support of ζ_1) with the Euclidean Sobolev norm on the same domain; in the second one we have exploited Hörmander's condition.

We want to show that, for $F \in L^2\left((0,T), W_{X,loc}^{s^2,2}(\mathbb{G})\right)$,

$$
\left\| \zeta_1 F_\varepsilon - \zeta_1 F \right\|_{L^2\left((0,T), W_X^{s^2,2}(\mathbb{G})\right)} \to 0. \tag{2.17}
$$

Now:

$$\begin{aligned}&\left\|\zeta_1 F_\varepsilon-\zeta_1 F\right\|^2_{L^2\left((0,T),W_X^{s^2,2}(\mathbb{G})\right)}\\&=\int_0^T\left\|\zeta_1 F_\varepsilon(t,\cdot)-\zeta_1 F(t,\cdot)\right\|^2_{W_X^{s^2,2}(\mathbb{G})}dt\equiv\int_0^T g_\varepsilon(t)\,dt\end{aligned}$$

where by (2.15) we already know that

$$g_\varepsilon(t)\to 0 \text{ for } a.e.t\in[0,T]\,,\text{ as }\varepsilon\to 0.$$

To apply Lebesgue theorem and conclude the desired result we need to bound g_ε with an integrable function independent of ε. Now:

$$\begin{aligned}\left\|\zeta_1 F_\varepsilon(t,\cdot)-\zeta_1 F(t,\cdot)\right\|_{W_X^{s^2,2}(\mathbb{G})}&\leqslant\left\|\zeta_1 F_\varepsilon(t,\cdot)\right\|_{W_X^{s^2,2}(\mathbb{G})}+\left\|\zeta_1 F(t,\cdot)\right\|_{W_X^{s^2,2}(\mathbb{G})}\\\left\|\zeta_1 F_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{G})}&\leqslant\left\|F_\varepsilon(t,\cdot)\right\|^2_{L^2(K)}\leqslant\left\|F(t,\cdot)\right\|^2_{L^2(K')}\in L^1(0,T)\end{aligned}$$

where $K\Subset K'\Subset\mathbb{G}$ and ε small enough (see (2.12)). By (1.3), we have $X_i(F_\varepsilon)=(X_iF)_\varepsilon$, then

$$\begin{aligned}X_i(\zeta_1F_\varepsilon)&=(X_i\zeta_1)F_\varepsilon+\zeta_1(X_iF)_\varepsilon\,,\\\left\|X_i(\zeta_1F_\varepsilon(t,\cdot))\right\|^2_{L^2(\mathbb{G})}&\leqslant c\left(\left\|F_\varepsilon(t,\cdot)\right\|^2_{L^2(K)}+\left\|(X_iF)_\varepsilon(t,\cdot)\right\|^2_{L^2(K)}\right)\\&\leqslant c\left(\left\|F(t,\cdot)\right\|^2_{L^2(K')}+\left\|X_iF(t,\cdot)\right\|^2_{L^2(K')}\right)\in L^1(0,T)\,,\end{aligned}$$

and an interative reasoning allows to conclude (2.17) Recalling (2.16) and the fact that

$$\left\|\zeta_1u_\varepsilon\right\|_{L^2(\mathbb{G}_T)}\to\left\|\zeta_1u\right\|_{L^2(\mathbb{G}_T)}\,,$$

we conclude that the right hand side of (2.13) is bounded. Hence the sequence ζu_ε is bounded in $L^2\left((0,T)\,,W^{1,2}_{X^R}(\mathbb{G})\right)$, and there exists a subsequence of ζu_ε weakly converging in $L^2\left((0,T)\,,W^{1,2}_{X^R}(\mathbb{G})\right)$ to some g and in particular weakly converging in $L^2(\mathbb{G}_T)$ to ζu. This is enough to say that $\zeta u\in L^2\left((0,T)\,,W^{1,2}_{X^R}(\mathbb{G})\right)$. Moreover,

$$\begin{aligned}\left\|\zeta u\right\|_{L^2\left((0,T),W^{1,2}_{X^R}(\mathbb{G})\right)}&\leqslant\liminf\left\|\zeta u_\varepsilon\right\|_{L^2\left((0,T),W^{1,2}_{X^R}(\mathbb{G})\right)}\\&\leqslant c\left\{\left\|\zeta_1F\right\|_{L^2\left((0,T),W^{s,2}_{X^R}(\mathbb{G})\right)}+\left\|\zeta_1u\right\|_{L^2(\mathbb{G}_T)}\right\}\end{aligned}$$

hence (2.11) holds. ■

Theorem 2.14 (Regularity estimates in x) *Let $u\in W^{1,2}\left((0,T)\,,L^2_{loc}(\mathbb{G})\right)$, $u(0,\cdot)=0$, be a weak solution to $\mathcal{L}u=F\in L^2\left((0,T)\,,L^2_{loc}(\mathbb{G})\right)$ in the sense of (2.10) and let $\zeta,\zeta_1\in C_0^\infty(\mathbb{G})\,,\zeta\prec\zeta_1$. Then for any $k=1,2,3,...$, there exists*

$c = c(k, \zeta, \zeta_1, \mathbb{G}, \nu) > 0$ *such that whenever* $\zeta_1 F \in L^2\left((0,T), W^{k+s^2-1,2}_{X^R}(\mathbb{G})\right)$ *then* $\zeta u \in L^2\left((0,T), W^{k,2}_{X^R}(\mathbb{G})\right)$ *and*

$$\|\zeta u\|_{W^{k,2}_{X^R}(\mathbb{G}_T)} \leqslant c\left\{\|\zeta_1 F\|_{L^2\left((0,T), W^{k+s-1,2}_{X^R}(\mathbb{G})\right)} + \|\zeta_1 u\|_{L^2(\mathbb{G}_T)}\right\}. \tag{2.18}$$

Proof. We will prove (2.18) by induction on k. For $k = 1$ this is exactly Proposition 2.13. Assume that (2.18) holds up to an integer k and let $u \in \mathcal{H}_0$ such that $\mathcal{L}u \in L^2\left((0,T), W^{k+s^2,2}_{X^R}(\mathbb{G})\right)$. By the inductive assumption, $\zeta u \in L^2\left((0,T), W^{k,2}_{X^R}(\mathbb{G})\right)$. Let X^R_I be a right invariant differential operator with $|I| \leqslant k$, then $\zeta X^R_I u \in L^2(\mathbb{G}_T)$. We would like to apply Proposition 2.13 to $X^R_I u$, but in order to do that we would need to know that $X^R_I u \in W^{1,2}\left((0,T), L^2_{loc}(\mathbb{G})\right)$ with $X^R_I u(0,\cdot) = 0$, which is unclear. Then, let u_ε be the mollified version of u as in the proof of Proposition 2.13, so that:

$$X^R_I(u_\varepsilon)(t,x) = \int_{\mathbb{G}} \left(X^R_I \phi_\varepsilon\right)\left(x \circ z^{-1}\right) u(t,z)\, dz$$

which is a smooth function in x, and since $X^R_I \phi_\varepsilon$ is integrable (although its $L^1(\mathbb{G})$ norm is not uniformly bounded with respect to ε) we have

$$X^R_I(u_\varepsilon) \in L^2\left((0,T), L^2_{loc}(\mathbb{G})\right)$$

(see (2.12)) and since $\partial_t u \in L^2\left((0,T), L^2_{loc}(\mathbb{G})\right)$, the same is true for $\partial_t X^R_I(u_\varepsilon)$, which equals $X^R_I(\partial_t u)_\varepsilon$. Then

$$X^R_I(u_\varepsilon) \in W^{1,2}\left((0,T), L^2_{loc}(\mathbb{G})\right)$$

which also implies

$$X^R_I(u_\varepsilon)(0,x) = \int_{\mathbb{G}} \left(X^R_I \phi_\varepsilon\right)\left(x \circ z^{-1}\right) u(0,z)\, dz = 0$$

since $u(0,\cdot) = 0$ in $L^2(\mathbb{G})$. We claim that

$$\mathcal{L}\left(X^R_I(u_\varepsilon)\right) = X^R_I(\mathcal{L}(u_\varepsilon)) = X^R_I(F_\varepsilon)$$

at least in weak sense. Actually, noting that $\mathcal{L}$ and X^R_I commute,

$$\begin{aligned}\int_{\mathbb{G}} \mathcal{L}\left(X^R_I(u_\varepsilon)\right)(t,x)\,\varphi(x)\, dx &= \int_{\mathbb{G}} X^R_I(\mathcal{L}(u_\varepsilon))(t,x)\,\varphi(x)\, dx \\ &= -\int_{\mathbb{G}} \mathcal{L}(u_\varepsilon)(t,x)\left(X^R_I\varphi\right)(x)\, dx\end{aligned}$$

since $X^R_I\varphi \in C^\infty_0(\mathbb{G})$ and $\mathcal{L}(u_\varepsilon) = F_\varepsilon$ for a.e. t and x (see (2.14))

$$= -\int_{\mathbb{G}} F_\varepsilon(t,x)\left(X^R_I\varphi\right)(x)\, dx = \int_{\mathbb{G}} X^R_I(F_\varepsilon)(t,x)\,\varphi(x)\, dx$$

for a.e. t. Therefore we can apply Proposition 2.13 to $X_I^R(u_\varepsilon)$ getting

$$\begin{aligned}&\left\|\zeta X_I^R(u_\varepsilon)\right\|_{L^2\left((0,T),W^{1,2}_{X^R}(\mathbb{G})\right)}\\&\leqslant c\left\{\left\|\zeta_1 X_I^R(F_\varepsilon)\right\|_{L^2\left((0,T),W^{s,2}_{X^R}(\mathbb{G})\right)}+\left\|\zeta_1 X_I^R(u_\varepsilon)\right\|_{L^2(\mathbb{G}_T)}\right\}.\end{aligned}$$

Noting that

$$\left\|\zeta_1 X_I^R(u_\varepsilon)\right\|_{L^2(\mathbb{G}_T)}\leqslant\left\|\zeta_1 X_{I'}^R(u_\varepsilon)\right\|_{L^2\left((0,T),W^{1,2}_{X^R}(\mathbb{G})\right)}$$

for some I' with $|I'|=|I|-1$, we can proceed iteratively getting, for some different cutoff function $\zeta_2\succ\zeta_1$,

$$\left\|\zeta X_I^R(u_\varepsilon)\right\|_{L^2\left((0,T),W^{1,2}_{X^R}(\mathbb{G})\right)}\leqslant c\left\{\left\|\zeta_2 X_I^R(F_\varepsilon)\right\|_{L^2\left((0,T),W^{s,2}_{X^R}(\mathbb{G})\right)}+\left\|\zeta_2 u_\varepsilon\right\|_{L^2(\mathbb{G}_T)}\right\}. \tag{2.19}$$

From this bound, which is uniform with respect to ε, reasoning like in the proof of Proposition 2.13 we read that, under the assumption $X_I^R F\in L^2\left((0,T),W^{s^2,2}_{X^R}(\mathbb{G})\right)$, which is true as soon as $F\in L^2\left((0,T),W^{k+s^2,2}_{X^R}(\mathbb{G})\right)$, we have the uniform boundedness of

$$\left\|\zeta X_I^R(u_\varepsilon)\right\|_{L^2\left((0,T),W^{1,2}_{X^R}(\mathbb{G})\right)},$$

which implies the weak convergence in $L^2\left((0,T),W^{1,2}_{X^R}(\mathbb{G})\right)$ of (a subsequence of) $\zeta X_I^R(u_\varepsilon)$ to some g. In particular the convergence is in $L^2(\mathbb{G}_T)$, which implies that for every $\eta\in L^2(0,T)$ and $\phi\in C_0^\infty(\mathbb{G})$

$$\int_0^T\eta(t)\int_{\mathbb{G}}\zeta(x)X_I^R(u_\varepsilon)(t,x)\phi(x)\,dxdt\to\int_0^T\eta(t)\int_{\mathbb{G}}g(t,x)\phi(x)\,dxdt.$$

Pick the cutoff function $\zeta(x)=1$ on some bounded open set Ω, then for every $\phi\in C_0^\infty(\Omega)$ we have

$$\int_0^T\eta(t)\int_{\mathbb{G}}X_I^R(u_\varepsilon)(t,x)\phi(x)\,dxdt\to\int_0^T\eta(t)\int_{\mathbb{G}}g(t,x)\phi(x)\,dxdt.$$

On the other hand,

$$\begin{aligned}&\int_0^T\eta(t)\int_{\mathbb{G}}X_I^R(u_\varepsilon)(t,x)\phi(x)\,dxdt\\&=(-1)^{|I|}\int_0^T\eta(t)\int_{\mathbb{G}}u_\varepsilon(t,x)X_I^R\phi(x)\,dxdt\\&\to(-1)^{|I|}\int_0^T\eta(t)\int_{\mathbb{G}}u(t,x)X_I^R\phi(x)\,dxdt,\end{aligned}$$

hence

$$\int_0^T \eta(t) \int_{\mathbb{G}} g(t,x)\,\phi(x)\,dxdt = (-1)^{|I|} \int_0^T \eta(t) \int_{\mathbb{G}} u(t,x)\,X_I^R\phi(x)\,dxdt$$

which implies, for a.e. t and a.e. $x \in \Omega$,

$$g(t,x) = X_I^R u(t,x)$$

in the sense of weak derivatives. This means that $\zeta X_I^R u \in L^2\left((0,T), W_{X^R}^{1,2}(\mathbb{G})\right)$ and $\zeta X_I^R(u_\varepsilon) \to \zeta X_I^R u$ weakly in $L^2\left((0,T), W_{X^R}^{1,2}(\mathbb{G})\right)$, which also implies, by (2.19),

$$\left\|\zeta X_I^R u\right\|_{L^2\left((0,T),W_{X^R}^{1,2}(\mathbb{G})\right)} \leqslant c\left\{\left\|\zeta_2 X_I^R F\right\|_{L^2\left((0,T),W_{X^R}^{s,2}(\mathbb{G})\right)} + \|\zeta_2 u\|_{L^2(\mathbb{G}_T)}\right\}.$$

So we are done. ■

Next, we want to derive from the previous result the fact that, for F smooth enough, weak solutions to $\mathcal{L}u = F$ are actually strong solutions. Also, we want to establish Hölder continuity with respect to time of solutions (and their space derivatives):

Theorem 2.15 *Let $u \in W^{1,2}\left((0,T), L^2_{loc}(\mathbb{G})\right), u(0,\cdot) = 0$, be a weak solution to $\mathcal{L}u = F \in L^2\left((0,T), L^2_{loc}(\mathbb{G})\right)$ in the sense of (2.10).*

(i) For any $k = 0,1,2,3,...$

$$\text{if } F \in L^2\left((0,T), W_{X^R,loc}^{k+s^2+2s-1,2}(\mathbb{G})\right) \text{ then } u \in W^{1,2}\left((0,T), W_{X^R,loc}^{k+2s,2}(\mathbb{G})\right)$$

and u is also a strong solution to $\mathcal{L}u = F$. In particular, for every multiindex I with $|I| \leqslant k$ we have

$$X_I^R u \in C^0\left([0,T], L^2_{loc}(\mathbb{G})\right) \text{ and } X_I^R u(0,\cdot) = 0.$$

(ii) For every (cartesian) derivative ∂_x^α and $\zeta, \zeta_1 \in C_0^\infty(\mathbb{G}), \zeta \prec \zeta_1$, there exists $c = c(\alpha, \zeta, \zeta_1, \mathbb{G}, \nu) > 0$ and a positive integer h such that whenever $F \in L^2\left((0,T), W_{X^R,loc}^{h,2}(\mathbb{G})\right)$ then

$$\sup_{0<t_1<t_2<T} \sup_{x\in\mathbb{G}} \frac{\left|\zeta(x)\left[\partial_x^\alpha u(t_2,x) - \partial_x^\alpha u(t_1,x)\right]\right|}{|t_2-t_1|^{1/2}} \leqslant c\left\{\|\zeta_1 F\|_{L^2\left((0,T),W_{X^R}^{h,2}(\mathbb{G})\right)} + \|\zeta_1 u\|_{L^2(\mathbb{G}_T)}\right\}$$

and

$$\sup_{x\in\mathbb{G}} \left|\zeta(x)\,\partial_x^\alpha u(t,x)\right| \leqslant c\,|t|^{1/2}\left\{\|\zeta_1 F\|_{L^2\left((0,T),W_{X^R}^{h,2}(\mathbb{G})\right)} + \|\zeta_1 u\|_{L^2(\mathbb{G}_T)}\right\} \ \forall t \in [0,T].$$

(iii) In particular, if

$$\zeta_1 F \in L^2\left((0,T), C^\infty(\mathbb{G})\right)$$

then

$$\zeta u \in C^0\left([0,T], C^\infty(\mathbb{G})\right) \text{ and } \zeta u_t \in L^2\left((0,T), C^\infty(\mathbb{G})\right).$$

Proof. Let $\zeta \in C_0^\infty(\mathbb{G})$ and $u \in W^{2s,2}_{X^R,loc}(\mathbb{G})$. Inequalities

$$\|\zeta u\|_{W_X^{2,2}(\mathbb{G})} \leqslant c\|\zeta u\|_{W^{2,2}(\mathbb{G})} \leqslant c\|\zeta u\|_{W^{2s,2}_{X^R}(\mathbb{G})}$$

show that

$$W^{2s,2}_{X^R,loc}(\mathbb{G}) \subset W^{2,2}_{X,loc}(\mathbb{G}).$$

Let $u \in W^{1,2}\left((0,T), L^2_{loc}(\mathbb{G})\right), u(0,\cdot) = 0$, be a weak solution to $\mathcal{L}u = F \in L^2\left((0,T), W^{h,2}_{X^R,loc}(\mathbb{G})\right)$. By Theorem 2.14, if $\zeta_1 F \in L^2\left((0,T), W^{k+s^2-1,2}_{X^R}(\mathbb{G})\right)$, then $\zeta u \in L^2\left((0,T), W^{k,2}_{X^R}(\mathbb{G})\right)$. In particular, if $h \geq 2s + s^2 - 1$ then $u \in L^2\left((0,T), W^{2,2}_{X,loc}(\mathbb{G})\right)$ and this implies that u is actually a strong solution to the equation $\mathcal{L}u = F$, so that for a.e. t and a.e. x we have

$$-u_t(t,x) + \sum_{i,j=1}^{q} a_{ij}(t) X_i X_j u(t,x) = F(t,x). \tag{2.20}$$

This identity allows to transfer further x-regularity of both F and u to u_t: if, for some $k = 1,2,3,...$, we know that $h \geq k + 2s + s^2 - 1$, then by Theorem 2.14 $u \in L^2\left((0,T), W^{k+2s,2}_{X^R,loc}(\mathbb{G})\right)$, so that $X_i X_j u \in L^2\left((0,T), W^{k,2}_{X^R,loc}(\mathbb{G})\right)$, hence by (2.20) $u_t \in L^2\left((0,T), W^{k,2}_{X^R,loc}(\mathbb{G})\right)$ and $u \in W^{1,2}\left((0,T), W^{k,2}_{X^R,loc}(\mathbb{G})\right)$.

This implies that for $|I| \leqslant k$, $X_I^R u \in C^0\left([0,T], L^2_{loc}(\mathbb{G})\right)$. Moreover we can write, for every $t_1, t_2 \in [0,T]$ and a.e. $x \in \mathbb{G}$,

$$u(t_2,x) - u(t_1,x) = \int_{t_1}^{t_2} \partial_t u(t,x)\,dt \tag{2.21}$$

$$X_I^R u(t_2,x) - X_I^R u(t_1,x) = \int_{t_1}^{t_2} \partial_t X_I^R u(t,x)\,dt. \tag{2.22}$$

Letting $t_1 = 0$ in (2.21) we get

$$u(t_2,x) = \int_0^{t_2} \partial_t u(t,x)\,dt,$$

an identity which can also be differentiated with respect to X_I^R, giving

$$X_I^R u(t_2,x) = \int_0^{t_2} X_I^R \partial_t u(t,x)\,dt,$$

which implies

$$X_I^R u(0,\cdot) = 0.$$

This completes the proof of (i). Next, multiplying both sides of (2.22) for $\zeta \in C_0^\infty(\mathbb{G})$ and taking $L^2(\mathbb{G})$-norms we get, recalling that X_I^R commutes with $\mathcal{L}$:

$$\begin{aligned}
&\int_{\mathbb{G}} \zeta(x)^2 \left|X_I^R u(t_2,x) - X_I^R u(t_2,x)\right|^2 dx \\
&\leqslant \int_{\mathbb{G}} \zeta(x)^2 \left| \int_{t_1}^{t_2} \left\{ -X_I^R \mathcal{L} u(t,x) + \sum_{i,j=1}^q a_{ij}(t) X_i X_j X_I^R u(t,x) \right\} dt \right|^2 dx \\
&\leqslant \int_{\mathbb{G}} \zeta(x)^2 \left(\int_{t_1}^{t_2} \left\{ \left|X_I^R F(t,x)\right| + c_\nu \sum_{i,j=1}^q \left|X_i X_j X_I^R u(t,x)\right| \right\} dt \right)^2 dx \\
&\leqslant \int_{\mathbb{G}} \zeta(x)^2 |t_2 - t_1| \left\{ \int_0^T \left|X_I^R F(t,x)\right|^2 dt + c_\nu \sum_{i,j=1}^q \int_0^T \left|X_i X_j X_I^R u(t,x)\right|^2 dt \right\} dx \\
&= |t_2 - t_1| \left\{ \left\|\zeta X_I^R F\right\|^2_{L^2(\mathbb{G}_T)} + c_\nu \sum_{i,j=1}^q \left\|\zeta X_i X_j X_I^R u\right\|^2_{L^2(\mathbb{G}_T)} \right\}.
\end{aligned}$$

By Theorem 2.14 this implies that

$$\begin{aligned}
&\sup_{0<t_1<t_2<T} \frac{\int_{\mathbb{G}} \zeta(x)^2 \left|X_I^R u(t_2,x) - X_I^R u(t_2,x)\right|^2 dx}{|t_2 - t_1|} \\
&\leqslant c \left\{ \|\zeta_1 F\|_{L^2\left((0,T), W^{h,2}_{X^R}(\mathbb{G})\right)} + \|\zeta_1 u\|_{L^2(\mathbb{G}_T)} \right\}^2
\end{aligned}$$

for some h large enough and any cutoff function ζ_1 such that $\zeta \prec \zeta_1$. On the other hand, letting

$$v(x) = u(t_2,x) - u(t_2,x)$$

and noting that every cartesian derivative $\partial_x^\alpha v(x)$ can be bounded, uniformly on a compact set of $\mathbb{G}$ by a suitable linear combination of $X_I^R v$, we arrive to a bound

$$\begin{aligned}
&\sup_{0<t_1<t_2<T} \frac{\|\zeta\left[\partial_x^\alpha u(t_2,\cdot) - \partial_x^\alpha u(t_1,\cdot)\right]\|_{L^2(\mathbb{G})}}{|t_2 - t_1|^{1/2}} \\
&\leqslant c \left\{ \|\zeta_1 F\|_{L^2\left((0,T), W^{h_1,2}_{X^R}(\mathbb{G})\right)} + \|\zeta_1 u\|_{L^2(\mathbb{G}_T)} \right\}
\end{aligned}$$

for some integer $h_1 > h$. And since also the sup of $|\partial_x^\alpha u(t_2,\cdot) - \partial_x^\alpha u(t_1,\cdot)|$ can be bounded, by Sobolev embeddings, by suitable L^2 norms of higher order derivatives, we also have a control

$$\sup_{0<t_1<t_2<T} \sup_{x\in\mathbb{G}} \frac{\left|\zeta(x)\left[\partial_x^\alpha u(t_2,x) - \partial_x^\alpha u(t_1,x)\right]\right|}{\left|t_2-t_1\right|^{1/2}}$$
$$\leqslant c\left\{\left\|\zeta_1 F\right\|_{L^2\left((0,T),W_{X^R}^{h_2,2}(\mathbb{G})\right)} + \left\|\zeta_1 u\right\|_{L^2(\mathbb{G}_T)}\right\}$$

for some integer $h_2 > h_1$. Also, since $\partial_x^\alpha u(0,x) = 0$, this implies

$$\sup_{x\in\mathbb{G}} \left|\zeta(x)\,\partial_x^\alpha u(t,x)\right| \leqslant c\,|t|^{1/2}\left\{\left\|\zeta_1 F\right\|_{L^2\left((0,T),W_{X^R}^{h_2,2}(\mathbb{G})\right)} + \left\|\zeta_1 u\right\|_{L^2(\mathbb{G}_T)}\right\},$$

This ends the proof of (ii). The previous result also shows that

$$\zeta_1 F \in L^2(0,T), C^\infty(\mathbb{G}) \Longrightarrow \zeta u \in C^0([0,T], C^\infty(\mathbb{G})).$$

Then the equality

$$u_t = \sum_{i,j=1}^q a_{ij}(t)\,X_i X_j u - F$$

also implies that

$$\zeta u_t \in L^2((0,T), C^\infty(\mathbb{G})).$$

■

We end this section with an easy example showing that the regularity properties of the solution cannot be improved for bounded measurable coefficients $a_{ij}(t)$.

Example 2.16 *Let us consider the uniformly parabolic operator*

$$\mathcal{L}u = -u_t + a(t)\,u_{xx}$$

with $a \in L^\infty(\mathbb{R})$, $a(t) \geq \nu > 0$. *The function*

$$u(t,x) = \exp\left(-\int_0^t a(\tau)\,d\tau\right)\sin x$$

satisfies $\mathcal{L}u = 0$; u *is smooth w.r.t.* x *and only Lipschitz continuous w.r.t.* t. *Let*

$$U(t,x) = t^\alpha u(t,x) \text{ for some } \alpha \in \left(\frac{1}{2},1\right).$$

Then U *solves the problem*

$$\begin{cases} \mathcal{L}U = F \text{ for } x\in\mathbb{R}, t>0 \\ U(0,x) = 0 \end{cases}$$

with $F(t,x) = -\alpha t^{\alpha-1} u(t,x)$, *so that, as soon as* $\alpha > \frac{1}{2}$,

$$F \in L^2((0,1)\times\mathbb{R}).$$

Moreover,

$$U_t(t,x) = \alpha t^{\alpha-1} u(t,x) - t^{\alpha} a(t) u(t,x) \in L^2((0,T), C^{\infty}(\mathbb{R}))$$

Hence

$$U \in W^{1,2}((0,T), C^{\infty}(\mathbb{R})) \cap C^{0,\alpha}([0,T], C^{\infty}(\mathbb{R})).$$

Since $\alpha > \frac{1}{2}$ can be chosen as close to $1/2$ as we want, this shows that the regularity with respect to t expressed by Theorem 2.15 cannot be improved. Also, note that the Hölder continuity w.r.t. t cannot be improved to Lipschitz continuity just remaing far off $t = 0$: if we multiply the above $U(t,x)$ for $|t-t_0|^{\alpha}$ we get a similar example exhibiting a α-Hölder continuity w.r.t. t near $t = t_0$.

3 Regularization of distributional solutions

In this section we want to extend our smoothness result, established in Theorem 2.15 (iii) for functions in $W^{1,2}\left((0,T), L^2_{loc}(\mathbb{G})\right)$, to more general distributions. First of all, we need to make precise the distributional notions that we will use.

Definition 3.1 *Let $\Omega \subseteq \mathbb{G}$ be an open set. We will say that $u \in L^2((0,T), \mathcal{D}'(\Omega))$ if $u \in \mathcal{D}'(\Omega_T)$ and for every $\phi \in \mathcal{D}(\Omega)$ there exists a function $h_\phi \in L^2(0,T)$ such that for every $\psi \in \mathcal{D}(0,T)$,*

$$\langle u, \phi \otimes \psi \rangle = \int_0^T h_\phi(t) \psi(t)\, dt.$$

In this case we will write, more transparently, $h_\phi(t) = \langle u(t,\cdot), \phi \rangle$ and

$$\langle u, \phi(x) \psi(t) \rangle = \int_0^T \langle u(t,\cdot), \phi \rangle \psi(t)\, dt$$

for every $\phi \in \mathcal{D}(\Omega)$ and $\psi \in \mathcal{D}(0,T)$ (and therefore also for every $\psi \in L^2(0,T)$).

Analogously, we will say that $u \in W^{1,2}((0,T), \mathcal{D}'(\Omega))$ if $u \in \mathcal{D}'(\Omega_T)$ with both u and its distributional derivative $\partial_t u$ belonging to $L^2((0,T), \mathcal{D}'(\Omega))$.

We will say that u is a distributional solution to $\mathcal{L}u = F$ in Ω_T, with $F \in L^2((0,T), \mathcal{D}'(\Omega))$ if $u \in W^{1,2}((0,T), \mathcal{D}'(\Omega))$ and:

$$\langle -\partial_t u(t,\cdot), \phi \rangle + \sum_{i,j=1}^{q} a_{ij}(t) \langle X_i X_j u(t,\cdot), \phi \rangle = \langle F(t,\cdot), \phi \rangle$$

for every $\phi \in \mathcal{D}(\Omega)$ and a.e. $t \in (0,T)$, or equivalently:

$$\int_0^T \left\{ \langle -\partial_t u(t,\cdot), \phi \rangle + \sum_{i,j=1}^{q} a_{ij}(t) \langle u(t,\cdot), X_i X_j \phi \rangle \right\} \psi(t)\, dt$$
$$= \int_0^T \langle F(t,\cdot), \phi \rangle \psi(t)\, dt$$

$\forall \phi \in \mathcal{D}(\Omega), \psi \in L^2(0,T)$.

The proof of a regularity result for distributional solutions usually begins identifying the given distribution, locally, with some derivative of a continuous function, in view of the classical result about the local structure of distributions. For distributions in the class $L^2((0,T),\mathcal{D}'(\Omega))$ we could not find in the literature any reference for a similar result. So we will explicitly assume that our distribution could be seen, on a fixed domain compactly contained in Ω, as a space derivative of a suitable function:

Definition 3.2 *Let $u \in L^2((0,T),\mathcal{D}'(\Omega))$ for some open set $\Omega \subseteq \mathbb{G}$. We will say that u satisfies the x-finite order assumption on Ω if:*

there exists a function $h \in L^2\left((0,T),L^1_{loc}(\Omega)\right)$ and a multiindex α such that

$$u = \frac{\partial^\alpha h}{\partial x^\alpha} \text{ in } \mathcal{D}'(\Omega_T) \tag{3.1}$$

that is

$$\langle u, \phi(x)\psi(t)\rangle = \int_0^T \left((-1)^{|\alpha|}\int_{\Omega'} h(t,x)\frac{\partial^\alpha \phi}{\partial x^\alpha}(x)\,dx\right)\psi(t)\,dt$$

$\forall \phi \in \mathcal{D}(\Omega), \psi \in L^2(0,T)$.

If $u \in W^{1,2}((0,T),\mathcal{D}'(\Omega))$, we will say that u satisfies the x-finite order assumption on Ω if (3.1) holds with $h \in W^{1,2}\left((0,T),L^1_{loc}(\Omega)\right)$.

Note that if $u \in W^{1,2}((0,T),\mathcal{D}'(\Omega))$ satisfies the x-finite order assumption on Ω', then $h \in C^0\left([0,T],L^1_{loc}(\Omega)\right)$. In particular, saying that $u(0,\cdot) = 0$ means that $h(0,\cdot) = 0$ a.e. in Ω.

The aim of this section is to prove that:

Theorem 3.3 *For some bounded domain $\Omega \subset \mathbb{G}$, let u be a distributional solution to $\mathcal{L}u = F$ in Ω_T with $F \in L^2((0,T),\mathcal{D}'(\Omega))$. Assume that u satisfies the x-finite order assumption (see Definition 3.2) and $u(0,\cdot) = 0$ in Ω. Then, for every domain $\Omega' \Subset \Omega$, if*

$$F \in L^2\left((0,T),C^\infty\left(\overline{\Omega}\right)\right)$$

then

$$u \in C^0\left([0,T],C^\infty\left(\overline{\Omega'}\right)\right) \text{ and } u_t \in L^2\left((0,T),C^\infty\left(\overline{\Omega'}\right)\right).$$

In order to prove Theorem 3.3 we will adapt the technique used in [3, §4] for sublaplacians.

Let us consider the second order differential operator

$$\mathcal{L}^R = \sum_{j=1}^N \left(X_j^R\right)^2$$

built using the whole canonical base of right invariant vector fields. This is a right-invariant (but no longer homogeneous) uniformly elliptic operator in $\mathbb{G}$, which at the origin coincides with the standard Laplacian. The fundamental solution of the Laplacian can be proved to be a parametrix for $\mathcal{L}^R$:

Proposition 3.4 (see [3, Prop. 4.2.]) *Let $V \subset \mathbb{G}$ be a neighborhood of the origin. There exist $\widetilde{\gamma} \in C^{\infty}(\mathbb{G} \setminus \{0\})$ and $\omega \in C^{\infty}(\mathbb{G} \setminus \{0\})$, both supported in V, satisfying*

$$|\widetilde{\gamma}(x)| \leqslant \frac{c}{|x|^{N-2}} \tag{3.2}$$

$$|\partial_{x_i}\widetilde{\gamma}(x)| \leqslant \frac{c}{|x|^{N-1}} \quad i = 1, 2, ..., N \tag{3.3}$$

$$|\omega(x)| \leqslant \frac{c}{|x|^{N-2}}$$

and such that in the sense of distributions

$$\mathcal{L}^{R}\widetilde{\gamma} = -\delta + \omega.$$

(Here δ is the Dirac mass as a distribution in $\mathbb{R}^N$).

Let us now consider three open sets in $\mathbb{G}$, $\Omega' \Subset \Omega'' \Subset \Omega$ and let V be a neighborhood of the origin such that $V^{-1} \circ \Omega' \subset \Omega''$. Define $\widetilde{\gamma}$ as in Proposition 3.4, with $\widetilde{\gamma}$ supported in V. The convolution with $\widetilde{\gamma}$ defines a regularizing operator that acts on functions $u \in L^1_{loc}(\mathbb{G}_T)$ as follows. For every $x \in \Omega'$ and $t \in [0, T]$ we set

$$T_V u(t, x) = (\widetilde{\gamma} * u(t, \cdot))(x) = \int_{\mathbb{G}} \widetilde{\gamma}\left(x \circ y^{-1}\right) u(t, y)\, dy. \tag{3.4}$$

The subscript V in T_V recalls that the definition of the operator depends on the choice of the neighborhood V used to define $\widetilde{\gamma}$.

Note that

$$T_V : L^2\left((0, T), L^1(\Omega'')\right) \longrightarrow L^2\left((0, T), L^1(\Omega')\right).$$

Namely, for $x \in \Omega'$ and $x \circ y^{-1} \in \operatorname{sprt}\widetilde{\gamma}$, the point $y = \left(x \circ y^{-1}\right)^{-1} \circ x$ ranges in $V^{-1} \circ \Omega' \subset \Omega''$, hence introducing characteristic functions,

$$\chi_{\Omega'}(x)\, T_V u(t, x) = \int_{\mathbb{G}} (\widetilde{\gamma}\chi_V)\left(x \circ y^{-1}\right) u(t, y)\, \chi_{\Omega''}(y)\, dy,$$

or

$$\chi_{\Omega'} T_V u(t, \cdot) = \widetilde{\gamma}\chi_V * u(t, \cdot)\, \chi_{\Omega''} \tag{3.5}$$

which by Young's inequality gives, at least for a.e. t,

$$\|T_V u(t, \cdot)\|_{L^1(\Omega')} \leqslant \|\widetilde{\gamma}\|_{L^1(V)} \|u(t, \cdot)\|_{L^1(\Omega'')}$$

and hence

$$\|T_V u\|_{L^2((0,T), L^1(\Omega'))} \leqslant \|\widetilde{\gamma}\|_{L^1(V)} \|u\|_{L^2((0,T), L^1(\Omega''))}.$$

Also, T_V acts on distributions $u \in L^2((0,T), \mathcal{D}'(\Omega))$ as follows. For every $\varphi \in \mathcal{D}(\Omega')$ we set

$$\langle T_V u(t,\cdot), \varphi \rangle = \langle u(t,\cdot), T_V^* \varphi \rangle \tag{3.6}$$

where

$$T_V^* \varphi(y) = \int_{\mathbb{G}} \widetilde{\gamma}\left(x \circ y^{-1}\right) \varphi(x)\, dx.$$

Observe that the assumption on V implies that $T_V^* \varphi$ is a test function in Ω'. Namely, for $x \in \operatorname{sprt} \varphi$ and $x \circ y^{-1} \in \operatorname{sprt} \widetilde{\gamma}$ the point y ranges in $\Omega'' \Subset \Omega$. The function $T_V^* \varphi$ is smooth, as one can see writing

$$T_V^* \varphi(y) = \int_{\mathbb{G}} \widetilde{\gamma}(z)\, \varphi(z \circ y)\, dx$$

and computing left invariant derivatives

$$X_I (T_V^* \varphi)(y) = \int_{\mathbb{G}} \widetilde{\gamma}(z)\, (X_I \varphi)(z \circ y)\, dx.$$

Therefore the pairing (3.6) is well defined. Also, from the previous identity we easily read that if $\varphi_k \to 0$ in $\mathcal{D}(\Omega)$ then $T_V^* \varphi_k \to 0$ in $\mathcal{D}(\Omega')$. Hence $T_V u(t,\cdot) \in \mathcal{D}'(\Omega')$. Moreover

$$\int_0^T |\langle T_V u(t,\cdot), \varphi \rangle|^2\, dt = \int_0^T |\langle u(t,\cdot), T_V^* \varphi \rangle|^2\, dt < \infty$$

(just by definition of $L^2((0,T), \mathcal{D}'(\Omega))$), so that

$$T_V : L^2((0,T), \mathcal{D}'(\Omega)) \longrightarrow L^2((0,T), \mathcal{D}'(\Omega')).$$

Next, we need to prove the regularizing properties of T_V. The following result is an adaptation of [3, Prop. 4.4.].

Proposition 3.5 (Regularizing properties of T_V) *Let $\Omega' \Subset \Omega'' \Subset \Omega$. There exists a neighborhood V of the origin such that the operator T_V defined in (3.6) has the following properties.*

(1) Let $u \in \mathcal{D}'((0,T) \times \Omega)$ such that $u = \frac{\partial^\alpha}{\partial x^\alpha} g$, for some $g \in L^2\left((0,T), L^1_{loc}(\Omega)\right)$ and multiindex α. Then $T_V u \in L^2((0,T), \mathcal{D}'(\Omega'))$ and

$$T_V u = \sum_{|\beta| \leqslant |\alpha|-1} \partial_x^\beta A_\beta \text{ in } (0,T) \times \Omega'$$

for suitable $A_\beta \in L^2\left((0,T), L^1_{loc}(\Omega')\right)$.

(2) Let $u \in L^2((0,T), L^p_{loc}(\Omega))$ for some $1 \leqslant p < \frac{N}{2}$, then

$$T_V u \in L^2\left((0,T), L^{p'}(\Omega')\right) \text{ for } \frac{1}{p'} > \frac{1}{p} - \frac{2}{N}$$

and

$$\|T_V u\|_{L^2\left((0,T), L^{p'}(\Omega')\right)} \leqslant c\, \|u\|_{L^2\left((0,T), L^p_{loc}(\Omega)\right)}.$$

(3) Let $u \in L^2\left((0,T), L^2_{loc}(\Omega)\right)$ then $T_V u \in L^2\left((0,T), W_X^{1,2}(\Omega')\right)$.

(4) Let $u \in L^2\left((0,T), C^\infty\left(\overline{\Omega}\right)\right)$, then $T_V u \in L^2\left((0,T), C^\infty\left(\overline{\Omega'}\right)\right)$.

Remark 3.6 *Throughout the next proof, and also in other deductions in the following, all the stated equalities of the kind $A(t) = B(t)$ hold for a.e. t. Rigorously speaking, we should write chains of equalities of the kind*

$$\int_0^T A(t)\,\psi(t)\,dt = \int_0^T B(t)\,\psi(t)\,dt \text{ for every } \psi \in \mathcal{D}(0,T)$$

and then deduce that $A(t) = B(t)$ a.e.

Proof. This proof is an adaptation of the proof of [3, Prop. 4.4].

(1) Let $u = \partial_{x_i}\partial_x^{\alpha'} g$ for some α' with $|\alpha'| = |\alpha| - 1$. Then, for $\varphi \in C_0^\infty(\Omega')$

$$\begin{aligned}\langle T_V u(t,\cdot), \varphi\rangle &= \left\langle \partial_{y_i}\partial_y^{\alpha'} g(t,y), \int_{\mathbb{G}} \widetilde{\gamma}\left(x \circ y^{-1}\right)\varphi(x)\,dx \right\rangle \\ &= \left\langle \partial_y^{\alpha'} g(t,y), \int_{\mathbb{G}} -\partial_{y_i}\left[\widetilde{\gamma}\left(x \circ y^{-1}\right)\right]\varphi(x)\,dx \right\rangle.\end{aligned}$$

We can write

$$\begin{aligned}-\partial_{y_i}\left[\widetilde{\gamma}\left(x \circ y^{-1}\right)\right] &= -\sum_{k=1}^N \left(\partial_k\widetilde{\gamma}\right)\left(x \circ y^{-1}\right)\partial_{y_i}\left[x \circ y^{-1}\right]_k \\ &= \sum_{k=1}^N h_k\left(x \circ y^{-1}\right) Z_k(x,y)\end{aligned}$$

where by (3.3) $h_k(z)$ are locally integrable functions, smooth outside the pole, and Z_k are polynomials (these polynomials also depend on the index i corresponding to ∂_{y_i}, but for simplicity we suppress this unimportant index). Hence

$$\begin{aligned}\langle T_V u(t,\cdot), \varphi\rangle &= \left\langle \partial_y^{\alpha'} g(t,y), \int_{\mathbb{G}} \sum_{k=1}^N h_k\left(x \circ y^{-1}\right) Z_k(x,y)\,\varphi(x)\,dx \right\rangle \\ &= \left\langle \partial_y^{\alpha'} g(t,y), \int_{\mathbb{G}} \sum_{k=1}^N h_k(w)\, Z_k(w \circ y, y)\,\varphi(w \circ y)\,dw \right\rangle\end{aligned}$$

since the function $y \mapsto \int_{\mathbb{G}} \sum_{k=1}^N h_k(w)\, Z_k(w \circ y, y)\,\varphi(w \circ y)\,dw$ belongs to $\mathcal{D}(\Omega)$

$$= \left\langle g(t,y), \int_{\mathbb{G}} \sum_{k=1}^N h_k(w)\,(-1)^{|\alpha'|}\,\partial_y^{\alpha'}\left[Z_k(w \circ y, y)\,\varphi(w \circ y)\right] dw \right\rangle$$

Now,

$$(-1)^{|\alpha'|}\,\partial_y^{\alpha'}\left[Z_k(w \circ y, y)\,\varphi(w \circ y)\right] = \sum_{|\beta| \leqslant |\alpha'|} \left(D^\beta \varphi\right)(w \circ y)\, a_{\beta,k}(w,y)$$

for suitable polynomials $a_{\beta,k}$, hence

$$\langle T_V u(t,\cdot),\varphi\rangle = \int_{\mathbb{G}} g(t,y)\left(\int_{\mathbb{G}}\sum_{k=1}^{N} h_k(w)\sum_{|\beta|\leqslant|\alpha'|}\left(D^\beta\varphi\right)(w\circ y)\,a_{\beta,k}(w,y)\,dw\right)dy$$

$$= \int_{\mathbb{G}} g(t,y)\left(\int_{\mathbb{G}}\sum_{k=1}^{N} h_k\left(x\circ y^{-1}\right)\sum_{|\beta|\leqslant|\alpha'|}\left(D^\beta\varphi\right)(x)\,a_{\beta,k}\left(x\circ y^{-1},y\right)dx\right)dy$$

$$= \int_{\mathbb{G}}\sum_{|\beta|\leqslant|\alpha'|}\left(D^\beta\varphi\right)(x)\sum_{k=1}^{N}\left(\int_{\mathbb{G}} g(t,y)\,h_k\left(x\circ y^{-1}\right)a_{\beta,k}\left(x\circ y^{-1},y\right)dy\right)dx$$

Next, observe that

$$b_\beta(t,x) = \sum_{k=1}^{N}\int_{\mathbb{G}} g(t,y)\,h_k\left(x\circ y^{-1}\right)a_{\beta,k}\left(x\circ y^{-1},y\right)dy$$

belongs to $L^2\left((0,T),L^1_{loc}(\Omega')\right)$, since $g\in L^2\left((0,T),L^1_{loc}(\Omega)\right)$, $h_k\in L^1$ and is compactly supported in V, $a_{\beta,k}$ are polynomials: for every $K\Subset\Omega'$ there exist V and K' such that $K\Subset K'\Subset\Omega'$ such that

$$\int_K |b_\beta(t,x)|\,dx \leqslant \sum_{k=1}^{N}\int_K\int_{\mathbb{G}}\left|g(t,y)\,h_k\left(x\circ y^{-1}\right)a_{\beta,k}\left(x\circ y^{-1},y\right)\right|dydx$$
$$\leqslant c\int_{K'}|g(t,y)|\,dy$$

so that

$$\int_0^T \|b_\beta(t,\cdot)\|^2_{L^1(K)}\,dt \leqslant c\int_0^T \|g(t,\cdot)\|^2_{L^1(K')}\,dt$$

Hence, letting

$$A_\beta(t,x) = (-1)^{|\beta|}\,b_\beta(t,x)$$

we can write

$$\langle T_V u(t,\cdot),\varphi\rangle = \int_{\mathbb{G}}\sum_{|\beta|\leqslant|\alpha'|}(-1)^{|\beta|}A_\beta(t,x)\left(\partial_x^\beta\varphi\right)(x)\,dx$$
$$= \left\langle\sum_{|\beta|\leqslant|\alpha|-1}\partial_x^\beta A_\beta(t,\cdot),\varphi\right\rangle$$

with $A_\beta\in L^2\left((0,T),L^1_{loc}(\Omega')\right)$, hence $T_V u$ has the desired structure.

(2) Inequality

$$\|T_V u(t,\cdot)\|_{L^{p'}(\Omega')} \leqslant c\,\|u(t,\cdot)\|_{L^p(\Omega)} \text{ for a.e. } t\in[0,T]$$

follows from (3.5) and Young's inequality since, by (3.2), $\widetilde{\gamma}\in L^r(\mathbb{G})$ for $1\leqslant r<\frac{N}{N-2}$. Taking $L^2(0,T)$ norms we get (2).

(3) We know that any derivative $\partial_{x_i}\widetilde{\gamma}$ $(i=1,2,...,N)$ is integrable and supported in V, hence each function $X_i^R\widetilde{\gamma}$ $(i=1,2,...,N)$ is a linear combination with polynomial coefficients of integrable functions, compactly supported in V, so that $X_i^R\widetilde{\gamma}\in L^1(\mathbb{G})$. Also, for a.e. $t\in[0,T]$, $u\chi_{\Omega''}(t,\cdot)\in L^2(\mathbb{G})$ hence by Young's inequality

$$\chi_{\Omega'}T_V u(t,\cdot)=\widetilde{\gamma}*(u(t,\cdot)\chi_{\Omega''})\in L^2(\mathbb{G}),$$

that is $T_V u(t,\cdot)\in L^2(\Omega')$, with

$$\|T_V u(t,\cdot)\|_{L^2(\Omega')}\leqslant\|\widetilde{\gamma}\|_{L^1}\|u(t,\cdot)\|_{L^2(\Omega'')}$$

and

$$\chi_{\Omega'}X_i^R T_V u(t,\cdot)=\left(X_i^R\widetilde{\gamma}\right)*(u\chi_{\Omega''})\in L^2(\mathbb{G}),$$

that is $X_i^R T_V u\in L^2(\Omega')$. This holds for $i=1,2,...,N$ (not just for the first q derivatives). Now, let us recall that the left invariant vector fields X_i $(i=1,2,...,N)$ can be written as linear combinations with polynomial coefficients of the right invariant vector fields X_i^R. Hence by the boundedness of Ω' we also have

$$X_i T_V u(t,\cdot)\in L^2(\Omega')\text{ for }i=1,2,...,N$$

with

$$\|X_i T_V u(t,\cdot)\|_{L^2(\Omega')}\leqslant c\sum_{j=1}^N\left\|X_j^R\widetilde{\gamma}\right\|_{L^1}\|u(t,\cdot)\|_{L^2(\Omega'')}$$

in particular $T_V u\in L^2\left((0,T),W_X^{1,2}(\Omega')\right)$ with

$$\|T_V u\|_{L^2\left((0,T),W_X^{1,2}(\Omega')\right)}\leqslant\|\widetilde{\gamma}\|_{L^1}\|u\|_{L^2((0,T)\times\Omega'')}+c\sum_{j=1}^N\left\|X_j^R\widetilde{\gamma}\right\|_{L^1}\|u\|_{L^2((0,T)\times\Omega'')}.$$

(4). Let $u\in C^\infty\left(\overline{\Omega}\right)$. From

$$T_V u(t,x)=\int_{\mathbb{G}}\widetilde{\gamma}\left(x\circ y^{-1}\right)u(t,y)\,dy=\int_{\mathbb{G}}\widetilde{\gamma}(w)\,u\left(t,w^{-1}\circ x\right)dw$$

we read that for $x\in\Omega'$ and any left invariant differential operator $\mathcal{P}$ we can write

$$\mathcal{P}T_V u(t,x)=\int_V\widetilde{\gamma}(w)\,\mathcal{P}u\left(t,w^{-1}\circ x\right)dw,$$

showing that $\mathcal{P}T_V u(t,\cdot)\in C^\infty(\Omega')$. Moreover,

$$\max_{x\in\Omega'}|\mathcal{P}T_V u(t,x)|\leqslant c\max_{x\in\Omega}|\mathcal{P}u(t,x)|$$

so that, for every $k=0,1,2,...$

$$\|T_V u(t,\cdot)\|_{C^k(\Omega')}\leqslant\|u(t,\cdot)\|_{C^k(\Omega)}$$

and also

$$\int_0^T \|T_V u(t,\cdot)\|^2_{C^k(\Omega')} dt \leqslant \int_0^T \|u(t,\cdot)\|^2_{C^k(\Omega)} dt.$$

Hence $T_V u \in L^2\left((0,T), C^\infty\left(\overline{\Omega'}\right)\right)$. ■

Corollary 3.7 *Let $\Omega' \Subset \Omega \Subset \mathbb{G}$. For every distribution $u \in \mathcal{D}'((0,T)\times\Omega)$ such that $u = \partial_x^\alpha g$ for some multindex α and $g \in L^2\left((0,T), L^1_{loc}(\Omega)\right)$ there exist a neighborhood of the origin V and an integer K such that $(T_V)^K u \in L^2\left((0,T), W^{1,2}_X(\Omega')\right)$.*

The proof follows exactly that of [3, Corollary 4.5].

Proposition 3.8 *Let $\Omega' \Subset \Omega$ and V small enough so that $V \circ \Omega' \Subset \Omega$. Then for any distribution $u \in L^2((0,T), \mathcal{D}'(\Omega))$ and every left invariant operator $\mathcal{P}$ on $\mathbb{G}$ we have*

$$\mathcal{P}T_V u = T_V \mathcal{P}u \text{ in } L^2((0,T), \mathcal{D}'(\Omega')) \tag{3.7}$$

Also, if $u \in W^{1,2}((0,T), \mathcal{D}'(\Omega))$ then

$$\mathcal{L}T_V u = T_V \mathcal{L}u \text{ in } L^2((0,T), \mathcal{D}'(\Omega')) \tag{3.8}$$

Remark 3.9 *The previous proposition can be obviously iterated writing*

$$\mathcal{P}T_V^K u = T_V^K \mathcal{P}u \text{ in } L^2((0,T), \mathcal{D}'(\Omega'))$$
$$\mathcal{L}T_V^K u = T_V^K \mathcal{L}u \text{ in } L^2((0,T), \mathcal{D}'(\Omega'))$$

for any fixed positive integer K, provided V is chosen small enough to have

$$\underbrace{V \circ V \circ \ldots \circ V}_{K \text{ times}} \circ \Omega' \Subset \Omega.$$

Proof. Let $u \in L^2((0,T), \mathcal{D}'(\Omega))$, then $T_V u \in L^2((0,T), \mathcal{D}'(\Omega'))$ and for every $\varphi \in \mathcal{D}(\Omega')$ we can write, denoting by $\mathcal{P}^*$ the transpose operator of $\mathcal{P}$ and recalling that $\mathcal{P}^*$ is still left invariant,

$$\begin{aligned}
\langle \mathcal{P}T_V u(t,\cdot), \varphi\rangle = \langle T_V u(t,\cdot), \mathcal{P}^*\varphi\rangle &= \left\langle u(t,y), \int_{\mathbb{G}} \widetilde{\gamma}\left(x \circ y^{-1}\right) \mathcal{P}^*\varphi(x)\,dx \right\rangle \\
&= \left\langle u(t,y), \int_{\mathbb{G}} \widetilde{\gamma}(w)\, \mathcal{P}^*\varphi(w \circ y)\,dw \right\rangle \\
&= \left\langle u(t,y), \mathcal{P}^* \int_{\mathbb{G}} \widetilde{\gamma}(w)\, \varphi(w \circ y)\,dw \right\rangle \\
&= \left\langle \mathcal{P}u(t,y), \int_{\mathbb{G}} \widetilde{\gamma}(w)\, \varphi(w \circ y)\,dw \right\rangle \\
&= \left\langle \mathcal{P}u(t,y), \int_{\mathbb{G}} \widetilde{\gamma}\left(x \circ y^{-1}\right) \varphi(x)\,dx \right\rangle \\
&= \langle T_V \mathcal{P}u(t,\cdot), \varphi\rangle.
\end{aligned}$$

where the above equalities holds for a.e. t, as usual. This implies (3.7).

To prove (3.8) it is enough to show that

$$\partial_t T_V u = T_V \partial_t u \text{ for } u \in W^{1,2}\left((0,T), \mathcal{D}'(\Omega)\right).$$

Actually, for every $\psi \in \mathcal{D}(0,T)$ and $\varphi \in \mathcal{D}(\Omega')$ we have

$$\begin{aligned}
&\int_0^T \psi(t) \langle \partial_t T_V u(t,\cdot), \varphi\rangle \, dt = \langle \partial_t T_V u, \varphi \otimes \psi\rangle \\
&= -\langle T_V u, \varphi \otimes \partial_t \psi\rangle = -\int_0^T \partial_t \psi(t) \langle T_V u(t,\cdot), \varphi\rangle \, dt \\
&= -\int_0^T \partial_t \psi(t) \left\langle u(t,y), \int_{\mathbb{G}} \widetilde{\gamma}\left(x \circ y^{-1}\right) \varphi(x)\, dx \right\rangle dt \\
&\quad = -\langle u, T_V^* \varphi \otimes \partial_t \psi\rangle = \langle \partial_t u, T_V^* \varphi \otimes \psi\rangle \\
&\quad = \int_0^T \psi(t) \left\langle \partial_t u(t,y), \int_{\mathbb{G}} \widetilde{\gamma}\left(x \circ y^{-1}\right) \varphi(x)\, dx \right\rangle dt \\
&\quad = \int_0^T \psi(t) \langle T_V \partial_t u(t,\cdot), \varphi\rangle \, dt.
\end{aligned}$$

Hence $\partial_t T_V u = T_V \partial_t u$. ■

Lemma 3.10 *Let $\Omega' \Subset \Omega'' \Subset \Omega$ and $u \in L^2\left((0,T), \mathcal{D}'(\Omega)\right)$ satisfying the x-finite order assumption in Ω. There exists V small enough so that if*

$$T_V u \in L^2\left((0,T), C^\infty\left(\overline{\Omega''}\right)\right)$$

then $u \in L^2\left((0,T), W_X^{1,2}(\Omega')\right)$.

Proof. For fixed $\Omega' \Subset \Omega$ and positive integer K to be chosen later, there exists a neighborhood V of the origin such that

$$\underbrace{V \circ V \circ ... \circ V}_{K \text{ times}} \circ \Omega' \Subset \Omega.$$

Let

$$\begin{aligned}
\Omega_j &= \underbrace{V \circ V \circ ... \circ V}_{j \text{ times}} \circ \Omega' \text{ for } j = 1, 2, ..., K \\
\Omega_0 &= \Omega'.
\end{aligned}$$

so that $\Omega_K \Subset \Omega$. Let $\varphi \in C_0^\infty(\Omega_K)$, using the definition of T_V and Lemma 3.4 we obtain

$$\begin{aligned}
\mathcal{L}^R T_V u(t,\cdot) &= \mathcal{L}^R\left(\widetilde{\gamma} * u(t,\cdot)\right) = \mathcal{L}^R \widetilde{\gamma} * u(t,\cdot) \\
&= (-\delta + \omega) * u(t,\cdot) = -u(t,\cdot) + \omega * u(t,\cdot)
\end{aligned}$$

We know that $u=\partial_x^\alpha g$ for some multindex α and $g\in L^2\left((0,T),L^1_{loc}(\Omega)\right)$. Note that the kernel ω satisfies the same properties of $\widetilde{\gamma}$ in terms of support and growth estimate. Then, arguing as in the proof of Proposition 3.5 we see that

$$\omega * u(t,\cdot)=\sum_{|\beta|\leqslant A-1} D_x^\beta A_\beta(t,\cdot)$$

with $A_\beta\in L^2\left((0,T),L^1_{loc}(\Omega)\right)$ so that

$$u=\sum_{|\beta|\leqslant A-1} D^\beta A_\beta-\mathcal{L}^R T_V u \text{ in } L^2\left((0,T),\mathcal{D}'(\Omega_K)\right),$$

with $\mathcal{L}^R T_V u\in L^2\left((0,T),C^\infty\left(\overline{\Omega_K}\right)\right)$ since $T_V u\in L^2\left((0,T),C^\infty\left(\overline{\Omega_K}\right)\right)$ by assumption. Actually, for every $k=0,1,2,...$,

$$\left\|\mathcal{L}^R T_V u(t,\cdot)\right\|_{C^k(\Omega_K)}\leqslant c\left\|T_V u(t,\cdot)\right\|_{C^{k+2}(\Omega_K)}$$

hence for every $k=0,1,2,...$

$$\int_0^T\left\|\mathcal{L}^R T_V u(t,\cdot)\right\|^2_{C^k(\Omega_K)}dt\leqslant c\int_0^T\left\|T_V u(t,\cdot)\right\|^2_{C^{k+2}(\Omega_K)}dt<\infty.$$

We can then start again with the identity

$$\mathcal{L}^R T_V u(t,\cdot)=-u(t,\cdot)+\omega * u(t,\cdot)$$

where now we know in advance that

$$u=\sum_{|\beta|\leqslant A-1} D^\beta A_\beta \text{ in } L^2\left((0,T),\mathcal{D}'(\Omega_K)\right)$$

(the smooth function $\mathcal{L}^R T_V u$ can be absorbed in this expression) with $A_\beta\in L^2\left((0,T),L^1_{loc}(\Omega)\right)$ and, applying iteratively the above argument, in k_1 steps we eventually conclude $u\in L^2\left((0,T),L^1_{loc}(\Omega_{K-k_1})\right)$. Hence

$$\mathcal{L}^R T_V u=-u+\omega * u \text{ in } L^2\left((0,T),\mathcal{D}'(\Omega_{K-k_1})\right)$$

that is u coincides with $\omega * u$ in $L^2\left((0,T),\mathcal{D}'(\Omega_{K-k_1})\right)$, modulo the smooth function $\mathcal{L}^R T_V u$.

Let us reason again like in the proof of Proposition 3.5: since by Proposition 3.4, $\omega\in L^{\frac{N-1}{N-2}}(\mathbb{G})$ we see that $u\in L^2\left((0,T),L^{\frac{N-1}{N-2}}(\Omega_{K-k_1-1})\right)$; then with k_2 iterations of this argument we conclude that $u\in L^2\left((0,T),L^2(\Omega_{K-k_1-1-k_2})\right)$ and with one more iteration $u\in L^2\left((0,T),W^{1,2}_X(\Omega_{K-k_1-1-k_2-1})\right)$. Picking finally $K=k_1+k_2+2$ we get the desired assertion. ■

Proof of Theorem 3.3. Fix $\Omega'\Subset\Omega''\Subset\Omega'''\Subset\Omega$. By Corollary 3.7, there exist a positive integer K and a neighborhood V of the origin such that

$T_V^K u \in L^2\left((0,T), W_X^{1,2}(\Omega''')\right)$. Applying Corollary 3.7 also to $\partial_t u$, and possibly choosing a larger integer K and a smaller neighborhood V, we can also assume

$$T_V^K \partial_t u = \partial_t T_V^K u \in L^2\left((0,T), W_X^{1,2}(\Omega''')\right),$$

so that

$$T_V^K u \in W^{1,2}\left((0,T), W_X^{1,2}(\Omega''')\right).$$

Let now $U \subseteq V$ a neighborhood of the origin such that

$$\underbrace{U \circ U \circ ... \circ U}_{2K \text{ times}} \circ \Omega'' \Subset \Omega'''.$$

Let

$$\Omega_j = \underbrace{U \circ U \circ ... \circ U}_{j \text{ times}} \circ \Omega'' \text{ for } j = 1, 2, ..., 2K$$
$$\Omega_0 = \Omega'';$$

so that $\Omega_{2K} \Subset \Omega'''$. Clearly, it is still true that

$$T_U^K u \in L^2\left((0,T), W_X^{1,2}(\Omega''')\right)$$

(having replaced the operator T_V with T_U, based on a smaller neighborhood).

By Proposition 3.8 and Remark 3.9 we have

$$\mathcal{L}\left(T_U^K u\right) = T_U^K F \text{ in } L^2\left((0,T), \mathcal{D}'(\Omega_{2K})\right). \tag{3.9}$$

Since, $F \in L^2\left((0,T), C^\infty\left(\overline{\Omega}\right)\right)$, by point (4) in Proposition 3.5 we have $T_U^K F \in L^2\left((0,T), C^\infty\left(\overline{\Omega_{2K}}\right)\right)$. By (3.9) then $\mathcal{L}\left(T_U^K u\right) \in L^2\left((0,T), C^\infty\left(\overline{\Omega_{2K}}\right)\right)$ and, since $T_U^K u \in W^{1,2}\left((0,T), W_X^{1,2}(\Omega_{2K})\right)$, we can apply Theorem 2.15 to conclude that $T_U^K u \in C^0\left([0,T], C^\infty\left(\overline{\Omega_{2K-1}}\right)\right)$ and

$$T_U^K u_t \in L^2\left((0,T), C^\infty\left(\overline{\Omega_{2K-1}}\right)\right).$$

Applying Lemma 3.10 to u and $\partial_t u$ we see that $T_U^{K-1} u \in W^{1,2}\left((0,T), W_X^{1,2}(\Omega_{2K-2})\right)$. Iterating this argument K times shows that $u \in W^{1,2}\left((0,T), W_X^{1,2}(\Omega'')\right)$. Since $F \in L^2\left((0,T), C^\infty\left(\overline{\Omega}\right)\right)$ we can apply again Theorem 2.15 to conclude $u \in C^0\left([0,T], C^\infty\left(\overline{\Omega'}\right)\right)$ and $u_t \in L^2\left((0,T), C^\infty\left(\overline{\Omega'}\right)\right)$. ■

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