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Two Characterization of BV Functions on Carnot Groups via the Heat Semigroup

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In this paper we provide two different characterizations of sets with finite perimeter and functions of bounded variation in Carnot groups, analogous to those that hold in Euclidean spaces, in terms of the short-time behavior of the heat semigroup. The second one holds under the hypothesis that the reduced boundary of a set of finite perimeter is rectifiable, a result that presently is known in Step 2 Carnot groups.

1 Introduction

There are various definitions of *variation* of a function, and the related classes BV of *functions of bounded variation*, that make sense in different contexts and are known to be equivalent in wide generality. In the Euclidean framework, that is, $\mathbb{R}^n$ endowed with the Euclidean metric and the Lebesgue measure, the variation of $f \in L^1(\mathbb{R}^n)$ can be defined as

$$|Df|(\mathbb{R}^n) = \sup\left\{\int_{\mathbb{R}^n} f \operatorname{div} g \, dx : \ g \in C_0^1(\mathbb{R}^n, \mathbb{R}^n), \ \|g\|_{L^\infty(\mathbb{R}^n)} \le 1\right\}, \tag{1}$$

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and $|Df|(\mathbb{R}^n) < +\infty$ is equivalent to saying that $f \in \mathrm{BV}(\mathbb{R}^n)$, that is, the distributional gradient of f is an ($\mathbb{R}^n$-valued) finite Radon measure. This approach can be generalized to ambients where a measure and a coherent differential structure, which allows to define a divergence operator, are defined. Alternatively, the variation of f can be defined through a relaxation procedure,

$$|Df|(\mathbb{R}^n) = \inf\left\{\liminf_{k\to\infty}\int_{\mathbb{R}^n}|\nabla f_k|\,\mathrm{d}x:\ f_k \in \mathrm{Lip}(\mathbb{R}^n),\ \ f_k \to f \text{ in } L^1(\mathbb{R}^n)\right\}, \tag{2}$$

and the space BV can be defined, accordingly, as the finiteness domain of the relaxed functional in $L^1(\mathbb{R}^n)$. To generalize (2), no differential structure is needed, apart from Lipschitz functions and a suitable substitute of the modulus of the gradient. Definitions (1) and (2) have been extended to several contexts, such as manifolds and metric measure spaces (see, e.g., [2, 20, 22]), and in particular Carnot–Carathéodory spaces and Carnot groups (see [6, 10, 17, 19]). However, we point out that the original definition of variation of a function, and in particular of a set of finite perimeter, has been given by De Giorgi (see [13, 14]) by a regularization procedure based on the heat kernel. He proved that

$$|Df|(\mathbb{R}^n) = \lim_{t\to 0}\int_{\mathbb{R}^n}|\nabla T_t f|\,\mathrm{d}x, \tag{3}$$

(where ∇ denotes the gradient with respect to the space variables $x \in \mathbb{R}^n$ and $T_t f(x) = \int_{\mathbb{R}^n} h(t, x-y) f(y)\,\mathrm{d}y$ is the heat semigroup given by the Gauss–Weierstrass kernel h) and that if the above quantity is finite, then $f \in \mathrm{BV}(\mathbb{R}^n)$. Equality (3) has been recently proved for Riemannian manifolds M in [11], with the only restriction that the Ricci curvature of M be bounded from below, thus generalizing the result in [23], where further bounds on the geometry of M were assumed.

In this paper, the framework is that of Carnot groups, where it is known that formulae (1) and (2) (where of course the intrinsic differential structure is used) give equivalent definitions of $|Df|$, see [17]; we show here that (3) holds in a weaker sense, all the quantities involved being again the intrinsic ones; namely, the two sides of (3) are equivalent, although we do not know whether they are equal (see Theorem 2.11 for the precise statement).

We think that it is interesting to comment on the proof of (3) in the various situations. Thinking of the left-hand side as defined by (1), the $\leq$ inequality follows easily by lower semicontinuity of the total variation and the L^1 convergence of $T_t f$ to f, so that the more difficult part is the $\geq$ inequality. In $\mathbb{R}^n$ it follows from the (trivial) commutation relation $D_i T_t f = T_t D_i f$ between the heat semigroup and the partial derivatives, whereas in the Riemannian case, as treated in [11], it follows from the estimate $\|dT_t f\|_{L^1(M)} \le e^{K^2 t}|df|(M)$, where d denotes the exterior differential on M, which replaces the gradient in (3), and $-K^2$ is a lower bound for the Ricci curvature of M. In the sub-Riemannian case we are dealing with, the Euclidean commutation does not hold, and we are not able to prove that equality (3) still holds, but only that the semigroup approach identifies the BV class. The proof relies on algebraic properties and Gaussian estimates on the heat kernel and its derivatives which allow to estimate the commutator $[D, T_t]$. Still different arguments are used in the infinite-dimensional case of Wiener spaces.

Recently, inspired by [21], another connection between the heat semigroup and BV functions in $\mathbb{R}^n$ has been pointed out in [24], where the following equality is proved:

$$|Df|(\mathbb{R}^n) = \lim_{t\to 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(t, x-y)|f(x) - f(y)|\,dy\,dx, \tag{4}$$

which means that $f \in \mathrm{BV}(\mathbb{R}^n)$ if and only if the right-hand side is finite, and equality holds. Equality (4) is first proved for characteristic functions and then extended to the general case using the coarea formula. The proof for characteristic functions of sets of finite perimeter, in turn, is based on a blow-up argument on the points of the reduced boundary and uses the rectifiability of the reduced boundary. In this paper, we extend (4) to Carnot groups (see Theorem 2.14) but, according to the preceding comments, our proof depends upon the rectifiability of the reduced boundary, which is presently known to be true only in groups of Step 2, see [18]. Therefore, our proof would immediately generalize to more general groups, if the rectifiability of the reduced boundary were proved. Note also that in general the heat kernel does not have the symmetries that the Gaussian kernel has, such as rotation invariance. As a consequence, Equation (4) assumes a different form in general Carnot groups of step two (see (29) in Theorem 2.14 below); however, in the special but important case of groups of Heisenberg type, (29) simplifies to a form very similar to (4) (see Remark 2.15).

We point out that in the Euclidean case both (3) and (4) hold not only in the whole of $\mathbb{R}^n$, but also in a localized form and with the heat semigroup replaced by the semigroup generated by a general uniformly elliptic operator, under suitable boundary conditions; see [4]. Finally, let us mention that the short-time behavior of the heat semigroup in $\mathbb{R}^n$ has been shown to be useful also to describe further geometric properties of boundaries; see [3].

2 Preliminaries and Main Results

We collect in the first two subsections the notions on Carnot groups and BV functions needed in the paper. After that, we devote the last subsection to the statement of the main results and some comments.

2.1 Carnot groups and heat kernels

2.1.1 *Basic definitions*

Here we recall some basic known facts about Carnot groups, whose precise definition we shall give after introducing some notation. We refer to [7, §1.4] for the proofs and further details.

Let $\mathbb{R}^n$ be equipped with a Lie group structure by the composition law $\circ$ (which we call *translation*) such that 0 is the identity and $x^{-1} = -x$ (i.e., the group inverse is the Euclidean opposite); let $\mathbb{R}^n$ be also equipped with a family $\{D(\lambda)\}_{\lambda>0}$ of group automorphisms of $(\mathbb{R}^n, \circ)$ (called *dilations*) of the following form:

$$D(\lambda) : (x_1, \dots, x_n) \mapsto (\lambda^{\omega_1} x_1, \dots, \lambda^{\omega_n} x_n),$$

where $\omega_1 \le \omega_2 \le \dots \le \omega_n$ are positive integers, with $\omega_1 = \omega_2 = \dots = \omega_q = 1$, $\omega_i > 1$ for $i > q$ for some $q < n$.

If L_x is the left translation operator acting on functions, $(L_x f)(y) = f(x \circ y)$, we say that a *differential operator P is left invariant* if $P(L_x f) = L_x(P f)$ for every smooth function f. Also, we say that *a differential operator P is homogeneous of degree* $\delta > 0$ if

$$P\left(f(D(\lambda))\right)(x) = \lambda^{\delta}\,(P f)(D(\lambda)x)$$

for every test function f, $\lambda > 0$, $x \in \mathbb{R}^n$, and *a function f is homogeneous of degree* $\delta \in \mathbb{R}$ if

$$f(D(\lambda)x) = \lambda^{\delta} f(x) \text{ for every } \lambda > 0,\ x \in \mathbb{R}^n.$$

Now, for $i = 1, 2, \dots, q$, let

$$X_i = \sum_{j=1}^{n} q_i^j(x)\,\partial_{x_j} \tag{5}$$

be the unique left-invariant vector field (with respect to $\circ$) which agrees with ∂_{x_i} at the origin, and assume that the Lie algebra generated by $X_1, X_2, \dots, X_q$ coincides with the

Lie algebra of $\mathbb{G}$. Then we say that $\mathbb{G} = (\mathbb{R}^n, \circ, \{D(\lambda)\}_{\lambda>0})$ is a *(homogeneous) Carnot group* or a *homogeneous stratified group.*

More explicitly, setting $V_1 = \mathrm{span}\{X_1, X_2, \ldots, X_q\}$, $V_{i+1} = [V_1, V_i]$ (the space generated by the commutators $[X, Y]$, with $X \in V_1, Y \in V_i$), there is $k \in \mathbb{N}$ such that $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k$, with $V_k \neq \{0\}$ and $V_i = \{0\}$ for $i > k$. The integer k is called the *step* of $\mathbb{G}$. Since V_1 at 0 can be identified with $\mathbb{R}^q$, by the left invariance of the vector fields X_i we can identify V_1 with $\mathbb{R}^q$ for every $x \in \mathbb{R}^n$. Note that all vector fields in V_j are j-homogeneous and call

$$Q = \sum_{j=1}^{k} j \dim V_j = \sum_{j=1}^{n} \omega_j,$$

the *homogeneous dimension* of $\mathbb{G}$. The operator

$$L = \sum_{j=1}^{q} X_j^2$$

is called *sub-Laplacian*; it is left invariant, homogeneous of degree 2 and, by Hörmander's Theorem, hypoelliptic.

2.1.2 *Structure of left- and right-invariant vector fields*

It is sometimes useful to consider also the *right-invariant vector fields* X_j^R $(j = 1, \ldots, n)$, which agree with ∂_{x_j} (and therefore with X_j) at 0; also these X_i^R are homogeneous of degree 1 for $i \leq q$. (For the following properties of invariant vector fields see [27, pp. 606–621] or [8]). As to the *structure of the left- (or right-) invariant vector fields*, it can be proved that the systems $\{X_i\}$ and $\{X_i^R\}$ have the following "triangular form" with respect to Cartesian derivatives:

$$X_i = \partial_{x_i} + \sum_{k=i+1}^{n} q_i^k(x)\, \partial_{x_k}, \tag{6}$$

$$X_i^R = \partial_{x_i} + \sum_{k=i+1}^{n} \bar{q}_i^k(x)\, \partial_{x_k}, \tag{7}$$

where $q_i^k, \bar{q}_i^k$ are polynomials, homogeneous of degree $\omega_k - \omega_i$ (the ω_i's are the dilation exponents). When the triangular form of the X_i's is not important, we shall keep writing (5) more compactly.

The identities (6)–(7) imply that any Cartesian derivative ∂_{x_k} can be written as a linear combination of the X_i's (and, analogously, of the X_j^R's). In particular, any homogeneous differential operator can be rewritten as a linear combination of left-invariant (or, similarly, right-invariant) homogeneous vector fields, with polynomial coefficients. The above structure of the X_i's also implies that the L^2 transpose X_i^* of X_i is just $-X_i$ (as in a standard integration by parts). From the above equations we also find that

$$X_i = \sum_{k=i}^{n} c_i^k(x)\, X_k^R,$$

where $c_i^k(x)$ are polynomials, homogeneous of degree $\omega_k - \omega_i$. In particular, since $\omega_k - \omega_i < \omega_k$, $c_i^k(x)$ does not depend on x_h for $h \geq k$ and therefore commutes with X_k^R, that is

$$X_i f = \sum_{k=i}^{n} X_k^R\, (c_i^k(x)\, f) \quad (i = 1, \dots, n)$$

for every test function f. The above identity can be sharpened as follows; taking its value at the origin, we get

$$\partial_{x_i} = \sum_{k=i}^{q} c_i^k \partial_{x_k},$$

since, for $k > q$, the $c_i^k(x)$ are homogeneous polynomials of positive degree, and therefore vanish at the origin. Hence $c_i^k = \delta_{ik}$ for $i, k = 1, 2, \dots, q$, that is:

$$X_i f = X_i^R f + \sum_{k=q+1}^{n} X_k^R\, (c_i^k(x)\, f) \quad \text{for } i = 1, \dots, q. \tag{8}$$

Moreover, the operators X_k^R, with $k > q$, can be expressed in terms of the $X_1^R, \dots, X_q^R$, that is, for every $k = q+1, \dots, n$, there are constants $\vartheta^k_{i_1 i_2 \cdots i_{\omega_k}}$ such that

$$X_k^R = \sum_{1 \leq i_j \leq q} \vartheta^k_{i_1 i_2 \cdots i_{\omega_k}} X_{i_1}^R X_{i_2}^R \cdots X_{i_{\omega_k}}^R. \tag{9}$$

2.1.3 *Convolutions*

The *convolution* of two functions in $\mathbb{G}$ is defined as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x \circ y^{-1})\, g(y)\, \mathrm{d}y = \int_{\mathbb{R}^n} g(y^{-1} \circ x)\, f(y)\, \mathrm{d}y,$$

for every couple of functions for which the above integrals make sense. From this definition we see that if P is any left-invariant differential operator, then

$$P(f * g) = f * Pg$$

(provided the integrals converge). Note that, if $\mathbb{G}$ is not abelian, we cannot write $f * Pg = Pf * g$. Instead, if X and X^R are, respectively, a left-invariant and right-invariant vector field which agree at the origin, the following hold (see [27, p. 607]):

$$(Xf) * g = f * (X^R g); \quad X^R(f * g) = (X^R f) * g.$$

Explicitly this means

$$\int_{\mathbb{R}^n} X^R g(y^{-1} \circ x)\, f(y)\, \mathrm{d}y = \int_{\mathbb{R}^n} g(y^{-1} \circ x)\, Xf(y)\, \mathrm{d}y \tag{10}$$

for any f, g for which the above integrals make sense.

2.1.4 *Heat kernels*

If $\mathbb{G} = (\mathbb{R}^n, \circ, D(\lambda))$ is a Carnot group, we can naturally define its *parabolic version* setting, in $\mathbb{R}^{n+1}$:

$$(t, u) \circ_P (s, v) = (t + s, u \circ v);\ D_P(\lambda)(t, u) = (\lambda^2 t, D(\lambda)u).$$

If we define the *parabolic homogeneous group* $\mathbb{G}_P = (\mathbb{R}^{n+1}, \circ_P, D_P(\lambda))$, its homogeneous dimension is $Q + 2$. Let us now consider the heat operator in $\mathbb{G}_P$,

$$\mathcal{H} = \partial_t - \sum_{j=1}^{q} X_j^2 = \partial_t - L,$$

which is translation invariant, homogeneous of degree 2 and hypoelliptic. By a general result due to Folland (see [16]), $\mathcal{H}$ possesses a homogeneous fundamental solution $h(t, x)$, usually called *the heat kernel on* $\mathbb{G}$. The next theorem collects several well-known important facts about h, which are useful in the sequel; the statements (i)–(v) below can be found in [16, Section 1G], while the estimates (11)–(12) are contained in [28, Section IV.4].

Theorem 2.1. There exists a function $h(t,x)$ defined in $\mathbb{R}^{n+1}$ with the following properties:

(i) $h \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$;
(ii) $h(\lambda^2 t, D(\lambda)x) = \lambda^{-Q} h(t,x)$ for any $t > 0$, $x \in \mathbb{R}^n$, $\lambda > 0$;
(iii) $h(t,x) = 0$ for any $t < 0$, $x \in \mathbb{R}^n$;
(iv) $\int_{\mathbb{R}^n} h(t,x)\,\mathrm{d}x = 1$ for any $t > 0$;
(v) $h(t,x^{-1}) = h(t,x)$ for any $t > 0$, $x \in \mathbb{R}^n$.

Setting $h_t(x) = h(t,x)$, let us introduce the *heat semigroup*, defined as follows for any $f \in L^1(\mathbb{R}^n)$:

$$W_t f(x) = \int_{\mathbb{R}^n} h(t, y^{-1} \circ x) f(y)\,\mathrm{d}y = (f * h_t)(x).$$

Then, for any $f \in L^1(\mathbb{R}^n)$ and $t > 0$, we have $W_t f \in C^\infty(\mathbb{R}^n)$ and the function $u(t,x) = W_t f(x)$ solves the equation $\mathcal{H}u = 0$ in $(0,\infty) \times \mathbb{R}^n$. Moreover,

$$u(t,x) \to f(x) \text{ strongly in } L^1(\mathbb{R}^n) \text{ as } t \to 0.$$

Finally, for every nonnegative integers j, k, for every $x \in \mathbb{R}^n$, $t > 0$, the following Gaussian bounds hold:

$$\mathbf{c}^{-1} t^{-Q/2} \mathrm{e}^{-|x|^2/\mathbf{c}^{-1}t} \le h(t,x) \le \mathbf{c} t^{-Q/2} \mathrm{e}^{-|x|^2/\mathbf{c}t}, \tag{11}$$

$$|X_{i_1} \cdots X_{i_j} (\partial_t)^k h(t,x)| \le \mathbf{c}(j,k) t^{-(Q+j+2k)/2} \mathrm{e}^{-|x|^2/\mathbf{c}t} \tag{12}$$

for $i_1, i_2, \ldots, i_j \in \{1, 2, \ldots, q\}$. Here $\mathbf{c} \ge 1$ is a constant only depending on $\mathbb{G}$, while $\mathbf{c}(j,k)$ depends on $\mathbb{G}$, j, k. □

Remark 2.2. The Gaussian estimates in this context usually involve the so-called *homogeneous norm* and are more precise than those we use here, that are stated in terms of the Euclidean norm. We have stated the Gaussian bounds in the present form because we do not need the estimates in their full strength and (11), (12) follow immediately from the usual estimates. In this way we avoid the introduction of the homogeneous norm, which we do not need.

The bound on the derivatives $X_{i_1} \cdots X_{i_j} \partial_t^k h(t,x)$ still holds, in the same form, if the vector fields $X_{i_1}, \ldots, X_{i_j}$ are replaced with any family of j vector fields, homogeneous of degree 1; for instance, we will apply this bound to derivatives with respect to the right-invariant vector fields X_i^R. Moreover, a homogeneity argument shows that if $a(x)$ is a homogeneous function of degree j', then

$$|X_{i_1} \cdots X_{i_j} \partial_t^k [a(x) h(t,x)] \le c(j,k,a) t^{-(Q+j-j'+2k)/2} \mathrm{e}^{-|x|^2/\mathbf{c}t}. \qquad \square$$

2.2 BV functions

Let us define the Sobolev spaces $W^{1,p}(\mathbb{G})$, $1 \le p < \infty$, and the space $\mathrm{BV}(\mathbb{G})$ of functions of bounded variation in $\mathbb{G}$ and list their main properties. We remark that the definition of $\mathrm{BV}(\mathbb{G})$ goes back to [10] and refer to [1] and to [18] for more information on the Euclidean and the sub-Riemannian case, respectively. We start from the Sobolev case.

Definition 2.3. For $1 \le p < \infty$ we say that $f \in L^p(\mathbb{R}^n)$ belongs to the Sobolev space $W^{1,p}(\mathbb{G})$ if there are $f_1, \dots, f_q \in L^p(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f(x) X_i g(x)\,\mathrm{d}x = -\int_{\mathbb{R}^n} g(x) f_i(x)\,\mathrm{d}x, \quad i = 1, \dots, q,$$

for all $g \in C_0^1(\mathbb{R}^n)$. In this case, we denote f_i as $X_i f$ and set

$$\nabla_X f = (X_1 f, X_2 f, \dots, X_q f).$$

We also let

$$\|\nabla_X f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla_X f|\,\mathrm{d}x = \int_{\mathbb{R}^n} \sqrt{\sum_{i=1}^q |X_i f|^2}\,\mathrm{d}x. \qquad \square$$

We now consider functions of bounded variation.

Definition 2.4. For $f \in L^1(\mathbb{R}^n)$ we say that $f \in \mathrm{BV}(\mathbb{G})$ if there are finite Radon measures μ_i such that

$$\int_{\mathbb{R}^n} f(x) X_i g(x)\,\mathrm{d}x = -\int_{\mathbb{R}^n} g(x)\,\mathrm{d}\mu_i, \quad i = 1, \dots, q,$$

for all $g \in C_0^1(\mathbb{R}^n)$. In this case, we denote μ_i by $X_i f$ and note that the total variation of the $\mathbb{R}^q$-valued measure $D_{\mathbb{G}} f = (X_1 f, \dots, X_q f)$ is given by

$$|D_{\mathbb{G}} f|(\mathbb{R}^n) = \sup\left\{ \int_{\mathbb{R}^n} f(x) \mathrm{div}_{\mathbb{G}} g(x)\,\mathrm{d}x : g \in C_0^1(\mathbb{R}^n, \mathbb{R}^q),\ \|g\|_\infty \le 1 \right\}, \tag{13}$$

where

$$\mathrm{div}_{\mathbb{G}} g(x) = \sum_{i=1}^{q} X_i g_i(x), \text{ if } g(x) = (g_1(x), \ldots, g_q(x)),$$

and $\|g\|_\infty = \sup_{x\in\mathbb{R}^n} |g(x)|$. □

With the same proof contained, for example in [1, Prop. 3.6], it is possible to show that if f belongs to $\mathrm{BV}(\mathbb{G})$, then its total variation $|D_{\mathbb{G}} f|$ is a finite positive Radon measure and there is a $|D_{\mathbb{G}} f|$-measurable function $\sigma_f : \mathbb{R}^n \to \mathbb{R}^q$ such that $|\sigma_f(x)| = 1$ for $|D_{\mathbb{G}} f|$-almost everywhere (a.e.) $x \in \mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} f(x) \mathrm{div}_{\mathbb{G}} g(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \langle g, \sigma_f\rangle\,\mathrm{d}|D_{\mathbb{G}} f| \tag{14}$$

for all $g \in C_0^1(\mathbb{G}, \mathbb{R}^q)$ where, here and in the following, we denote by $\langle\cdot,\cdot\rangle$ the usual inner product in $\mathbb{R}^q$.

We denote by $D_{\mathbb{G}} f$ the vector measure $-\sigma_f |D_{\mathbb{G}} f|$, so that $X_i f$ is the measure $(-\sigma_f)_i |D_{\mathbb{G}} f|$ and the following integration by parts holds true:

$$\int_{\mathbb{R}^n} f(x) X_i g(x)\,\mathrm{d}x = -\int_{\mathbb{R}^n} g(x)\,\mathrm{d}(X_i f)(x) \tag{15}$$

for all $g \in C_0^1(\mathbb{G})$. Note also that

$$|X_i f|(\mathbb{R}^n) \le |D_{\mathbb{G}} f|(\mathbb{R}^n). \tag{16}$$

To visualize the above definitions, we note that whenever f is a smooth function, by the Euclidean divergence theorem we have

$$\sigma_f = -\frac{\nabla_X f}{|\nabla_X f|}; \quad d|D_{\mathbb{G}} f|(x) = |\nabla_X f(x)|\,\mathrm{d}x;$$

(whenever $|\nabla_X f| \neq 0$) and

$$\mathrm{d}(X_i f)(x) = X_i f(x)\,\mathrm{d}x.$$

It is clear that $W^{1,1}(\mathbb{G})$ functions are $\mathrm{BV}(\mathbb{G})$ functions whose measure gradient is absolutely continuous with respect to Lebesgue measure. Moreover, since it is the supremum of L^1-continuous functionals, the total variation is $L^1(\mathbb{R}^n)$ lower semicontinuous (see [18, Theorem 2.17]), that is, $f_k \to f$ in $L^1(\mathbb{R}^n)$ implies that

$$|D_{\mathbb{G}} f|(\mathbb{R}^n) \le \liminf_{k\to\infty} |D_{\mathbb{G}} f_k|(\mathbb{R}^n); \tag{17}$$

namely

$$\int_{\mathbb{R}^n} f_k(x)\mathrm{div}_{\mathbb{G}} g(x)\,\mathrm{d}x \le |D_{\mathbb{G}} f_k|(\mathbb{R}^n), \quad \forall g \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^q),\ \|g\|_\infty \le 1.$$

Passing to the liminf as $k \to \infty$, we get

$$\int_{\mathbb{R}^n} f(x)\mathrm{div}_{\mathbb{G}} g(x)\,\mathrm{d}x \le \liminf_{k\to\infty} |D_{\mathbb{G}} f_k|(\mathbb{R}^n),$$

and taking the supremum over all possible g's we get (17).

Definition 2.5 (Sets of finite $\mathbb{G}$-perimeter)**.** If χ_E is the characteristic function of a measurable set $E \subset \mathbb{R}^n$ and $|D_{\mathbb{G}}\chi_E|$ is finite, we say that E is a set of finite $\mathbb{G}$-perimeter and we write $P_{\mathbb{G}}(E)$ instead of $|D_{\mathbb{G}}\chi_E|$, $P_{\mathbb{G}}(E, F)$ instead of $|D_{\mathbb{G}}\chi_E|(F)$ for F Borel. Also, we call (generalized inward) $\mathbb{G}$-normal the q-vector

$$\nu_E(x) = -\sigma_{\chi_E}(x).$$

□

Recall that $|\nu_E(x)| = 1$ for $P_{\mathbb{G}}(E)$-a.e. $x \in \mathbb{R}^n$. In this case (14) takes the form

$$\int_E \mathrm{div}_{\mathbb{G}} g(x)\,\mathrm{d}x = -\int_{\mathbb{R}^n} \langle g, \nu_E\rangle\,\mathrm{d}P_{\mathbb{G}}(E). \tag{18}$$

Moreover, note that if E has finite perimeter, then, by the isoperimetric inequality in Carnot groups (see, e.g., [10, Theorem 1.18 and Remark 1.19]), either E or its complement E^c has finite measure.

Remark 2.6. To help the reader to visualize the above definitions, let us specialize them to the case of a bounded smooth domain E; see [18, Proposition 2.22]. Let n_E be the Euclidean unit inner normal at ∂E, and consider the q-vector v whose ith component is defined by

$$v_i = \sum_{j=1}^n q_i^j(x)(n_E)_j(x)$$

(where the q_i^j are the coefficients of the vector fields X_i, defined in (5)). Then

$$\int_E \operatorname{div}_{\mathbb{G}} g(x)\,\mathrm{d}x = -\int_{\partial E} \langle g, v\rangle\,\mathrm{d}H^{n-1}(x)$$

from which we read that in this case

$$\nu_E = \frac{v}{|v|}, \quad \mathrm{d}P_{\mathbb{G}}(E) = |v|\,\mathrm{d}(H^{n-1} \llcorner \partial E)(x) \tag{19}$$

at least at those points of the boundary where $v \neq 0$ (noncharacteristic points). Here H^{n-1} is the Euclidean $(n-1)$-dimensional Hausdorff measure and $\llcorner$ denotes the restriction of the measure. □

Identity (10) can be extended to the case $g \in C^1(\mathbb{G})$, $f \in \mathrm{BV}(\mathbb{G})$ as follows:

$$\int_{\mathbb{R}^n} X_i^R g(y^{-1} \circ x)\, f(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} g(y^{-1} \circ x)\,\mathrm{d}(X_i f)(y), \tag{20}$$

where the last integral is made with respect to the measure defined by $X_i f$. To see this, it is enough to take a sequence $f_k \in W^{1,1}(\mathbb{G})$ such that $f_k \to f$ in $L^1(\mathbb{R}^n)$ and $X_i f_k$ is weakly* convergent to $X_i f$ (the existence of such a sequence is proved in [17]), so that

$$\begin{aligned}\lim_{k\to\infty} \int_{\mathbb{R}^n} X_i^R g(y^{-1} \circ x)\, f_k(y)\,\mathrm{d}y &= \lim_{k\to\infty} \int_{\mathbb{R}^n} g(y^{-1} \circ x)\, X_i f_k(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} g(y^{-1} \circ x)\,\mathrm{d}(X_i f)(y).\end{aligned}$$

Let us come to some finer properties of BV functions that we need only in the proof of Theorem 2.14. We refer to [17, Theorem 2.3.5] for a proof of the following statement.

Proposition 2.7 (Coarea formula). If $f \in \mathrm{BV}(\mathbb{G})$, then, for a.e. $\tau \in \mathbb{R}$, the set $E_\tau = \{x \in \mathbb{R}^n : f(x) > \tau\}$ has finite $\mathbb{G}$-perimeter and

$$|D_{\mathbb{G}} f|(\mathbb{R}^n) = \int_{-\infty}^{+\infty} |D_{\mathbb{G}} \chi_{E_\tau}|(\mathbb{R}^n)\,\mathrm{d}\tau. \tag{21}$$

Conversely, if $f \in L^1(\mathbb{R}^n)$ and $\int_{-\infty}^{+\infty} |D_{\mathbb{G}}\chi_{E_\tau}|(\mathbb{R}^n)\,\mathrm{d}\tau < +\infty$, then $f \in \mathrm{BV}(\mathbb{G})$ and equality (21) holds. Moreover, if $g : \mathbb{R}^n \to \mathbb{R}$ is a Borel function, then

$$\int_{\mathbb{R}^n} g(x)\,\mathrm{d}|D_{\mathbb{G}} f|(x) = \int_{-\infty}^{+\infty} \int_{\mathbb{R}^n} g(x)\,\mathrm{d}|D_{\mathbb{G}}\chi_{E_\tau}|(x)\,\mathrm{d}\tau. \tag{22}$$

□

Next we have to introduce the reduced boundary. This definition relies on a particular notion of balls in $\mathbb{G}$. We will denote by $B_\infty(x, r)$ the d_∞-balls of center x and radius r, where $d_\infty(x, y) = d_\infty(0, y^{-1} \circ x)$ and $d_\infty(0, x)$ is the homogeneous norm in $\mathbb{G}$ which is considered in [18, Theorem 5.1]. We do not recall its explicit analytical definition since we shall never need it. The distance d_∞ can be used to define the spherical Hausdorff measures $\mathcal{S}^k_\infty$ through the usual Carathéodory construction; see, for example, [15, Section 2.10.2].

Definition 2.8. Let $E \subset \mathbb{R}^n$ be measurable with finite $\mathbb{G}$-perimeter. We say that $x \in \partial^*_{\mathbb{G}} E$ (reduced boundary of E) if the following conditions hold:

(i) $P_{\mathbb{G}}(E, B_\infty(x, r)) > 0$ for all $r > 0$;

(ii) the following limit exists

$$\lim_{r\to 0} \frac{D_{\mathbb{G}}\chi_E(B_\infty(x, r))}{|D_{\mathbb{G}}\chi_E|(B_\infty(x, r))}$$

and equals $\nu_E(x)$;

(iii) the equality $|\nu_E(x)| = 1$ holds. □

In order to state the next result, let us introduce some notation. For $E \subset \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$, $r > 0$ we consider the translated and dilated set E_{r,x_0} defined as

$$E_{r,x_0} = \{x \in \mathbb{R}^n :\ x_0 \circ D(r)x \in E\} = D(r^{-1})(x_0^{-1} \circ E).$$

For any vector $\nu \in \mathbb{R}^q$ we define the sets

$$S^+_{\mathbb{G}}(\nu) = \{x \in \mathbb{R}^n :\ \langle \pi x, \nu\rangle \geq 0\}, \qquad T_{\mathbb{G}}(\nu) = \{x \in \mathbb{R}^n : \langle \pi x, \nu\rangle = 0\},$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^q$ is the projection which reads the first q components. Hence the set $T_{\mathbb{G}}(\nu)$ is a Euclidean ($(n-1)$-dimensional) "vertical plane" in $\mathbb{R}^n$; it is also a subgroup of $\mathbb{G}$, since on the first q components the Lie group operation is just the Euclidean sum.

We are now in a position to state the following result (see [18, Theorem 3.1]).

Theorem 2.9 (blow-up on the reduced boundary). Let $\mathbb{G}$ be of Step 2. If $E \subset \mathbb{R}^n$ is a set of finite $\mathbb{G}$-perimeter and $x_0 \in \partial^*_{\mathbb{G}} E$, then

$$\lim_{r \to 0} \chi_{E_{r,x_0}} = \chi_{S^+_{\mathbb{G}}(\nu_E(x_0))} \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n). \tag{23}$$

(Note that $\nu_E(x_0)$ is pointwise well defined for $x_0 \in \partial^*_{\mathbb{G}} E$, by point (ii) in Definition 2.8). Moreover, $P_{\mathbb{G}}(E)$-a.e. $x \in \mathbb{R}^n$ belongs to $\partial^*_{\mathbb{G}} E$. □

The last statement in the above theorem allows us to rewrite formula (18) as an integral on the reduced boundary with respect to the $Q-1$ spherical Hausdorff measure as follows:

$$\int_E \operatorname{div}_{\mathbb{G}} g(x)\, \mathrm{d}x = -\theta_\infty \int_{\partial^*_{\mathbb{G}} E} \langle g, \nu_E \rangle \, \mathrm{d}\mathcal{S}^{Q-1}_\infty, \tag{24}$$

(here θ_∞ is a constant depending on $\mathbb{G}$; see [18, Theorem 3.10]), which looks much closer to the classical divergence theorem. Moreover, the following analog of (19) holds:

$$P_{\mathbb{G}}(E, \cdot) = \theta_\infty \mathcal{S}^{Q-1}_\infty \llcorner (\partial^*_{\mathbb{G}} E). \tag{25}$$

In order to apply the L^1_{loc} convergence of (23), we will need also the following.

Remark 2.10. If $E, \{E_k\}_{k=1}^\infty \subset \mathbb{R}^n$ are measurable, $\chi_{E_k} \to \chi_E$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then $f\chi_{E_k} \to f\chi_E$ in $L^1(\mathbb{R}^n)$. In fact, given $\varepsilon > 0$, there is a compact set K such that

$$\int_{\mathbb{R}^n \setminus K} |f| \, \mathrm{d}x < \varepsilon,$$

whence

$$\int_{\mathbb{R}^n} |(\chi_{E_k} - \chi_E) f| \, \mathrm{d}x \le \varepsilon + \|f\|_\infty \int_K |\chi_{E_k} - \chi_E| \, \mathrm{d}x,$$

and the last integral tends to 0 as $k \to \infty$. □

2.3 Main results

We state here the main results of this paper, namely Theorems 2.11 and 2.14.

Theorem 2.11. Let $f \in L^1(\mathbb{R}^n)$. The quantity

$$\limsup_{t\to 0} \|\nabla_X W_t f\|_{L^1(\mathbb{R}^n)}$$

is finite if and only if $f \in \mathrm{BV}(\mathbb{G})$. In this case the following holds:

$$\begin{aligned} |D_{\mathbb{G}} f|(\mathbb{R}^n) &\leq \liminf_{t\to 0} \|\nabla_X W_t f\|_{L^1(\mathbb{R}^n)} \leq \limsup_{t\to 0} \|\nabla_X W_t f\|_{L^1(\mathbb{R}^n)} \\ &\leq (1+c)|D_{\mathbb{G}} f|(\mathbb{R}^n), \end{aligned}$$

where $c \geq 0$ is a constant depending only on $\mathbb{G}$. □

Theorem 2.11 shows that even in Carnot groups Definition 2.4 is equivalent to the analog of De Giorgi's original definition (3) given in the Euclidean case, even though we do not know if the limit exists. Section 3 is devoted to the proof of Theorem 2.11.

The next result requires the additional hypothesis that $\mathbb{G}$ is a Carnot group of Step 2. As we explain in Section 4, the reason is that our proof uses the rectifiability of the reduced boundary of a set of finite perimeter in $\mathbb{G}$, which is known only in the Step 2 case; see [18]. Our argument works whenever the rectifiability theorem is true, and then would extend immediately to all cases where the rectifiability result is extended.

We start by defining the function

$$\phi_{\mathbb{G}}(\nu) = \int_{T_{\mathbb{G}}(\nu)} h(1, x)\, \mathrm{d}x, \tag{26}$$

for vectors $\nu \in \mathbb{R}^q$. Here the integral is taken over $T_{\mathbb{G}}(\nu)$, which is a hyperplane in $\mathbb{R}^n$; to simplify notation, we keep denoting by $\mathrm{d}x$ the $(n-1)$-dimensional Lebegue measure over $T_{\mathbb{G}}(\nu)$.

The function $\phi_{\mathbb{G}}$ is continuous and, using the Gaussian bounds (11), there are constants c_1, c_2, depending on the group, such that

$$0 < c_1 \leq \phi_{\mathbb{G}}(\nu) \leq c_2. \tag{27}$$

Remark 2.12 (Rotation invariance of heat kernels)**.** Note that if the heat kernel is invariant under horizontal rotations (i.e., under Euclidean rotations in the space $\mathbb{R}^q$ of the first variables $x_1, x_2, \ldots, x_q$ of $\mathbb{R}^n$), then $\phi_{\mathbb{G}}$ is actually a constant (that is, independent of ν). This happens, for instance, in all groups of Heisenberg type (briefly called H-type groups), in view of the known formula assigning the heat kernel in that context (for a discussion of H-type groups see, for instance, [7, Chapter 18]; for the computation of the heat kernel in this context, see [26]). By the way, we point out that in the Heisenberg groups with more than one vertical direction the heat kernel is invariant for horizontal rotations, but the sub-Laplacian is not. This can be seen through a direct computation based on (3.14) in [7]. On the other hand, from the general formula assigning the heat kernel in Carnot groups of step 2, proved by [12] (see also [5]), one can expect the existence of groups of step 2 (more general than H-type groups) where the heat kernel is not invariant under horizontal rotations and the function $\phi_{\mathbb{G}}$ is not constant. □

Let us first state our next result in the case of perimeters.

Theorem 2.13. Let $\mathbb{G}$ be of Step 2. If $E \subset \mathbb{R}^n$ is a set of finite $\mathbb{G}$-perimeter, the following equality holds:

$$\lim_{t\to 0} \frac{1}{2\theta_\infty \sqrt{t}} \int_{E^c} W_t \chi_E(x)\, \mathrm{d}x = \int_{\partial^*_{\mathbb{G}} E} \phi_{\mathbb{G}}(\nu_E)\, \mathrm{d}\mathcal{S}^{Q-1}_\infty. \tag{28}$$

Conversely, if either E or E^c has finite measure and

$$\liminf_{t\to 0} \frac{1}{\sqrt{t}} \int_{E^c} W_t \chi_E(x)\, \mathrm{d}x < +\infty,$$

then E has finite perimeter, and (28) holds. □

For general BV functions we have the following.

Theorem 2.14. Let $\mathbb{G}$ be of Step 2. Then, if $f \in \mathrm{BV}(\mathbb{G})$, the following equality holds:

$$\int_{\mathbb{R}^n} \phi_{\mathbb{G}}(\sigma_f)\, \mathrm{d}|D_{\mathbb{G}} f| = \lim_{t\to 0} \frac{1}{4\sqrt{t}} \int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x) - f(y)| h(t, y^{-1} \circ x)\, \mathrm{d}x\, \mathrm{d}y, \tag{29}$$

where σ_f is defined in (14). Conversely, if $f \in L^1(\mathbb{R}^n)$ and the liminf of the quantity on the right-hand side is finite, then $f \in \mathrm{BV}(\mathbb{G})$ and (29) holds. □

Remark 2.15. Note that, in view of (25), (27) and Remark 2.12, the right-hand side of (28) is always equivalent to, and in groups of Heisenberg-type is a multiple of, $P_{\mathbb{G}}(E)$; the left-hand side of (29) is always equivalent to, and in groups of Heisenberg-type is a multiple of, $|D_{\mathbb{G}} f|(\mathbb{R}^n)$. □

The proofs of the above results will be given in Section 4.

3 Proof of Theorem 2.11

In this section, we give the proof of Theorem 2.11; first of all, we note that $W_t f \to f$ in L^1 as $t \to 0$; moreover, $W_t f \in C^\infty(\mathbb{R}^n)$, hence

$$|D_{\mathbb{G}} W_t f|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |\nabla_X W_t f(x)| \, \mathrm{d}x.$$

Therefore by the lower semicontinuity of the variation, that is (17), the inequality

$$|D_{\mathbb{G}} f|(\mathbb{R}^n) \le \liminf_{t\to 0} \int_{\mathbb{R}^n} |\nabla_X W_t f(x)| \, \mathrm{d}x \tag{30}$$

holds, and then it remains to prove only the upper bound. We also note that inequality (30) implies that if $f \in L^1(\mathbb{R}^n)$ is such that the right-hand side of (30) is finite, then $f \in \mathrm{BV}(\mathbb{G})$. So, Theorem 2.11 will follow if we prove that

$$\limsup_{t\to 0} \int_{\mathbb{R}^n} |\nabla_X W_t f(x)| \, \mathrm{d}x \le (1+c)|D_{\mathbb{G}} f|(\mathbb{R}^n) \tag{31}$$

for all $f \in \mathrm{BV}(\mathbb{G})$.

We start with the following result, which will be useful in the proof of Theorem 2.11. It is a consequence of the algebraic properties of $\mathbb{G}$ and the Gaussian estimates on the heat kernel h (see Subsection 2.1.4).

Lemma 3.1. For $i, j \in \{1, \dots, q\}$ let G^i_j be the kernel defined by the identity

$$\begin{aligned}\sum_{k=q+1}^{n} X_k^R(c_i^k(\cdot)h(t,\cdot))(z) &= \sum_{j=1}^{q} X_j^R \sum_{k=q+1}^{n} \sum_{1\le i_l\le q} \vartheta^k_{ji_2\cdots i_{\omega_k}} X_{i_2}^R \cdots X_{i_{\omega_k}}^R (c_i^k(\cdot)\, h(t,\cdot))(z)\\ &= \sum_{j=1}^{q} X_j^R G^i_j(t,z),\end{aligned} \tag{32}$$

where the functions c_i^k and the constants $\vartheta^k_{i_1 i_2 \cdots i_{\omega_k}}$ are those introduced in (8) and (9), respectively. They have the following properties:

(i) $G^i_j(\lambda^2 t, D(\lambda)z) = \lambda^{-Q} G^i_j(t, z)$ for any $\lambda > 0$, $t > 0$, $z \in \mathbb{R}^n$;

(ii) there is a positive constant c, independent of t, such that

$$\int_{\mathbb{R}^n} |G^i_j(t, z)| \, \mathrm{d}z \le c;$$

(iii) $\int_{\mathbb{R}^n} G^i_j(t, z) \, \mathrm{d}z = 0$ for any $t > 0$;

(iv) for every $\varepsilon > 0$, $t_0 > 0$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_R} |G^i_j(t, z)| \, \mathrm{d}z < \varepsilon \quad \text{for any } 0 < t \le t_0. \qquad \square$$

Proof. (i) holds because $c_i^k(\cdot)$ is homogeneous of degree $\omega_k - \omega_i = \omega_k - 1$ (for $i = 1, 2, \ldots, q$), $h(t, \cdot)$ is homogeneous of degree $-Q$, and $X^R_{i_2} \cdots X^R_{i_{\omega_k}}$ is a homogeneous differential operator of degree $\omega_k - 1$. For the same reason, the Gaussian estimates on h also imply, by Remark 2.2,

$$|G^i_j(t, z)| = |X^R_{i_2} \cdots X^R_{i_{\omega_k}} (c_i^k(\cdot) \, h(t, \cdot))(z)| \le \mathbf{c} t^{-Q/2} \mathrm{e}^{-|z|^2/\mathbf{c}t}. \tag{33}$$

To prove (ii), with the change of variables $z = D(\sqrt{t})w$ and using (i) and (33), we compute

$$\begin{aligned} \int_{\mathbb{R}^n} |G^i_j(t, z)| \, \mathrm{d}z &= \int_{\mathbb{R}^n} |G^i_j(t, D(\sqrt{t})w)| t^{Q/2} \, \mathrm{d}w \\ &= \int_{\mathbb{R}^n} |G^i_j(1, w)| \, \mathrm{d}w \le \int_{\mathbb{R}^n} \mathbf{c} \mathrm{e}^{-|w|^2/\mathbf{c}} \, \mathrm{d}w, \end{aligned} \tag{34}$$

which is a finite constant. (iii) holds because G^i_j is, by definition, a linear combination of derivatives of the kind

$$X^R_{i_2} [X^R_{i_3} \cdots X^R_{i_{\omega_k}} (c_i^k(\cdot) \, h(t, \cdot))] = X^R_{i_2} H(t, \cdot);$$

we know that $(X_i^R)^T = -X_i^R$ (where $()^{\mathrm{T}}$ denotes transposition) and therefore we may deduce that $\int_{\mathbb{R}^n} X_i^R f(x)\,\mathrm{d}x = 0$ for any f for which this integral makes sense. To prove (iv), it suffices to modify the computation (34) as follows:

$$\int_{\mathbb{R}^n \setminus B_R} |G_j^i(t,z)|\,\mathrm{d}z \le \int_{|w|>R/\sqrt{t}} \mathbf{c}\mathrm{e}^{-|w|^2/\mathbf{c}}\,\mathrm{d}w < \varepsilon$$

for R large enough and $t \le t_0$. ■

The following result contains the main estimate on the commutator between the derivative X_i and the heat semigroup $X_i(W_t f) - W_t(X_i f)$.

Proposition 3.2. For any $f \in \mathrm{BV}(\mathbb{G})$ and $i \in \{1, 2, \dots, q\}$, $t > 0$, there exists an L^1 function μ_t^i on $\mathbb{R}^n$ such that

$$X_i(W_t f)(x) = W_t(X_i f)(x) + \mu_t^i(x),$$

where $W_t(X_i f)$ is the function defined by

$$W_t(X_i f)(x) := \int_{\mathbb{R}^n} h(t, y^{-1} \circ x)\,\mathrm{d}X_i f(y).$$

Moreover, for any $t > 0$,

$$\|\mu_t^i\|_{L^1(\mathbb{R}^n)} \le cq|D_{\mathbb{G}} f|(\mathbb{R}^n), \tag{35}$$

where c is the constant in Lemma 3.1, (ii). □

Proof. Let $f \in \mathrm{BV}(\mathbb{G})$ and fix $i \in \{1, 2, \ldots, q\}$. By (8) and (20) we have

$$
\begin{aligned}
X_i W_t f(x) &= \int_{\mathbb{R}^n} X_i h(t, y^{-1} \circ x) f(y)\, \mathrm{d}y \\
&= \int_{\mathbb{R}^n} X_i^R h(t, y^{-1} \circ x) f(y)\, \mathrm{d}y + \int_{\mathbb{R}^n} \sum_{k=q+1}^{n} X_k^R (c_i^k(\cdot)\, h(t, \cdot))(y^{-1} \circ x) f(y)\, \mathrm{d}y \\
&= \int_{\mathbb{R}^n} X_i^R h(t, y^{-1} \circ x) f(y)\, \mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n} \sum_{k=q+1}^{n} \sum_{1 \le i_j \le q} \vartheta^k_{i_1 i_2 \cdots i_{\omega_k}} X_{i_1}^R X_{i_2}^R \cdots X_{i_{\omega_k}}^R (c_i^k(\cdot)\, h(t, \cdot))(y^{-1} \circ x) f(y)\, \mathrm{d}y \\
&= \int_{\mathbb{R}^n} h(t, y^{-1} \circ x)\, \mathrm{d}(X_i f)(y) \\
&\quad + \sum_{k=q+1}^{n} \sum_{1 \le i_j \le q} \vartheta^k_{i_1 i_2 \cdots i_{\omega_k}} \int_{\mathbb{R}^n} X_{i_2}^R \cdots X_{i_{\omega_k}}^R (c_i^k(\cdot) h(t, \cdot))(y^{-1} \circ x)\, \mathrm{d}(X_{i_1} f)(y) \\
&= W_t(X_i f)(x) + \sum_{j=1}^{q} \int_{\mathbb{R}^n} G_j^i(t, y^{-1} \circ x)\, \mathrm{d}(X_j f)(y),
\end{aligned}
\tag{36}
$$

where the kernels G_j^i are defined in Lemma 3.1. Set

$$
\mu_t^i(x) = \sum_{j=1}^{q} \int_{\mathbb{R}^n} G_j^i(t, y^{-1} \circ x)\, \mathrm{d}(X_j f)(y);
\tag{37}
$$

then, by Lemma 3.1 and (16), we have

$$
\|\mu_t^i\|_{L^1(\mathbb{R}^n)} \le c \sum_{j=1}^{q} \int_{\mathbb{R}^n} |\,\mathrm{d}(X_j f)|(y) \le cq |D_{\mathbb{G}} f|(\mathbb{R}^n),
$$

that is (35). By (36) and (37), we can write

$$
X_i(W_t f)(x) = W_t(X_i f)(x) + \mu_t^i(x).
\tag{38}
$$

■

Proof of Theorem 2.11. Let us conclude with the proof of (31). By Proposition 3.2 we have (with the supremum taken, according to Definition 2.4, on all the functions

$g \in C^1_0(\mathbb{R}^n, \mathbb{R}^q)$ such that $\|g\|_\infty \le 1$):

$$\int_{\mathbb{R}^n} |\nabla_X W_t f(x)|\,\mathrm{d}x = \sup_g \left\{ \int_{\mathbb{R}^n} \sum_{i=1}^q g_i(x) X_i(W_t f)(x)\,\mathrm{d}x \right\}$$

$$= \sup_g \left\{ \int_{\mathbb{R}^n} \sum_{i=1}^q g_i(x) \left(\int_{\mathbb{R}^n} h(t, y^{-1} \circ x)\,\mathrm{d}X_i f(y) \right) \mathrm{d}x + \int_{\mathbb{R}^n} \sum_{i=1}^q g_i(x) \mu_t^i(x)\,\mathrm{d}x \right\}$$

since $h(t, z^{-1}) = h(t, z)$

$$= \sup_g \left\{ \sum_{i=1}^q \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} h(t, x^{-1} \circ y) g_i(x)\,\mathrm{d}x \right) \mathrm{d}X_i f(y) + \int_{\mathbb{R}^n} \sum_{i=1}^q g_i(x) \mu_t^i(x)\,\mathrm{d}x \right\}$$

$$= \sup_g \left\{ \sum_{i=1}^q \int_{\mathbb{R}^n} W_t g_i(y)\,\mathrm{d}X_i f(y) + \int_{\mathbb{R}^n} \sum_{i=1}^q g_i(x) \mu_t^i(x)\,\mathrm{d}x \right\}$$

$$= \sup_g \left\{ -\sum_{i=1}^q \int_{\mathbb{R}^n} X_i(W_t g_i)(y) f(y)\,\mathrm{d}y + \int_{\mathbb{R}^n} \sum_{i=1}^q g_i(x) \mu_t^i(x)\,\mathrm{d}x \right\},$$

where in the last identity we have used (15). We now exploit the fact that the supremum on all compactly supported functions g in the above expression can be replaced by the supremum on functions ϕ rapidly decreasing at infinity (as $W_t g$ is, by the Gaussian estimates) verifying the constraint $\|\phi\|_\infty \le 1$; therefore the last expression is, by (35),

$$\le |D_{\mathbb{G}} f| + \sum_{i=1}^q \|\mu_t^i\|_{L^1(\mathbb{R}^n)} \le (1 + cq^2)|D_{\mathbb{G}} f|(\mathbb{R}^n).$$

We have therefore proved that, for any $t > 0$,

$$\int_{\mathbb{R}^n} |\nabla_X W_t f(x)|\,\mathrm{d}x \le (1 + cq^2)|D_{\mathbb{G}} f|(\mathbb{R}^n).$$

Passing to the limsup as $t \to 0$, we are done. ■

4 Proof of Theorem 2.14

Theorem 2.14, owing to the coarea formula, follows from Theorem 2.13, so we first prove the latter.

As a preliminary result, we present a characterization of BV($\mathbb{G}$) functions analogous to that in [9, Proposition 2.3].

Lemma 4.1. If $f \in L^1(\mathbb{R}^n)$, $z \in \mathbb{R}^n$ and

$$\liminf_{t\to 0} \frac{1}{t} \int_{\mathbb{R}^n} |f(x \circ D(t)z) - f(x)| \, \mathrm{d}x < \infty, \tag{39}$$

then the distributional derivative of f along $Z = \sum_{j=1}^q z_j X_j$ is a finite measure. □

Proof. The proof is essentially the same contained in [9], but we repeat it for the reader's convenience. We start by recalling the following identity (see [7, p. 17]): if we take $\phi \in C_c^1(\mathbb{R}^n)$ and Z is any vector field that at the origin equals z, then

$$Z\phi(x) = \frac{\mathrm{d}}{\mathrm{d}t}[\phi(x \circ tz)]_{/t=0}.$$

Let us apply this identity to the vector field $Z = \sum_{i=1}^q z_i X_i$, which at the origin takes the value

$$z^* = (z_1, z_2, \ldots, z_q, 0, 0, \ldots, 0).$$

For any $z \in \mathbb{R}^n$,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\phi(x \circ D(t)z)]_{/t=0} = \frac{\mathrm{d}}{\mathrm{d}t}[\phi(x \circ tz^*)]_{/t=0},$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}[\phi(x \circ D(t)z)]_{/t=0} = \sum_{i=1}^q z_i (X_i\phi)(x).$$

Since

$$\left| \int_{\mathbb{R}^n} f(x) \frac{\phi(x \circ D(t)z^{-1}) - \phi(x)}{t} \, \mathrm{d}x \right| = \left| \int_{\mathbb{R}^n} \frac{f(x \circ D(t)z) - f(x)}{t} \phi(x) \, \mathrm{d}x \right|$$
$$\leq \|\phi\|_\infty \frac{1}{|t|} \int_{\mathbb{R}^n} |f(x \circ D(t)z) - f(x)| \, \mathrm{d}x,$$

from (39) we deduce that the functional

$$T_f\phi := \lim_{t\to 0} \int_{\mathbb{R}^n} f(x) \frac{\phi(x \circ D(t)z^{-1}) - \phi(x)}{t} \, \mathrm{d}x = -\int_{\mathbb{R}^n} f(x) Z\phi(x) \, \mathrm{d}x$$

is well defined and satisfies the condition $|T_f\phi| \leq C\|\phi\|_\infty$. Then there exists a measure μ_Z such that

$$T_f\phi = \int_{\mathbb{R}^n} \phi(x) \, \mathrm{d}\mu_Z(x), \quad \forall \phi \in C_c^1(\mathbb{R}^n).$$

By the density of $C_c^1(\mathbb{R}^n)$ in $C_c(\mathbb{R}^n)$, we get the conclusion. ■

Proof of Theorem 2.13. Let us introduce the functions

$$g(t) = \int_{E^c} W_t \chi_E(x)\,\mathrm{d}x,$$

$$F(t) = g(t^2) = \int_{E^c} W_{t^2} \chi_E(x)\,\mathrm{d}x.$$

The statement we want to prove is

$$\lim_{t\to 0} \frac{g(t)}{\sqrt{t}} = 2\theta_\infty \int_{\partial^*_{\mathbb{G}} E} \phi_{\mathbb{G}}(\nu_E)\,\mathrm{d}\mathcal{S}_\infty^{Q-1}.$$

First, note that $g(t) \to \int_{E^c} \chi_E(x)\,\mathrm{d}x = 0$ as $t \downarrow 0$. Indeed,

$$W_t\chi_E - \chi_E = -(W_t\chi_{E^c} - \chi_{E^c}), \tag{40}$$

since

$$\chi_E + \chi_{E^c} = 1 = W_t 1 = W_t\chi_E + W_t\chi_{E^c}$$

and, since by the finiteness of the perimeter of E either E or E^c has finite measure, $\|W_t\chi_E - \chi_E\|_{L^1(\mathbb{R}^n)} \to 0$ as $t \downarrow 0$. By the De L'Hospital rule we can evaluate $\lim_{t\to 0} g(t)/\sqrt{t}$ as

$$\lim_{t\to 0} 2\sqrt{t}g'(t) = \lim_{t\to 0} 2tg'(t^2) = \lim_{t\to 0} F'(t).$$

Computing the derivative of F using the divergence theorem (24) and the properties of $h(t,\cdot)$, we get

$$\begin{aligned} F'(t) &= 2t\int_{E^c} L\,W_{t^2}\chi_E\,\mathrm{d}x = 2t\int_{E^c} \mathrm{div}_{\mathbb{G}}\,\nabla_X W_{t^2}\chi_E\,\mathrm{d}x \\ &= 2\theta_\infty t\int_{\partial^*_{\mathbb{G}} E} \langle \nabla_X W_{t^2}\chi_E, \nu_E\rangle\,\mathrm{d}\mathcal{S}_\infty^{Q-1} \\ &= 2\theta_\infty t\int_{\partial^*_{\mathbb{G}} E} \langle \nabla_X \left(\int_E h(t^2, y^{-1}\circ x)\,\mathrm{d}y\right), \nu_E(x)\rangle\,\mathrm{d}\mathcal{S}_\infty^{Q-1}(x). \end{aligned}$$

Now, taking into account the fact that $h(t, z^{-1}) = h(t, z)$, we have

$$\int_E h(t^2, y^{-1} \circ x)\,\mathrm{d}y = \int_E h(t^2, x^{-1} \circ y)\,\mathrm{d}y = \int_E t^{-Q} h\left(1, D\left(\frac{1}{t}\right)(x^{-1} \circ y)\right) \mathrm{d}y$$

$$= \int_E t^{-Q} h\left(1, D\left(\frac{1}{t}\right)(x^{-1}) \circ D\left(\frac{1}{t}\right)(y)\right) \mathrm{d}y$$

$$= \int_{D(\frac{1}{t})E} h\left(1, D\left(\frac{1}{t}\right)(x^{-1}) \circ z\right) \mathrm{d}z$$

$$= \int_{D(\frac{1}{t})E} h\left(1, z^{-1} \circ D\left(\frac{1}{t}\right)(x)\right) \mathrm{d}z,$$

hence

$$F'(t) = 2\theta_\infty t \int_{\partial^*_{\mathbb{G}} E} \left\langle \frac{1}{t} \int_{D(\frac{1}{t})E} \nabla_X h\left(1, z^{-1} \circ D\left(\frac{1}{t}\right)(x)\right) \mathrm{d}z, \nu_E(x) \right\rangle \mathrm{d}\mathcal{S}^{Q-1}_\infty(x)$$

$$= 2\theta_\infty \int_{\partial^*_{\mathbb{G}} E} \left\langle \int_{D(\frac{1}{t})E} \nabla_X h(1, z^{-1} \circ D\left(\frac{1}{t}\right)(x)\,\mathrm{d}z, \nu_E(x) \right\rangle \mathrm{d}\mathcal{S}^{Q-1}_\infty(x)$$

$$= 2\theta_\infty \int_{\partial^*_{\mathbb{G}} E} \left\langle \int_{D(1/t)(x^{-1}E)} \nabla_X h(1, z^{-1})\,\mathrm{d}z, \nu_E(x) \right\rangle \mathrm{d}\mathcal{S}^{Q-1}_\infty(x)$$

$$= 2\theta_\infty \sum_{i=1}^{q} \int_{\partial^*_{\mathbb{G}} E} \nu_E(x)_i \left(\int_{D(1/t)(x^{-1}E)} X_i h(1, z^{-1})\,\mathrm{d}z \right) \mathrm{d}\mathcal{S}^{Q-1}_\infty(x).$$

By Theorem 2.9 (blow-up of the reduced boundary), we have the L^1_{loc} convergence

$$E_{1/t, x^{-1}} = D(1/t)(x^{-1}E) \to S^+_{\mathbb{G}}(\nu_E(x)). \tag{41}$$

Since by the Gaussian estimates $X_i h(1, z^{-1}) \in L^1 \cap C^\infty(\mathbb{R}^n)$, the integral on $E_{1/t, x^{-1}}$ converges to the integral on $S^+_{\mathbb{G}}(\nu_E(x))$ as t goes to 0 (see Remark 2.10) and is continuous as a function of the point x, we can apply the dominated convergence theorem, getting

$$\lim_{t \to 0} F'(t) = 2\theta_\infty \int_{\partial^*_{\mathbb{G}} E} \sum_{i=1}^{q} \nu_E(x)_i \left(\int_{S^+_{\mathbb{G}}(\nu_E(x))} X_i h(1, z^{-1})\,\mathrm{d}z \right) \mathrm{d}\mathcal{S}^{Q-1}_\infty(x)$$

$$= 2\theta_\infty \int_{\partial^*_{\mathbb{G}} E} \sum_{i=1}^{q} \nu_E(x)_i \left(\int_{S^-_{\mathbb{G}}(\nu_E(x))} X_i h(1, z)\,\mathrm{d}z \right) \mathrm{d}\mathcal{S}^{Q-1}_\infty(x)$$

with the obvious meaning of the symbol $S^-_{\mathbb{G}}(\nu_E(x))$. Next, we perform a standard integration by parts in the inner integral, exploiting the fact that the (Euclidean) outer normal to the halfspace $S^-_{\mathbb{G}}(\nu_E(x))$ is

$$n_E(x) = (\nu_E(x)_1, \ldots, \nu_E(x)_q, 0, \ldots, 0),$$

while

$$X_i h(1, z) = \partial_{z_i} h(1, z) + \sum_{j=q+1}^{n} \partial_{z_j}(q^i_j(z) h(1, z)),$$

hence

$$\sum_{i=1}^{q} \nu_E(x)_i \left(\int_{S^-_{\mathbb{G}}(\nu_E(x))} X_i h(1, z)\, \mathrm{d}z \right) = \sum_{i=1}^{q} (\nu_E(x)_i)^2 \int_{T_{\mathbb{G}}(\nu_E(x))} h(1, z)\, \mathrm{d}z = \phi_{\mathbb{G}}(\nu_E)$$

for any $x \in \partial^*_{\mathbb{G}} E$, and we conclude that

$$\lim_{t\to 0} \frac{1}{2\theta_\infty \sqrt{t}} \int_{E^c} W_t \chi_E(x)\, \mathrm{d}x = \int_{\partial^*_{\mathbb{G}} E} \phi_{\mathbb{G}}(\nu_E)\, \mathrm{d}\mathcal{S}^{Q-1}_\infty. \tag{42}$$

In order to prove the converse, assume that E has finite measure and note that, for any z,

$$\int_{\mathbb{R}^n} \chi_E(x \circ D(\sqrt{t}) z)(1 - \chi_E(x))\, \mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t}) z) - \chi_E(x)|\, \mathrm{d}x,$$

and that the Lebesgue measure is right invariant as well. Therefore

$$\begin{aligned}
\int_{E^c} W_t \chi_E(x)\, \mathrm{d}x &= t^{-Q/2} \int_{\mathbb{R}^n} \int_E h(1, D(1/\sqrt{t})(y^{-1} \circ x))(1 - \chi_E(x))\, \mathrm{d}y\, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h(1, z) \chi_E(x \circ D(\sqrt{t}) z^{-1})(1 - \chi_E(x))\, \mathrm{d}z\, \mathrm{d}x \\
&= \frac{1}{2} \int_{\mathbb{R}^n} h(1, w^{-1}) \int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t}) w) - \chi_E(x)|\, \mathrm{d}x\, \mathrm{d}w \\
&= \frac{1}{2} \int_{\mathbb{R}^n} h(1, w) \int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t}) w) - \chi_E(x)|\, \mathrm{d}x\, \mathrm{d}w.
\end{aligned}$$

Then the finiteness of the liminf of the integral in the left-hand side of (42) implies that there is a sequence $t_k \downarrow 0$ such that

$$\frac{1}{2}\int_{\mathbb{R}^n} h(1, w)\int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t_k})w) - \chi_E(x)|\,\mathrm{d}x\,\mathrm{d}w \le C\sqrt{t_k}$$

for all k. On the other hand, by the lower Gaussian estimates on $h(1, w)$ we can also write

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^n} h(1, w)\int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t_k})w) - \chi_E(x)|\,\mathrm{d}x\,\mathrm{d}w \\
&\quad\ge \frac{1}{2}\int_{|w|\le 2} h(1, w)\int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t_k})w) - \chi_E(x)|\,\mathrm{d}x\,\mathrm{d}w \\
&\quad\ge c\int_{|w|\le 2}\int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t_k})w) - \chi_E(x)|\,\mathrm{d}x\,\mathrm{d}w,
\end{aligned}$$

hence

$$\int_{|w|\le 2}\int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t_k})w) - \chi_E(x)|\,\mathrm{d}x\,\mathrm{d}w \le C\sqrt{t_k}.$$

So, if we define the function

$$\Phi_{t_k}(w) = \int_{\mathbb{R}^n} \frac{|\chi_E(x \circ D(\sqrt{t_k})w) - \chi_E(x)|}{\sqrt{t_k}}\,\mathrm{d}x,$$

by Fatou's Lemma we deduce that

$$\Phi_0(w) = \liminf_{k\to\infty} \Phi_{t_k}(w)$$

is integrable on $B = \{|w| \le 2\}$. So, for any $\varepsilon > 0$, there are a set B_ε of measure less than ε and a constant $C_\varepsilon > 0$ such that, for every $w \in B \setminus B_\varepsilon$, $\Phi_0(w) \le C_\varepsilon$. In particular, we may choose n linearly independent directions $w_1, \ldots, w_n$ with $\Phi_0(w_i) \le C_\varepsilon$, and there is $k_0 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t_k})w_i) - \chi_E(x)|\,\mathrm{d}x \le (C_\varepsilon + 1)\sqrt{t_k}, \quad \forall\, i = 1, \ldots, n,\ \ k > k_0.$$

Then, by Lemma 4.1, we can conclude that E has finite perimeter. ■

We note that the above proof of Theorem 2.13 simplifies that of the analogous result in the Euclidean setting given in [24].

With an argument similar to that used in the proof of [25, Proposition 8], it is also possible to prove the following proposition.

Proposition 4.2. If E has finite perimeter, then

$$\frac{1}{4\sqrt{t}}\int_{\mathbb{R}^n}|W_t\chi_E-\chi_E|\,\mathrm{d}x\le c_{\mathbb{G}}P_{\mathbb{G}}(E),\qquad t>0 \tag{43}$$

with

$$c_{\mathbb{G}}=\int_{\mathbb{R}^n}|\nabla_X h(1,z)|\,\mathrm{d}z.$$

□

Proof. First of all we note that

$$\int_{\mathbb{R}^n}(W_t\chi_E(x)-\chi_E(x))\,\mathrm{d}x=0.$$

This follows by (iv) in Theorem 1 if E has finite measure; otherwise E^c must have finite measure, and the same identity holds in view of (40). Recalling that $0\le W_t\chi_E(x)\le 1$, we deduce that

$$\begin{aligned}\int_{E^c}W_t\chi_E(x)\,\mathrm{d}x&=\int_{E^c}(W_t\chi_E(x)-\chi_E(x))\,\mathrm{d}x=\int_{\mathbb{R}^n}(W_t\chi_E(x)-\chi_E(x))^+\,\mathrm{d}x\\&=\int_{\mathbb{R}^n}(W_t\chi_E(x)-\chi_E(x))^-\,\mathrm{d}x=\frac{1}{2}\int_{\mathbb{R}^n}|W_t\chi_E(x)-\chi_E(x)|\,\mathrm{d}x\end{aligned} \tag{44}$$

(note that the above equality holds under the hypothesis that either E or E^c has finite measure). We can also write

$$\int_{E^c}W_t\chi_E(x)\,\mathrm{d}x=\int_{E^c}(W_t\chi_E(x)-\chi_E(x))\,\mathrm{d}x=\int_0^t\mathrm{d}s\int_{E^c}L\,W_t\chi_E(x)\,\mathrm{d}x.$$

But then by the divergence theorem

$$\begin{aligned}\int_{E^c}L\,W_t\chi_E(x)\,\mathrm{d}x&=\int_{\partial^*_{\mathbb{G}}E}\mathrm{d}P_{\mathbb{G}}(E)(x)\int_{\mathbb{R}^n}\langle\nabla_X h(s,y^{-1}\circ x),\nu_E(x)\rangle\chi_E(y)\,\mathrm{d}y\\&\le\int_{\partial^*_{\mathbb{G}}E}\mathrm{d}P_{\mathbb{G}}(E)(x)\int_{\mathbb{R}^n}|\nabla_X h(s,y^{-1}\circ x)|\,\mathrm{d}y.\end{aligned} \tag{45}$$

By (44) and (45) the conclusion then follows since

$$\begin{aligned}\frac{1}{4\sqrt{t}}\int_{\mathbb{R}^n}|W_t\chi_E-\chi_E|\,\mathrm{d}x &= \frac{1}{2\sqrt{t}}\int_{E^c} W_t\chi_E(x)\,\mathrm{d}x = \frac{1}{2\sqrt{t}}\int_0^t \mathrm{d}s\int_{E^c} L W_s\chi_E(x)\,\mathrm{d}x\\ &\le \frac{1}{2\sqrt{t}}\int_0^t\left(\int_{\partial^*_{\mathbb{G}}E}\mathrm{d}P_{\mathbb{G}}(E)(x)\int_{\mathbb{R}^n}|\nabla_X h(s,y^{-1}\circ x)|\,\mathrm{d}y\right)\mathrm{d}s\\ &= \frac{1}{2\sqrt{t}}\int_0^t\left(\int_{\partial^*_{\mathbb{G}}E}\mathrm{d}P_{\mathbb{G}}(E)(x)\frac{1}{\sqrt{s}}\int_{\mathbb{R}^n}|\nabla_X h(1,z)|\,\mathrm{d}z\right)\mathrm{d}s\\ &\le c_{\mathbb{G}}P_{\mathbb{G}}(E).\end{aligned}$$

■

Proof of Theorem 2.14. The derivation of Theorem 2.14 from Theorem 2.13 is based on the coarea formula, and is similar to that of [24, Theorem 4.1]. Assume $f\in \mathrm{BV}(\mathbb{G})$ and set $E_\tau=\{f>\tau\}$. By Proposition 2.7, for a.e. $\tau\in\mathbb{R}$ the set E_τ has finite perimeter, and hence, by Theorem 2.13, we can write

$$\lim_{t\to 0}\frac{1}{2\theta_\infty\sqrt{t}}\int_{E^c_\tau} W_t\chi_{E_\tau}(x)\,\mathrm{d}x=\int_{\partial^*_{\mathbb{G}}E_\tau}\phi_{\mathbb{G}}(\nu_{E_\tau}(x))\,\mathrm{d}\mathcal{S}^{Q-1}_\infty(x)$$

for a.e. τ.

Moreover, using the same arguments as [1, Proposition 3.69] and the continuity of $\phi_{\mathbb{G}}$, it is easily checked that the function $\phi_{\mathbb{G}}(\sigma_f(x))$ is (obviously bounded and) Borel. Comparing f with $f\vee\tau\chi_{E_\tau}$ and using [1, Proposition 3.73], we see that, for a.e. $\tau\in\mathbb{R}$, the equality $\sigma_f(x)=\nu_{E_\tau}(x)$ holds for $\mathcal{S}^{Q-1}_\infty$-a.e. $x\in\partial^*_{\mathbb{G}}E_\tau$, whence

$$\phi_{\mathbb{G}}(\sigma_f(x))=\phi_{\mathbb{G}}(\nu_{E_\tau}(x))\ \text{for a.e.}\,\tau\in\mathbb{R},\ \mathcal{S}^{Q-1}_\infty\text{-a.e.}\,x\in\partial^*_{\mathbb{G}}E_\tau.$$

We point out that if x belongs to the boundary of two different level sets E_τ, E_σ (which happens whenever f has a jump at x), then the above equality holds for both the levels because $\nu_{E_\tau}(x)=\nu_{E_\sigma}(x)$ for $\sigma,\tau\in[f^-(x),f^+(x)]$. We also note that, for almost every x, $y\in\mathbb{R}^n$,

$$\int_{\mathbb{R}}|\chi_{E_\tau}(x)-\chi_{E_\tau}(y)|d\tau=|f(x)-f(y)|. \tag{46}$$

With the aid of the coarea formula (22) with $g(x) = \phi_{\mathbb{G}}(\sigma_f(x))$, by (44), (46) and using the dominated convergence theorem, we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi_{\mathbb{G}}(\sigma_f(x))\, \mathrm{d}|D_{\mathbb{G}} f| &= \theta_\infty \int_{\mathbb{R}} \int_{\partial^*_{\mathbb{G}} E_\tau} \phi_{\mathbb{G}}(\nu_{E_\tau}(x))\, \mathrm{d}\mathcal{S}^{Q-1}_\infty(x)\, \mathrm{d}\tau \\
&= \int_{\mathbb{R}} \lim_{t\to 0} \frac{1}{2\sqrt{t}} \int_{E^c_\tau} W_t \chi_{E_\tau}(x)\, \mathrm{d}x d\tau \\
&= \lim_{t\to 0} \int_{\mathbb{R}} \frac{1}{4\sqrt{t}} \int_{\mathbb{R}^n} |W_t \chi_{E_\tau} - \chi_{E_\tau}|\, \mathrm{d}x\, \mathrm{d}\tau \\
&= \lim_{t\to 0} \frac{1}{4\sqrt{t}} \int_{\mathbb{R}} \int_{\mathbb{R}^n\times\mathbb{R}^n} |\chi_{E\tau}(y) - \chi_{E_\tau}(x)| h(t, y^{-1}\circ x)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\tau \\
&= \lim_{t\to 0} \frac{1}{4\sqrt{t}} \int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x) - f(y)| h(t, y^{-1}\circ x)\, \mathrm{d}x\, \mathrm{d}y. \qquad (47)
\end{aligned}$$

Conversely, assume that $f \in L^1(\mathbb{R}^n)$ and that

$$\liminf_{t\to 0} \frac{1}{\sqrt{t}} \int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x) - f(y)| h(t, y^{-1}\circ x)\, \mathrm{d}x\, \mathrm{d}y$$

is finite. Then

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x) - f(y)|\, h(t, y^{-1}\circ x)\, \mathrm{d}x\, \mathrm{d}y = \int_{\mathbb{R}} \int_{\mathbb{R}^n\times\mathbb{R}^n} |\chi_{E_\tau}(x) - \chi_{E_\tau}(y)|\, h(t, y^{-1}\circ x)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\tau.$$

Since $\int_{E^c} W_t \chi_E \geq 0$, by the above equality and (44), which we are allowed to exploit because, for $f \in L^1(\mathbb{R}^n)$, either E_τ or its complement has finite measure, we get

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}} \left(\liminf_{t\to 0} \frac{1}{\sqrt{t}} \int_{E^c_\tau} W_t \chi_{E_\tau}\, \mathrm{d}x \right) \mathrm{d}\tau \leq \liminf_{t\to 0} \int_{\mathbb{R}} \frac{1}{\sqrt{t}} \int_{E^c_\tau} W_t \chi_{E_\tau}\, \mathrm{d}x\, \mathrm{d}\tau \\
&\leq \liminf_{t\to 0} \frac{1}{2\sqrt{t}} \int_{\mathbb{R}^n\times\mathbb{R}^n} \int_{\mathbb{R}} |\chi_{E_\tau}(x) - \chi_{E_\tau}(y)|\, h(t, y^{-1}\circ x)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\tau < +\infty.
\end{aligned}$$

In particular, by Theorem 2.13, for a.e. $\tau \in \mathbb{R}$ the set E_τ has finite perimeter and the limit $\lim_{t\to 0} \frac{1}{\sqrt{t}} \int_{E^c_\tau} W_t \chi_{E_\tau}\, \mathrm{d}x$ exists. As a consequence, $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{G})$ and we may use the above identity (47) to get

$$|D_{\mathbb{G}} f|(\mathbb{R}^n) \leq \frac{1}{\min \phi_{\mathbb{G}}} \int_{\mathbb{R}^n} \phi_{\mathbb{G}}(\sigma_f)\, \mathrm{d}|D_{\mathbb{G}} f|$$

$$\leq \frac{1}{\min \phi_{\mathbb{G}}} \liminf_{t\to 0} \frac{1}{4\sqrt{t}} \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)|\, h(t, y^{-1} \circ x)\, \mathrm{d}x\, \mathrm{d}y < +\infty,$$

that is, $f \in \mathrm{BV}(\mathbb{G})$. ■

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