Annali di Matematica
DOI 10.1007/s10231-011-0186-1

Global $W^{2,p}$ estimates for nondivergence elliptic operators with potentials satisfying a reverse Hölder condition

M. Bramanti · L. Brandolini · E. Harboure · B. Viviani

Received: 5 May 2010 / Accepted: 14 January 2011

Abstract In this article, we give some a priori $L^p(\mathbb{R}^n)$ estimates for elliptic operators in nondivergence form with VMO coefficients and a potential V satisfying an appropriate reverse Hölder condition, generalizing previous results due to Chiarenza–Frasca–Longo to the scope of Schrödinger-type operators. In particular, our class of potentials includes unbounded functions such as nonnegative polynomials. We apply such a priori estimates to derive some global existence and uniqueness results under some additional assumptions on V.

Keywords Schrödinger operator · Global existence and uniqueness · Reverse Hölder condition · VMO coefficients · Global L^p estimates

Mathematics Subject Classification (2000) Primary 35J10; Secondary 35B45 · 35A05 · 42B35

All authors were partially supported by MTM 2008-06621-C02-01 from Ministerio de Ciencia e Innovación de España. Third and fourth authors were also supported by grants from Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) and Universidad Nacional del Litoral (UNL), Argentina.

M. Bramanti (✉)
Dipartimento di Matematica, Politecnico di Milano, Via Bonardi 9, 20133 Milan, Italy
e-mail: marco.bramanti@polimi.it

L. Brandolini
Dipartimento di Ingegneria dell'Informazione e Metodi Matematici, Università degli Studi di Bergamo, Viale Marconi 5, 24044 Dalmine, BG, Italy
e-mail: luca.brandolini@unibg.it

E. Harboure · B. Viviani
Instituto de Matemática Aplicada del Litoral, CONICET-Universidad Nacional del Litoral, Güemes 3450, 3000 Santa Fe, Republic of Argentina
e-mail: harbour@santafe-conicet.gov.ar

B. Viviani
e-mail: viviani@santafe-conicet.gov.ar

Published online: 17 February 2011

1 Introduction

Let us consider the linear, second-order elliptic operator

$$Lu \equiv Au + Vu \equiv -a_{ij}u_{x_i x_j} + Vu$$

where (for $i, j = 1, 2, \ldots, n$) $a_{ij} \in L^{\infty}(\mathbb{R}^n)$, $a_{ij} = a_{ji}$,

$$\mu |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \leq \frac{1}{\mu} |\xi|^2 \quad \forall x, \quad \xi \in \mathbb{R}^n, \text{ for some } \mu > 0, \tag{1}$$

$$a_{ij} \in VMO\left(\mathbb{R}^n\right) \tag{2}$$

which means that for $i, j = 1, 2, \ldots, n$

$$\eta_{ij}(r) = \sup_{\rho \leq r} \sup_{x \in \mathbb{R}^n} \left(\frac{1}{\left|B_\rho(x)\right|} \int_{B_\rho(x)} \left|a_{ij}(y) - a_{ij}^B\right| \mathrm{d}y \right)$$

(which is finite for every r since a_{ij} is bounded) vanishes for $r \to 0^+$. Here $a_{ij}^B = \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} a_{ij}(y)\, \mathrm{d}y$.

As to the potential V, we assume that it is not identically zero and that

$$V \in B_q \quad \text{for some } q \geq \frac{n}{2}, \tag{3}$$

which by definition means that $V \in L^q_{loc}$, $V \geq 0$, and there exists a constant $C > 0$ such that the *reverse Hölder inequality*

$$\left(\frac{1}{|B|} \int_B V(x)^q \, \mathrm{d}x \right)^{1/q} \leq C \left(\frac{1}{|B|} \int_B V(x)\, \mathrm{d}x \right)$$

holds for every ball B in $\mathbb{R}^n$.

An important property of the B_q class, proved in [6, Lemma 3], assures that the condition $V \in B_q$ also implies $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$ and that the $B_{q+\varepsilon}$ constant of V is controlled in terms of the one of B_q membership.

This in particular implies that $V \in L^q_{loc}$ for some q strictly greater than $n/2$. However, V in general will not be bounded, nor belonging to $L^p(\mathbb{R}^n)$ for any p. As a model example, we could take $V(x) = |x|^2$. More generally, as noted in [16], if V is any nonnegative polynomial, then V satisfies the stronger condition

$$\max_{x \in B} V(x) \leq C \frac{1}{|B|} \int_B V(x)\, \mathrm{d}x,$$

which implies $V \in B_q$ for every $q \in (1, \infty)$ with a uniform constant.

Another property of the B_q class that will be useful is the fact that the measure $\mathrm{d}\mu(y) = V(y)\, \mathrm{d}y$ is doubling (see e.g., [18, chap. V]).

We are mainly interested in proving global a priori L^p estimates of the kind

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} \leq C \left\{ \|Lu\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \right\} \tag{4}$$

for any $p \in (1, q]$, $u \in C_0^{\infty}(\mathbb{R}^n)$. When A is the Laplacian, these bounds have been proved by Shen [16]. Related results when A is the Laplacian and $V(x) = |x|^2$ (Hermite operator) have been proved by Thangavelu [19]. For a nondivergence operator A with VMO coefficients but with null potential, the result is due to Chiarenza–Frasca–Longo [3,4]; see also Vitanza [20] where the operator $A+V$, with a nonnegative $V \in L^q$ with $q > n/2$, and VMO leading coefficients is considered, and [9]. For operators $A+V$ with V satisfying (3) and A in divergence form, with coefficients satisfying (1) (or even a weaker weighted condition allowing degeneracy), Dziubanski [5] has obtained some bounds for the fundamental solution, which will be useful also to us.

We will prove the global bound (4) under assumptions (1–3), with a constant C only depending on the quantities involved in these assumptions (see Theorem 1 in Sect. 2). In particular, this means that C does not depend on any L^p_{loc} norm of V.

We are also interested in deriving from this global bound an existence and uniqueness result for the equation $Lu = f \in L^p(\mathbb{R}^n)$, $p \in (1, q]$. This will be accomplished in Sect. 3. Actually this requires an extra assumption on V, namely

$$V(x) \geq \delta > 0 \quad \text{for any } x \in \mathbb{R}^n \tag{5}$$

(see Theorem 17). The necessity of some extra assumption of this kind is clear since the simple equation $-\Delta u = f$ is not solvable in $W^{2,p}(\mathbb{R}^n)$ for any $f \in L^p(\mathbb{R}^n)$. On the other hand, in analogy with the fact that $-\Delta u + Vu = f$ is solvable in any bounded domain Ω as soon as $V \geq 0$ and $V \in L^p(\Omega)$ with $p > n/2$, we can expect that (5) can be relaxed. Actually, we will prove (see Theorem 23) that (5) can be replaced by the weaker condition:

There exist positive constants δ, R such that

$$V(x) \geq \delta > 0 \quad \text{for } |x| \geq R \tag{6}$$

so that our existence and uniqueness result applies for instance to the model equation of Hermite type

$$Au + |x|^2 u = f \in L^p(\mathbb{R}^n) \quad \text{for any } p \in (1, \infty)$$

or, for that matter, when V is any nonnegative polynomial, providing that the principal part A satisfies (1, 2).

The strategy we adopt in order to prove the global L^p bound is the following. Thanks to the a priori estimates proved by Chiarenza–Frasca–Longo [3] for the principal part operator, we are reduced to prove an L^p bound on Vu in terms of Lu. Freezing the coefficients a_{ij} at some point, we write a representation formula for u by means of the fundamental solution of a constant coefficient operator of type $A_0 + V$, for which the global estimates proved by Dziubanski [5] are available. Unfreezing now the coefficients, we get a representation formula for Vu ; this formula involves suitable (nonsingular) integral operators with positive kernels, applied to Lu, and their commutators, applied to the second-order derivatives of u.

In this way, after using the estimate in [5], we are led to prove that certain integral operators with positive kernels and their positive commutators are bounded on $L^p$with the L^p norms of the commutators small enough, in terms of the VMO moduli of the coefficients a_{ij}. This idea is borrowed by the papers by Chiarenza–Frasca–Longo but applied in the context of integral operators with positive kernels, on the whole $\mathbb{R}^n$, where the behavior of the kernel at infinity has to be carefully handled.

Once the global L^p bounds are established, an existence and uniqueness theorem is not yet a straightforward result, because of the unboundedness of both the domain and the $L^p(\mathbb{R}^n)$ norm of V, which makes troublesome the use of standard compactness arguments. To achieve the result, we have to revise some arguments carried out by Krylov [9] in the case of continuous or VMO coefficients, adapting the arguments to our weaker assumptions.

2 Global L^p estimates

Our main result is the following:

Theorem 1 *Under the assumptions (1–3), for every $p \in (1, q]$, there exists a constant $C > 0$ such that*

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} \le C \left\{ \|Lu\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \right\} \tag{7}$$

for any $u \in C_0^\infty(\mathbb{R}^n)$. The constant C depends on n, p, q, the ellipticity constant μ, the VMO moduli of the leading coefficients, and the B_q constant of V.

The bound (7) immediately extends to all functions $u \in W_V^{2,p}(\mathbb{R}^n)$, the closure of $C_0^\infty(\mathbb{R}^n)$ in the norm

$$\|u\|_{W_V^{2,p}(\mathbb{R}^n)} = \|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)}.$$

By the way, let us note that if $u \in W^{2,p}(\mathbb{R}^n)$ solves $Lu = f$ with $f \in L^p(\mathbb{R}^n)$, then automatically $Vu \in L^p(\mathbb{R}^n)$ (since $Au \in L^p(\mathbb{R}^n)$ by the boundedness of the coefficients); however, $u \in W^{2,p}(\mathbb{R}^n)$ does not imply in general that $Lu \in L^p(\mathbb{R}^n)$.

In order to prove (7), the first step is the following local result:

Theorem 2 *Under the assumptions (1–3), for any $p \in (1, q]$ there exist positive constants C, r such that for any $z_0 \in \mathbb{R}^n$, $u \in C_0^\infty(B_r(z_0))$*

$$\|Vu\|_{L^p(B_r(z_0))} \le C \|Lu\|_{L^p(B_r(z_0))}.$$

The constants C, r depend on n, p, q, the ellipticity constant μ, the VMO moduli of the leading coefficients, and the B_q constant of V.

We shall also use the following basic result proved in [3]:

Theorem 3 *Under the assumptions (1, 2), for any $p \in (1, \infty)$ there exist positive constants C, r such that for any $z_0 \in \mathbb{R}^n$, $u \in C_0^\infty(B_r(z_0))$*

$$\left\|D^2 u\right\|_{L^p(B_r(z_0))} \le C \|Au\|_{L^p(B_r(z_0))}.$$

The constants C, r depend on n, p, the ellipticity constant μ, and the VMO moduli of the leading coefficients.

Proof of Theorem 1 by Theorems 2 and 3 Let $\{\phi_i\}_{i=1}^\infty$ be a partition of unity of non negative functions in $\mathbb{R}^n$ such that $\phi_i \in C_0^\infty(B(z_i, r))$ with r as in Theorem 2 and such that the family of balls $B_i = B(z_i, r)$ has the finite overlapping property. Then, for any $u \in C_0^\infty(\mathbb{R}^n)$, since at any point the sum $\sum_i V\phi_i u$ has actually a finite and uniformly bounded number of terms, we can write:

$$\begin{aligned}
\|Vu\|_{L^p(\mathbb{R}^n)}^p &= \left\| \sum_i V\phi_i u \right\|_{L^p(\mathbb{R}^n)}^p \\
&\le C \sum_i \|V\phi_i u\|_{L^p(B(z_i,r))}^p \le C \sum_i \|L(\phi_i u)\|_{L^p(B(z_i,r))}^p \\
&\le C \sum_i \left\{ \|Lu\|_{L^p(B(z_i,r))}^p + \|Du\|_{L^p(B(z_i,r))}^p + \|u\|_{L^p(B(z_i,r))}^p \right\} \\
&\le C \left\{ \|Lu\|_{L^p(\mathbb{R}^n)}^p + \|Du\|_{L^p(\mathbb{R}^n)}^p + \|u\|_{L^p(\mathbb{R}^n)}^p \right\} \\
&\le C \left\{ \|Lu\|_{L^p(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \right\}^p.
\end{aligned} \tag{8}$$

Note that the finite overlapping property has also been used in the up to last inequality, to assure that $\sum_i \|Lu\|^p_{L^p(B(z_i,r))} \le C\|Lu\|^p_{L^p(\mathbb{R}^n)}$ etc.

Analogously, Theorem 3 implies

$$\begin{aligned}\left\|D^2u\right\|_{L^p(\mathbb{R}^n)} &\le C\left\{\|Au\|_{L^p(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}\right\}\\ &\le C\left\{\|Lu\|_{L^p(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}\right\}\end{aligned}$$

which, together with (8), gives

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} \le C\left\{\|Lu\|_{L^p(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}\right\}.$$

Then, the classical interpolation inequality (see e.g., [7, Thm. 7.27 p. 171])

$$\|Du\|_{L^p(\mathbb{R}^n)} \le \varepsilon\left\|D^2u\right\|_{L^p(\mathbb{R}^n)} + \frac{C}{\varepsilon}\|u\|_{L^p(\mathbb{R}^n)}$$

allows to write

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} \le C\left\{\|Lu\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}\right\}.$$

□

To prove Theorem 2, we pick a ball $B_r(z_0)$ with r to be chosen later, a point $x_0 \in B_r(z_0)$, and freeze the coefficients of A at x_0, getting the operator

$$L_0u = -a_{ij}(x_0)\,u_{x_ix_j} + V(x)\,u.$$

This operator can be rewritten in divergence form

$$L_0u = -\left(a_{ij}(x_0)\,u_{x_i}\right)_{x_j} + V(x)\,u,$$

which allows us to apply the results proved by Dziubanski [5] to deduce the following.

Theorem 4 (*See* [5, Proposition 4.9]) *The operator L_0 has a fundamental solution $\Gamma(x_0; x, y)$ satisfying the following bound: for any positive integer k there exists a constant c_k (independent of x_0) such that*

$$\Gamma(x_0; x, y) \le \frac{c_k}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^k}\cdot\frac{1}{|x-y|^{n-2}} \quad \textit{for any} \quad x, y \in \mathbb{R}^n, \quad x \ne y$$

where $\rho(x)$ is the "critical radius" associated to V, defined by:

$$\rho(x) = \sup\left\{r > 0 : \frac{r^2}{|B(x,r)|}\int_{B(x,r)} V(y)\,dy \le 1\right\}.$$

The function ρ has been introduced in [17]; it plays a central role also in [5] and [16]. Let us note that $\rho(x)$ is finite almost everywhere under our assumptions on V.

For any $u \in C_0^\infty(B_r(z_0))$, $x \in B_r(z_0)$, we can write:

$$\begin{aligned}u(x) &= \int \Gamma(x_0; x, y)\,L_0u(y)\,dy\\ &= \int \Gamma(x_0; x, y)\,Lu(y)\,dy + \int \Gamma(x_0; x, y)\left[A_0u(y) - Au(y)\right]dy.\end{aligned}$$

Letting $x_0 = x$, we get the representation formula:

$$u(x) = \int \Gamma(x; x, y)\, Lu(y)\, dy + \sum_{i,j=1}^{n} \int \Gamma(x; x, y) \left[a_{ij}(y) - a_{ij}(x)\right] u_{x_i x_j}(y)\, dy$$

which allows us to write the following pointwise bound, for every positive integer k:

$$\begin{aligned} |V(x)\, u(x)| &\leq c_k V(x) \int \frac{1}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^k} \cdot \frac{1}{|x-y|^{n-2}} \\ &\quad \cdot \left\{ |Lu(y)| + \sum_{i,j=1}^{n} \left|a_{ij}(y) - a_{ij}(x)\right| \left|u_{x_i x_j}(y)\right| \right\} dy \end{aligned}$$

for any $x \in \mathbb{R}^n$ and any positive integer k. Let us introduce the integral operators:

$$S_k f(x) = V(x) \int \frac{1}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^k} \cdot \frac{1}{|x-y|^{n-2}} f(y)\, dy$$
$$S_{k,a} f(x) = V(x) \int \frac{1}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^k} \cdot \frac{1}{|x-y|^{n-2}} |a(y) - a(x)|\, f(y)\, dy$$

for $a \in L^\infty \cap VMO(\mathbb{R}^n)$, so that our representation formula rewrites in compact form as

$$|Vu(x)| \leq c_k S_k(|Lu|)(x) + \sum_{i,j=1}^{n} S_{k,a_{ij}}\left(\left|u_{x_i x_j}\right|\right)(x). \tag{9}$$

We will prove that for any $p \in (1, q]$ and for k large enough

$$\|S_k f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \tag{10}$$

and that for any $\varepsilon > 0$ there exists r, depending on the VMO modulus of the function a, such that

$$\left\|S_{k,a} f\right\|_{L^p(B_r(z_0))} \leq \varepsilon \|f\|_{L^p(B_r(z_0))}. \tag{11}$$

Now, by (9–11) and Theorem 3 , for any $u \in C_0^\infty(B_r(z_0))$, r small enough, we have

$$\begin{aligned} \|Vu\|_{L^p} &\leq C \|Lu\|_{L^p} + \varepsilon \left\|u_{x_i x_j}\right\|_{L^p} \\ &\leq C \|Lu\|_{L^p} + C\varepsilon \|Au\|_{L^p} \leq (C + C\varepsilon) \|Lu\|_{L^p} + C\varepsilon \|Vu\|_{L^p} \end{aligned}$$

whence we get Theorem 2.

We now have to prove the L^p estimates (10, 11). Let us point out that inequality (10) has been basically proved by Shen (see proof of Theorem 3.1 in [16]). Nevertheless, we include here a shorter proof, based on the technique used for proving Theorem 4.3 in the same article. In order to do that, it is more convenient to consider the transposed operators:

$$S_k^* f(x) = \int \frac{V(y)}{\left(1 + \frac{|x-y|}{\rho(y)}\right)^k} \cdot \frac{1}{|x-y|^{n-2}} f(y)\, dy;$$
$$S_{k,a}^* f(x) = \int \frac{V(y)}{\left(1 + \frac{|x-y|}{\rho(y)}\right)^k} \cdot \frac{1}{|x-y|^{n-2}} |a(y) - a(x)|\, f(y)\, dy.$$

Theorem 5 *For k large enough, the operator* S_k^* *is continuous on* $L^p(\mathbb{R}^n)$ *for* $p \in [q', \infty]$ *(where* q' *is the conjugate exponent of* q*, and* $V \in B_q$*).*

Theorem 6 *For k large enough, the operator* $S_{k,a}^*$ *is continuous on* $L^p(\mathbb{R}^n)$ *for* $p \in [q', \infty)$ *and for any* $\varepsilon > 0$ *there exists* $r > 0$*, depending on the* VMO *modulus of* a*, such that*

$$\left\| S_{k,a}^* f \right\|_{L^p(B_r(z_0))} \leq \varepsilon \left\| f \right\|_{L^p(B_r(z_0))}.$$

By duality, the above two theorems imply (10, 11), and therefore, Theorem 2. The rest of this section will be devoted to the proof of Theorems 5, 6. We start with the following

Remark 7 In the study of the integral operators S_k^*, $S_{k,a}^*$ we may replace $\rho(y)$ by $\rho(x)$ in the kernel, whenever this is useful. The reason is that, in view of [16, Corollary 1.5], the following inequalities hold

$$C\left\{1 + \frac{|x-y|}{\rho(y)}\right\}^{1/k_0} \leq 1 + \frac{|x-y|}{\rho(x)} \leq C\left\{1 + \frac{|x-y|}{\rho(y)}\right\}^{k_0}$$

for some positive integer k_0, any $x, y \in \mathbb{R}^n$. Hence we can replace $\rho(y)$ with $\rho(x)$, possibly changing the integer k which appears in the kernel; nevertheless, since our aim is to prove L^p-boundedness for an integer k as large as we need, we keep calling it k.

Proof of Theorem 5 Since the kernel is positive, we can also assume $f \geq 0$. Also, we may assume $q > n/2$ because of the property $B_q \Rightarrow B_{q+\varepsilon}$ for some $\varepsilon > 0$.

We will prove the following pointwise bound

$$S_k^* f(x) \leq C\, M_{q'} f\ (x) \tag{12}$$

for any $x \in \mathbb{R}^n$, $f \in L^p(\mathbb{R}^n)$, $f \geq 0$, where $M_{q'} f$ is the maximal function of exponent q', i.e.,

$$M_{q'} f(x) = \sup_{B \ni x} \left(\frac{1}{|B|} \int_B f(y)^{q'}\, dy \right)^{1/q'}.$$

By the maximal inequality, for $p > q'$, (12) implies the theorem. When $p = q'$, we exploit again the fact that actually $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$, as already noted, so that (12) also holds with a smaller q'.

To prove (12), let us split

$$\begin{aligned} S_k^* f(x) &\leq C \int_{|x-y|<\rho(x)} \frac{V(y)}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^k} \cdot \frac{1}{|x-y|^{n-2}} f(y)\, dy \\ &\quad + C \int_{|x-y|\geq\rho(x)} \frac{V(y)}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^k} \cdot \frac{1}{|x-y|^{n-2}} f(y)\, dy \\ &\leq C \int_{|x-y|<\rho(x)} \frac{V(y)}{|x-y|^{n-2}} f(y)\, dy \\ &\quad + C \int_{|x-y|\geq\rho(x)} \left(\frac{\rho(x)}{|x-y|}\right)^k \frac{V(y)}{|x-y|^{n-2}} f(y)\, dy \\ &\equiv A(x) + B(x). \end{aligned}$$

Letting $B_j = B\left(x, 2^{-j}\rho(x)\right)$, we have (denoting by q' the conjugate exponent of q):

$$\begin{aligned}A(x) &\leq C\sum_{j=0}^{\infty}\frac{1}{\left(2^{-j}\rho(x)\right)^{n-2}}\int_{|x-y|\simeq 2^{-j}\rho(x)} V(y)\, f(y)\, dy\\ &\leq C\sum_{j=0}^{\infty}\left(2^{-j}\rho(x)\right)^{2}\left(\frac{1}{|B_j|}\int_{B_j} V(y)^q\, dy\right)^{1/q}\left(\frac{1}{|B_j|}\int_{B_j} f(y)^{q'}\, dy\right)^{1/q'}\\ &\leq CM_{q'}f(x)\sum_{j=0}^{\infty}\left(2^{-j}\rho(x)\right)^{2}\left(\frac{1}{|B_j|}\int_{B_j} V(y)\, dy\right),\end{aligned}$$

where in the last inequality we have applied the B_q condition on V. To show that the series appearing in the last expression is bounded by a constant, let us recall that, by [16, Lemma 1.2], we have

$$\frac{1}{r^n}\int_{B(x,r)} V(y)\, dy \leq C\left(\frac{R}{r}\right)^{\frac{n}{q}}\frac{1}{R^n}\int_{B(x,R)} V(y)\, dy \tag{13}$$

for any $0 < r < R < \infty$. Taking $R = \rho(x)$ and $r = 2^{-j}\rho(x)$ in (13), we get

$$\begin{aligned}A(x) &\leq CM_{q'}f(x)\sum_{j=0}^{\infty}\left(2^{-j}\rho(x)\right)^{2}\left(2^{j}\right)^{\frac{n}{q}}\left(\frac{1}{|B(x,\rho(x))|}\int_{B(x,\rho(x))} V(y)\, dy\right)\\ &\leq CM_{q'}f(x)\left(\frac{1}{\rho(x)^{n-2}}\int_{B(x,\rho(x))} V(y)\, dy\right)\sum_{j=0}^{\infty}\left(2^{-j}\right)^{2-\frac{n}{q}}\\ &\leq CM_{q'}f(x)\end{aligned}$$

since, by definition of ρ,

$$\frac{1}{\rho(x)^{n-2}}\int_{B(x,\rho(x))} V(y)\, dy \leq 1. \tag{14}$$

Analogously, letting $B_j = B\left(x, 2^{j}\rho(x)\right)$, we have:

$$\begin{aligned}B(x) &\leq C\sum_{j=0}^{\infty}\frac{2^{-jk}}{\left(2^{j}\rho(x)\right)^{n-2}}\int_{|x-y|\simeq 2^{j}\rho(x)} V(y)\, f(y)\, dy\\ &\leq C\sum_{j=0}^{\infty}\frac{\left(2^{j}\rho(x)\right)^{2}}{2^{jk}}\left(\frac{1}{|B_j|}\int_{B_j} V(y)^q\, dy\right)^{1/q}\left(\frac{1}{|B_j|}\int_{B_j} f(y)^{q'}\, dy\right)^{1/q'}\\ &\leq CM_{q'}f(x)\sum_{j=0}^{\infty}\frac{\left(2^{j}\rho(x)\right)^{2}}{2^{jk}}\left(\frac{1}{|B_j|}\int_{B_j} V(y)\, dy\right).\end{aligned}$$

Since $V(y)\,dy$ is doubling, for some positive constants α, C, and all j, we have

$$\int_{B_j} V(y)\,dy \le C2^{\alpha j} \int_{B(x,\rho(x))} V(y)\,dy, \tag{15}$$

so that

$$\begin{aligned} B(x) &\le CM_{q'}f(x) \sum_{j=0}^{\infty} \frac{\left(2^j \rho(x)\right)^2}{2^{jk}} \frac{2^{\alpha j}}{\left(2^j \rho(x)\right)^n} \int_{B(x,\rho(x))} V(y)\,dy \\ &= CM_{q'}f(x) \frac{1}{\rho(x)^{n-2}} \int_{B(x,\rho(x))} V(y)\,dy \sum_{j=0}^{\infty} \frac{1}{2^{j(k+n-\alpha-2)}} \\ &\le CM_{q'}f(x) \end{aligned}$$

where we used again (14) and we have chosen k large enough, to get $(k+n-\alpha-2)$ positive. This finishes the proof. □

Our next task is the proof of Theorem 6. It is convenient to settle this result in a suitably abstract framework. Namely, let

$$w(x,y) = \frac{V(y)}{\left(1+\frac{|x-y|}{\rho(y)}\right)^k} \cdot \frac{1}{|x-y|^{n-2}} \tag{16}$$

be the kernel of the integral operator S_k^* (note that now we have *not* replaced $\rho(y)$ by $\rho(x)$), so that

$$S_{k,a}^* f(x) = \int w(x,y)\,|a(y)-a(x)|\,f(y)\,dy.$$

We will deduce Theorem 6 from an abstract result. Before stating it, we need the following

Definition 8 We say that the kernel $W(x,y)$ satisfies "Hörmander's condition of order q" in the first variable, briefly $W \in H_1(q)$ if there exists a constant C such that for any $r>0$ and $x, x_0 \in \mathbb{R}^n$ such that $|x-x_0| \le r$, the following inequality holds:

$$\sum_{j=1}^{\infty} j\left(2^j r\right)^{n/q'} \left(\int_{2^j r \le |x_0-y| \le 2^{j+1} r} |W(x,y)-W(x_0,y)|^q\,dy\right)^{1/q} \le C.$$

Theorem 9 *Let $W(x,y)$ be a nonnegative kernel satisfying $H_1(q)$ for some $q>1$ and such that the integral operator*

$$Tf(x) = \int_{\mathbb{R}^n} W(x,y)\,f(y)\,dy$$

is continuous on $L^p(\mathbb{R}^n)$ for any $p \in (q', \infty)$. Then for $b \in BMO(\mathbb{R}^n)$ the operator ("positive commutator")

$$T_b f(x) = \int_{\mathbb{R}^n} |b(x)-b(y)|\,W(x,y)\,f(y)\,dy$$

is bounded on $L^p(\mathbb{R}^n)$ *for any* $p \in (q', \infty)$, *and*

$$\|T_b f\|_p \leq C \|b\|_{BMO} \|f\|_p$$

where $\|b\|_{BMO}$ *stands for the* BMO *seminorm.*

Remark 10 Condition $H_1(q)$ is weaker than the pointwise mean value inequality

$$|W(x, y) - W(x_0, y)| \leq C \frac{|x - x_0|}{|y - x_0|^{n+1}} \text{ for } |x - x_0| \leq r, \quad |y - x_0| \geq 2r, \tag{17}$$

and is stronger than the standard (integral) Hörmander's inequality

$$\int_{|x_0 - y| \geq 2|x_0 - x|} |W(x, y) - W(x_0, y)| \, dy \leq C.$$

We will show that our kernel actually satisfies $H_1(q)$, which is enough to get the desired result, while we cannot expect our kernel to satisfy the pointwise condition.

Remark 11 Conditions of the type $H_1(q)$ has been implicitly introduced in [10] in the context of weighted norm inequalities for multipliers; see also [11–13], among others. Commutator-type operators like $S_{k,a} f$ (that is, where the modulus $|b(x) - b(y)|$ appears inside the integrals) have been studied, in several contexts, under stronger smoothness conditions of the type (17), see [1] and [15].

Proof of Theorem 9 We start noting that the mere existence of the integral defining $T_b f$ is not obvious for $b \in BMO$. However, we can first assume that $b \in L^\infty(\mathbb{R}^n)$, which makes clear the existence of $T_b f \in L^p(\mathbb{R}^n)$ for any $f \in L^p(\mathbb{R}^n)$, and prove the theorem under this assumption, and then remove it by a standard truncation argument. Therefore, in the following, we will think $b \in L^\infty(\mathbb{R}^n)$, although our bounds will depend quantitatively on b only through its BMO seminorm.

We will prove the following pointwise inequality: for any $s > q'$, there exists a constant C such that

$$(T_b f)^{\#}(z) \leqslant C \|b\|_{BMO} [M_s(Tf)(z) + (M_s f)(z)], \tag{18}$$

with C independent of b and f, where

$$g^{\#}(z) = \sup_{B \ni z} \frac{1}{|B|} \int_B |g(x) - g_B| \, dx$$

is the sharp maximal function. This, by Fefferman-Stein's inequality (see e.g., [18, Thorem 2 p. 148]) together with the maximal theorem, will imply our result as soon as $T_b f \in L^p$, since we may always choose an appropriate s for a given $p > q'$.

To prove (18), let $B = B(x_0, r)$ be a ball such that $z \in B$. Let $f = f_1 + f_2$ with $f_1 = f\chi_{\widetilde{B}}$, $\widetilde{B} = 2B$, $f \geq 0$. For any $x \in B$,

$$\begin{aligned} |T_b f(x) - c_B| &= |T_b f_1(x) + T_b f_2(x) - c_B| \\ &\leqslant |T_b f_1(x)| + |T_b f_2(x) - c_B| = I + II. \end{aligned}$$

For the first term, we have

$$\begin{aligned} I &= \left| \int |b(x) - b(y)| \, W(x, y) \, f_1(y) \, dy \right| \\ &\leqslant |b(x) - b_B| \, T f_1(x) + T(|b - b_B| \, f_1)(x). \end{aligned}$$

Choose $c_B = \int |b(y) - b_B| W(x_0, y) f_2(y) \, dy$. Then,

$$
\begin{aligned}
II &= |T_b f_2(x) - c_B| \\
&= \left| T_b f_2(x) - \int |b(y) - b_B| W(x_0, y) f_2(y) \, dy \right| \\
&\leqslant \int ||b(x) - b(y)| W(x, y) - |b(y) - b_B| W(x_0, y)| f_2(y) \, dy \\
&\leqslant \int ||b(x) - b(y)| - |b(y) - b_B|| W(x, y) f_2(y) \, dy \\
&\quad + \int |b(y) - b_B| |W(x, y) - W(x_0, y)| f_2(y) \, dy \\
&\leqslant \int |b(x) - b_B| W(x, y) f_2(y) \, dy \\
&\quad + \int |b(y) - b_B| |W(x, y) - W(x_0, y)| f_2(y) \, dy \\
&= |b(x) - b_B| T f_2(x) + \int |b(y) - b_B| |W(x, y) - W(x_0, y)| f_2(y) \, dy
\end{aligned}
$$

so that

$$
\begin{aligned}
|T_b f(x) - c_B| \le\ & |b(x) - b_B| T f(x) + T(|b - b_B| f_1)(x) \\
& + \int |b(y) - b_B| |W(x, y) - W(x_0, y)| f_2(y) \, dy \\
\equiv\ & A(x) + B(x) + C(x).
\end{aligned}
$$

From this point, the proof follows similarly to that appearing in [8], but we include the details for the sake of completeness.

For the first term, by John-Nirenberg inequality, we get

$$
\begin{aligned}
\frac{1}{|B|} \int_B A(x) \, dx &= \frac{1}{|B|} \int_B |b(x) - b_B| T f(x) \, dx \\
&\leqslant \left(\frac{1}{|B|} \int |b(x) - b_B|^{s'} \right)^{1/s'} \left(\frac{1}{|B|} \int_B |Tf|^s(x) \, dx \right)^{1/s} \\
&\leqslant \|b\|_{BMO} M_s(Tf)(z).
\end{aligned}
$$

Next, we choose γ such that $s > \gamma > q'$. Then,

$$
\begin{aligned}
&\frac{1}{|B|} \int_B B(x) \, dx \\
&= \frac{1}{|B|} \int_B T(|b - b_B| f_1)(x) \, dx \\
&\leqslant \left(\frac{1}{|B|} \int_B T(|b - b_B| f_1)^{\gamma}(x) \, dx \right)^{1/\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\left(\frac{1}{|B|}\int_{\widetilde{B}}|b-b_B|^{\gamma}\,|f_1(x)|^{\gamma}\,\mathrm{d}x\right)^{1/\gamma}\\
&\leqslant C\left(\frac{1}{|B|}\int_{\widetilde{B}}|f(x)|^{s}\,\mathrm{d}x\right)^{1/s}\left(\frac{1}{|B|}\int_{\widetilde{B}}|b-b_B|^{\gamma\left(\frac{s}{\gamma}\right)'}\,\mathrm{d}x\right)^{1/\gamma\left(\frac{s}{\gamma}\right)'}\\
&\leqslant C\left(\frac{1}{|\widetilde{B}|}\int_{\widetilde{B}}|f(x)|^{s}\,\mathrm{d}x\right)^{1/s}\left\{\left(\frac{1}{|\widetilde{B}|}\int_{\widetilde{B}}\left|b-b_{\widetilde{B}}\right|^{\gamma\left(\frac{s}{\gamma}\right)'}\,\mathrm{d}x\right)^{1/\gamma\left(\frac{s}{\gamma}\right)'}+\left|b_B-b_{\widetilde{B}}\right|\right\}\\
&\leqslant C\,\|b\|_{BMO}\,M_s(f)(z),
\end{aligned}
$$

where in the last inequality we have used also $\left|b_B-b_{\widetilde{B}}\right|\le C\,\|b\|_{BMO}$.

Finally, if we now choose γ such that $1/\gamma+1/q+1/s=1$, which is possible since $s>q'$, we have for any $x\in B$

$$
\begin{aligned}
C(x) &= \int_{|x_0-y|\geq 2r}|b(y)-b_B|\,|W(x,y)-W(x_0,y)|\,f(y)\,\mathrm{d}y\\
&=\sum_{j=2}^{+\infty}\int_{|x_0-y|\approx 2^j r}|b(y)-b_B|\,|W(x,y)-W(x_0,y)|\,f(y)\,\mathrm{d}y\\
&\leqslant C\sum_{j=2}^{+\infty}\left(2^j r\right)^n\left(\frac{1}{|B_{2^j r}|}\int_{B_{2^j r}}|b(y)-b_B|^{\gamma}\,\mathrm{d}y\right)^{1/\gamma}\\
&\quad\times\left(\frac{1}{|B_{2^j r}|}\int_{|x_0-y|\approx 2^j r}|W(x,y)-W(x_0,y)|^{q}\,\mathrm{d}y\right)^{1/q}\left(\frac{1}{|B_{2^j r}|}\int_{|x_0-y|\approx 2^j r}|f(y)|^{s}\,\mathrm{d}y\right)^{1/s}\\
&\le C\sum_{j=2}^{+\infty}\left(2^j r\right)^n\left\{\left(\frac{1}{|B_{2^j r}|}\int_{B_{2^j r}}\left|b(y)-b_{B_{2^j r}}\right|^{\gamma}\,\mathrm{d}y\right)^{1/\gamma}+\left|b_B-b_{B_{2^j r}}\right|\right\}\\
&\quad\times\left(\frac{1}{|B_{2^j r}|}\int_{|x_0-y|\approx 2^j r}|W(x,y)-W(x_0,y)|^{q}\,\mathrm{d}y\right)^{1/q}M_s f(z)\\
&\leqslant C\,\|b\|_{BMO}\,M_s f(z)\sum_{j=2}^{+\infty}\left(2^j r\right)^{n/q'}j\left(\int_{|x_0-y|\approx 2^j r}|W(x,y)-W(x_0,y)|^{q}\,\mathrm{d}y\right)^{1/q}\\
&\leqslant C\,\|b\|_{BMO}\,M_s f(z)\,,
\end{aligned}
$$

where in the last inequality we have applied condition $H_1(q)$ and in the up to last inequality we have used

$$
\left|b_B-b_{B_{2^j r}}\right|\le Cj\,\|b\|_{BMO}\,.
$$

□

Next, we have to show that our kernel $w(x, y)$ appearing in (16) actually satisfies condition $H_1(q)$. This fact is contained in the following

Proposition 12 *The kernel $w(x, y)$ in (16) satisfies condition $H_1(q)$.*

Proof Because of the property $B_q \Rightarrow B_{q+\varepsilon}$ for some $\varepsilon > 0$, we may assume $q > n/2$. Let x, y, x_0 be such that $|x - x_0| \le r$ and $|y - x_0| \ge 2r$, so that in particular $|y - x_0| \simeq |y - x|$. Then

$$|w(x, y) - w(x_0, y)| \le c_k V(y) \left\{ \frac{1}{\left(1 + \frac{|x_0 - y|}{\rho(y)}\right)^k} \left| \frac{1}{|x - y|^{n-2}} - \frac{1}{|x_0 - y|^{n-2}} \right| + \frac{1}{|x - y|^{n-2}} \left| \frac{1}{\left(1 + \frac{|x-y|}{\rho(y)}\right)^k} - \frac{1}{\left(1 + \frac{|x_0-y|}{\rho(y)}\right)^k} \right| \right\} \equiv A + B.$$

$$A \le \frac{c_k V(y)}{\left(1 + \frac{|x_0 - y|}{\rho(y)}\right)^k} \cdot \frac{|x - x_0|}{|x_0 - y|^{n-1}};$$

$$B \le \frac{c_k V(y)}{|x - y|^{n-2}} \cdot \frac{|x - x_0|}{\rho(y)} \frac{1}{\left(1 + \frac{|x_0 - y|}{\rho(y)}\right)^{k+1}},$$

where in the last inequality we have exploited the bound

$$\left| \frac{1}{(1 + bt)^k} - \frac{1}{(1 + bt_0)^k} \right| \le \frac{kb}{(1 + b\bar{t})^{k+1}} |t - t_0|$$

for some $\bar{t} \in [t_0, t]$. Now,

$$\left(\int_{2^j r \le |x_0 - y| \le 2^{j+1} r} |w(x, y) - w(x_0, y)|^q \, dy \right)^{1/q}$$

$$\le \left(\int_{2^j r \le |x_0 - y| \le 2^{j+1} r} A^q dy \right)^{1/q} + \left(\int_{2^j r \le |x_0 - y| \le 2^{j+1} r} B^q dy \right)^{1/q}. \tag{19}$$

We start handling the first term, exploiting again the possibility of replacing $\rho(y)$ with $\rho(x)$ (leaving understood the possible change of the integer k).

$$\left(\int_{2^j r \le |x_0 - y| \le 2^{j+1} r} A^q dy \right)^{1/q} \le \frac{c_k}{\left(1 + \frac{2^j r}{\rho(x)}\right)^k} \cdot \frac{r}{\left(2^j r\right)^{n-1}} \left(\int_{2^j r \le |x_0 - y| \le 2^{j+1} r} V(y)^q \, dy \right)^{1/q}$$

by the reverse Hölder property of V

$$\le \frac{c_k}{\left(1 + \frac{2^j r}{\rho(x)}\right)^k} \cdot \frac{r}{(2^j r)^{n-1}} \left(2^j r\right)^{\frac{n}{q} - n} \int_{|x_0 - y| \le 2^{j+1} r} V(y) \, dy.$$

Next, to check condition $H_1(q)$ on the first term A, we add up in j as follows:

$$\sum_j j\left(2^j r\right)^{\frac{n}{q'}}\left(\int_{2^j r\leq|x_0-y|\leq 2^{j+1}r} A^q\,\mathrm{d}y\right)^{1/q} \tag{20}$$

$$\leq \sum_j j\frac{c_k}{\left(1+\frac{2^j r}{\rho(x)}\right)^k}\cdot\frac{r}{\left(2^j r\right)^{n-1}}\int_{|x_0-y|\leq 2^{j+1}r} V(y)\,\mathrm{d}y$$

$$=\sum_{j:2^j r<\rho(x)}(\ldots)+\sum_{j:2^j r\geq\rho(x)}(\ldots)\equiv A_I+A_{II}.$$

$$A_I\leq c_k\sum_{j:2^j r<\rho(x)}\frac{j}{2^j}\frac{1}{\left(2^j r\right)^{n-2}}\int_{|x_0-y|\leq 2^{j+1}r} V(y)\,\mathrm{d}y$$

applying (13) with r, R replaced by $2^j r$, $\rho(x)$, respectively

$$\leq c_k\sum_{j:2^j r<\rho(x)}\frac{j}{2^j}\left(\frac{\rho(x)}{2^j r}\right)^{\frac{n}{q}-2}\frac{1}{\rho(x)^{n-2}}\int_{|x_0-y|\leq\rho(x)} V(y)\,\mathrm{d}y$$

by definition of $\rho(x)$

$$\leq c_k\sum_{j:2^j r<\rho(x)}\frac{j}{2^j}\left(\frac{\rho(x)}{2^j r}\right)^{\frac{n}{q}-2}$$

since $2^j r<\rho(x)$ and $q>n/2$

$$\leq c_k\sum_{j:2^j r<\rho(x)}\frac{j}{2^j}\leq c_k.$$

On the other hand,

$$A_{II}\leq c_k\sum_{j:2^j r\geq\rho(x)}\frac{j}{2^j}\left(\frac{\rho(x)}{2^j r}\right)^k\cdot\frac{1}{\left(2^j r\right)^{n-2}}\int_{|x_0-y|\leq 2^{j+1}r} V(y)\,\mathrm{d}y$$

by the doubling condition on $V(y)\,\mathrm{d}y$

$$\leq c_k\sum_{j:2^j r\geq\rho(x)}\frac{j}{2^j}\left(\frac{\rho(x)}{2^j r}\right)^k\cdot\frac{1}{\left(2^j r\right)^{n-2}}\left(\frac{2^j r}{\rho(x)}\right)^{\alpha}\int_{|x_0-y|\leq\rho(x)} V(y)\,\mathrm{d}y$$

by definition of $\rho(x)$ and taking k large enough

$$\leq c_k\sum_{j:2^j r\geq\rho(x)}\frac{j}{2^j}\left(\frac{\rho(x)}{2^j r}\right)^k\cdot\frac{1}{\left(2^j r\right)^{n-2}}\left(\frac{2^j r}{\rho(x)}\right)^{\alpha}\rho(x)^{n-2}$$

$$=c_k\sum_{j:2^j r\geq\rho(x)}\frac{j}{2^j}\left(\frac{\rho(x)}{2^j r}\right)^{k+n-2-\alpha}$$

$$\leq c_k\sum_{j:2^j r\geq\rho(x)}\frac{j}{2^j}=c_k.$$

This shows that the left-hand side in (20) is bounded. To check the analogous condition on the term B, we use again the reverse Hölder assumption

$$\sum_j j\left(2^j r\right)^{\frac{n}{q'}}\left(\int\limits_{2^j r\le|x_0-y|\le 2^{j+1}r} B^q \,\mathrm{d}y\right)^{1/q}$$

$$\le \sum_j j\left(2^j r\right)^{\frac{n}{q'}} \frac{c_k}{\left(2^j r\right)^{n-2}}\cdot\frac{r}{\rho(x)}\frac{1}{\left(1+\frac{2^j r}{\rho(x)}\right)^{k+1}}\left(\int\limits_{2^j r\le|x_0-y|\le 2^{j+1}r} V(y)^q \,\mathrm{d}y\right)^{1/q}$$

$$\le \sum_j j\frac{c_k}{\left(2^j r\right)^{n-2}}\cdot\frac{r}{\rho(x)}\frac{1}{\left(1+\frac{2^j r}{\rho(x)}\right)^{k+1}}\int\limits_{|x_0-y|\le 2^{j+1}r} V(y)\,\mathrm{d}y$$

$$= \sum_{j:2^j r<\rho(x)}(\ldots)+\sum_{j:2^j r\ge\rho(x)}(\ldots)\equiv B_I+B_{II}.$$

$$B_I \le \sum_{j:2^j r<\rho(x)} j\frac{c_k}{\left(2^j r\right)^{n-2}}\cdot\frac{r}{\rho(x)}\int\limits_{|x_0-y|\le 2^{j+1}r} V(y)\,\mathrm{d}y$$

$$\le \sum_{j:2^j r<\rho(x)}\frac{j}{2^j}\frac{c_k}{\left(2^j r\right)^{n-2}}\int\limits_{|x_0-y|\le 2^{j+1}r} V(y)\,\mathrm{d}y$$

and from now on the estimate is the same as for the term A_I.

$$B_{II} \le \sum_{j:2^j r<\rho(x)} j\frac{c_k}{\left(2^j r\right)^{n-2}}\cdot\frac{r}{\rho(x)}\left(\frac{\rho(x)}{2^j r}\right)^{k+1}\int\limits_{|x_0-y|\le 2^{j+1}r} V(y)\,\mathrm{d}y$$

$$\le \sum_{j:2^j r<\rho(x)}\frac{j}{2^j}\frac{c_k}{\left(2^j r\right)^{n-2}}\left(\frac{\rho(x)}{2^j r}\right)^{k}\int\limits_{|x_0-y|\le 2^{j+1}r} V(y)\,\mathrm{d}y$$

and from now on the estimate is the same as for the term A_{II}.

This ends the proof. □

Now we are in position to complete the proof of Theorem 6.

Proof of Theorem 6 By Theorem 9 and Proposition 12 we get that for k large enough

$$\left\|S^*_{k,a}f\right\|_{L^p(\mathbb{R}^n)} \le C\,\|a\|_{BMO}\,\|f\|_{L^p(\mathbb{R}^n)} \tag{21}$$

for any $p\in\left(q',\infty\right)$; when $p=q'$ we still exploit the fact that actually $V\in B_{q+\varepsilon}$. In order to deduce Theorem 6 from this bound, one can apply the same localization argument firstly used in [3]. □

In this way, we finish the proof of Theorem 2, and therefore, Theorem 1 is completely proved.

We end this section pointing out some local estimates and regularity results, which follow from the previous ones. Besides being interesting in their own, these facts will be used in the next section to prove global existence theorems.

Theorem 13 (Local estimates) *Under the assumptions* (1–3)*, for any* $p \in (1, q]$ *the following local estimate holds, for any* $r > 0$ *and* $u \in W_V^{2,p}(B_r)$*:*

$$\|u\|_{W_V^{2,p}(B_{r/2})} \leq C \left\{ \|Lu\|_{L^p(B_r)} + \|u\|_{L^p(B_r)} \right\} \tag{22}$$

with C *only depending on the quantities involved in the assumptions.*

The proof of the previous result is essentially contained in the proof of Theorem 1, combined with standard techniques involving cutoff functions (see e.g., [7, §9.5 pp. 235–237]).

Let now Ω be any bounded domain. Since assumption (3) implies that $V \in L^q(\Omega)$ for some $q > n/2$, the following existence result can be applied to our situation:

Theorem 14 see [20] *Let* Ω *be any bounded domain and* $V \in L^q(\Omega)$ *for some* $q > n/2$*, then for any* $p \in (1, q]\, f \in L^p(\Omega)$*, the problem*

$$\begin{cases} Lu = f \text{ in } \Omega \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \end{cases}$$

has a unique solution.

Combining this existence result with the local estimate (22), one can also prove in a standard way (see for instance [9, Thm 3 p. 237]) the following regularity result:

Theorem 15 *Under the assumptions* (1–3)*, for any* $p \in (1, q]$ *there exists a constant* C *such that, for any* $r > 0$ *and* $u \in W_V^{2,p}(B_r)$*,*

$$\|u\|_{W_V^{2,q}(B_{r/2})} \leq C \left\{ \|Lu\|_{L^q(B_r)} + \|u\|_{L^p(B_r)} \right\}. \tag{23}$$

Remark 16 We want to stress again the fact that the constants C in (22) and (23) depend on V only through the constant involved in the reverse Hölder condition B_q. On the other hand, the existence result expressed by Theorem 14 is proved by means of an a priori estimate of the kind

$$\|u\|_{W_V^{2,q}(\Omega)} \leq C \|Lu\|_{L^q(\Omega)} \text{ for } u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), \tag{24}$$

where C depends also on $\|V\|_{L^q(\Omega)}$. We want to avoid any use of such quantitative dependence, so in the proof of (23) we have applied this existence result in a purely qualitative way, relying on the quantitative estimate (22).

3 Existence and uniqueness results

The aim of this section is to prove the following:

Theorem 17 *Assume that the operator* L *satisfies assumptions* (1–3) *and* (5)*. Then for any* $p \in (1, q]$ *there exists a constant* C *depending on* n*,* p*, and the quantities involved in the assumptions such that for every* $\lambda \geq 0$ *and* $u \in W_V^{2,p}(\mathbb{R}^n)$ *the following estimate holds:*

$$\|u\|_{W_V^{2,p}(\mathbb{R}^n)} \leq C \|Lu + \lambda u\|_{L^p(\mathbb{R}^n)}. \tag{25}$$

Moreover, for every $f \in L^p(\mathbb{R}^n)$ *there exists a unique* $u \in W_V^{2,p}(\mathbb{R}^n)$ *such that* $Lu + \lambda u = f$*.*

Our main task will be the proof of (25). Namely, applying the classical method of continuity (see e.g., [7, §5.2, pp. 74–75]) to the operators L and

$$L_0 = -\Delta + V + \lambda,$$

which is solvable in virtue of the results proved by [16], we can see that as soon as we know that (25) holds for some $p \in (1, q]$ and some λ, with a constant C depending only on the quantities specified above, existence and uniqueness for L follow for those p and λ.

We start with the following.

Remark 18 If $V \in B_q$ and λ is any positive constant, then $V + \lambda \in B_q$ with the same B_q constant. Then, since the constant in Theorem 1 depends on V only through its B_q constant, we immediately get the following:

$$\|u\|_{W_V^{2,p}(\mathbb{R}^n)} \leq C \left\{ \|Lu + \lambda u\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \right\} \tag{26}$$

for any $u \in C_0^\infty(\mathbb{R}^n)$ and $1 < p \leq q$.

Our next task is to remove the term $\|u\|_{L^p(\mathbb{R}^n)}$ on the right-hand side of (26), first when λ is large, and then for any $\lambda \geq 0$, provided V is bounded away from zero.

Theorem 19 *Under assumptions (1–3), there exist $\lambda_0, C > 0$ such that for any $p \in (1, q], \lambda \geq \lambda_0, u \in C_0^\infty(\mathbb{R}^n)$ the following estimate holds:*

$$\|u\|_{W_V^{2,p}(\mathbb{R}^n)} \leq C \|Lu + \lambda u\|_{L^p(\mathbb{R}^n)}.$$

Proof Let $\phi \in C_0^\infty(-1, 1)$ (not identically zero). We want to apply the global bound contained in Theorem 1 to the operator in $\mathbb{R}^{n+1}$

$$\widetilde{L} = L - \partial_{tt}^2$$

and the complex valued function

$$\widetilde{u}(x, t) = u(x) e^{i\sqrt{\lambda}t} \phi(t)$$

(where u is real valued) with $\lambda \geq 1$ to be chosen later. This technique is usually referred to as "Agmon's idea". We note that, being the operator L linear and with real valued coefficients, Theorem 1 can also be applied to complex functions. Then, a careful computation shows that the inequality

$$\left\|D^2\widetilde{u}\right\|_{L^p(\mathbb{R}^{n+1})} + \|D\widetilde{u}\|_{L^p(\mathbb{R}^{n+1})} + \|V\widetilde{u}\|_{L^p(\mathbb{R}^{n+1})} \leq C \left\{ \left\|\widetilde{L}\widetilde{u}\right\|_{L^p(\mathbb{R}^{n+1})} + \|\widetilde{u}\|_{L^p(\mathbb{R}^{n+1})} \right\}$$

gives

$$\begin{aligned} &\left\|D^2 u\right\|_{L^p(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} + \lambda \|u\|_{L^p(\mathbb{R}^n)} \\ &\leq C \left\{ \|Lu + \lambda u\|_{L^p(\mathbb{R}^n)} + \left(\sqrt{\lambda} + 1\right) \|u\|_{L^p(\mathbb{R}^n)} \right\}. \end{aligned}$$

Taking $C\left(\sqrt{\lambda} + 1\right) \leq \lambda/2$, that is $\lambda \geq \lambda_0$ large enough, yields

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} \leq C \|Lu + \lambda u\|_{L^p(\mathbb{R}^n)}$$

with C independent of λ. □

Next, we want to relax the condition $\lambda \geq \lambda_0$ appearing in Theorem 19 to $\lambda \geq 0$. In view of (26), it is enough to prove the following:

Proposition 20 *Under assumptions* (1–3, 5)*, for every* $p \in (1, q]$ *there exists a constant* C *only depending on the quantities involved in the assumptions such that*

$$\|u\|_{L^p(\mathbb{R}^n)} \leq C \, \|Lu + \lambda u\|_{L^p(\mathbb{R}^n)} \tag{27}$$

for any $\lambda \geq 0$, $u \in C_0^\infty(\mathbb{R}^n)$.

We start noting that it is enough to prove the estimate (27) under the additional (qualitative) assumption that a_{ij} and V are C^∞, but with a constant C only depending on the quantities involved in the assumptions. This is an easy consequence of the following:

Lemma 21 *Let*

$$a_{ij}^{(k)}, \quad V_k$$

be smooth functions obtained by a_{ij}, V *with the usual mollifiers; then* $a_{ij}^{(k)}$, V_k *satisfy conditions* (1–3, 5) *with constants independent of* k.

Proof For (1) and (5) this is trivial. For (2), this assertion depends on known properties of VMO functions (see [4, Thm.2.1]); for (3), let us write

$$\left(\int_{B_r(x_0)} V_k(x)^q \, dx \right)^{1/q} = \sup_{\substack{f \in C_0^\infty(B_r(x_0)) \\ \|f\|_{q'} \leq 1}} \left| \int V_k(x) f(x) \, dx \right|.$$

Then

$$\begin{aligned}
\left| \int V_k(x) f(x) \, dx \right| &= \left| \int f(x) \int V(x-y) \varphi_{1/k}(y) \, dy dx \right| \\
&= \left| \int \varphi_{1/k}(y) \, dy \int_{B_r(x_0)} V(x-y) f(x) \, dx \right| \\
&\leq \int \varphi_{1/k}(y) \, dy \, \|f\|_{q'} \, \|V(\cdot - y)\|_{L^q(B_r(x_0))} \\
&\leq \int \varphi_{1/k}(y) \, dy \, \|V\|_{L^q(B_r(x_0 - y))}
\end{aligned}$$

by the reverse Hölder condition on V

$$\begin{aligned}
&\leq \int \varphi_{1/k}(y) \, dy \, |B_r|^{1/q} \, C \left(\frac{1}{|B_r|} \int_{B_r(x_0-y)} V(x) \, dx \right) \\
&= C \, |B_r|^{1/q} \left(\frac{1}{|B_r|} \int_{B_r(x_0)} \left(\int V(x-y) \varphi_{1/k}(y) \, dy \right) dx \right) \\
&= C \, |B_r|^{1/q} \left(\frac{1}{|B_r|} \int_{B_r(x_0)} V_k(x) \, dx \right)
\end{aligned}$$

hence

$$\left(\frac{1}{|B_r|}\int_{B_r(x_0)} V_k(x)^q \,dx\right)^{1/q} \leq C\frac{1}{|B_r|}\int_{B_r(x_0)} V_k(x)\,dx$$

and we are done. □

Now, in order to prove (27) under the additional assumption of smooth coefficients and potential, we follow the line of Krylov [9]. (The following sentences, as well as the next Lemma and its proof, should be read keeping Krylov' book at hand).

Let us say that the property $A(\mu)$ holds if for any $\lambda \geq \mu$ and $p \in (1, q]$, there exists $C > 0$ such that:

$$\|u\|_{W_V^{2,p}(\mathbb{R}^n)} \leq C\,\|Lu + \lambda u\|_{L^p(\mathbb{R}^n)} \quad \text{for any } u \in C_0^\infty\left(\mathbb{R}^n\right). \tag{28}$$

Krylov' strategy consists in showing that whenever $A(\mu)$ holds for some μ, then the constant C in (28) is actually independent of μ. This, in turn, allows one to prove that $A(\mu) \Rightarrow A(\mu - \varepsilon)$ for some positive ε; an iterative argument then gives $A(0)$, which is (27).

By Theorem 19, we already know that $A(\lambda_0)$ holds for some $\lambda_0 > 0$; this in particular implies the solvability of $Lu + \lambda u = f \in L^p(\mathbb{R}^n)$ for any $\lambda \geq \lambda_0$, a fact that is used in Krylov' argument. Reading carefully the proof of [9, Thm. 2 p. 251], one can see that to get (27) in our context, it is enough to prove, under our assumptions, the following result:

Lemma 22 (*See* [9, Lemma 1 p. 249]). *Take a constant $\lambda \geq 0$ and bounded infinitely differentiable functions u and f with f vanishing outside $B_1(0)$. Assume further that the coefficients of L are infinitely differentiable and that $V \geq 1$. Finally, suppose that*

$$Lu + \lambda u = f.$$

Then there exists a constant $\gamma > 0$ depending only on the dimension n and the ellipticity constant in (1) *such that for any $p \in (1, q]$ and some constant C, depending only on p and the quantities involved in* (1–2) *the following inequality holds*

$$\|u/v\|_{L^p(\mathbb{R}^n)} \leq C\,\|f\|_{L^p(\mathbb{R}^n)}$$

where $v(x) = \exp(-\gamma\,|x|)$.

We stress the fact that, different from what is done in [9], we need to prove the above Lemma with a constant C which does not depend on any L^p_{loc} norm of V.

Proof of the Lemma Let h be the classical solution to

$$\begin{cases} Lh + \lambda h = 0 & \text{in } B_4(0) \\ h = u & \text{on } \partial B_4(0). \end{cases}$$

Hence $w = h - u$ is infinitely differentiable in $\overline{B}_4$, vanishes on $\partial B_4(0)$ and satisfies $Lw + \lambda w = -f$. The arguments on [9, p. 250] can be repeated in order to show that it is enough to prove

$$|w(x)| \leq C\,\|Lu + \lambda u\|_{L^p(\mathbb{R}^n)} \quad \text{for } |x| = 2. \tag{29}$$

Here, we just point out that the estimate

$$\|w\|_{L^p(B_4)} \leq C\,\|f\|_{L^p(\mathbb{R}^n)},$$

used in the argument of [9, p. 250], holds with a constant C independent of V, as one can see comparing w with the solution to $Ag = |f|$ in B_4, $g = 0$ on ∂B_4.

By the maximum principle (for smooth functions, see [9, Thm.3 p. 233]), it is enough to prove (29) for $\lambda = 0$; to do that, let us define ψ as the unique solution to

$$\begin{cases} L\psi = |f| & \text{in } B_4(0) \\ \psi \in W^{2,q}(B_4) \cap W_0^{1,q}(B_4). \end{cases}$$

This solution exists in view of Theorem 14, since $V \in L^q(B_4)$ with $q > n/2$. As in [9, p. 251], to prove (29) it is enough to show that

$$|\psi(x)| \leq C \|L\psi\|_{L^p(\mathbb{R}^n)} \quad \text{for } |x| = 2.$$

To do this, take a point x_0 with $|x_0| = 2$ and observe that by embedding theorems we have, since $q > n/2$,

$$\begin{aligned} |\psi(x_0)| &\leq C \|\psi\|_{W^{2,q}(B_{1/2}(x_0))} \\ &\leq C \left\{ \|L\psi\|_{L^q(B_1(x_0))} + \|\psi\|_{L^p(B_1(x_0))} \right\} \end{aligned}$$

where we have applied our estimate (23). Since f vanishes outside $B_1(0)$, the first term in the right-hand side vanishes. Moreover,

$$\|\psi\|_{L^p(B_1(0))} \leq C \|f\|_{L^p(B_1(0))}$$

with C independent of V, again by the maximum principle for smooth functions. This finishes the proof. □

We are now interested in relaxing the assumption $V(x) \geq \delta > 0$. As noted in the introduction, this kind of restriction cannot be completely removed. However, we can prove the following:

Theorem 23 *Assume that the operator L satisfies assumptions (1–3) and (6). Let K be a constant such that*

$$\|V\|_{L^1(B_R)} \leq K \tag{30}$$

where R is as in (6). Then for any $p \in (1, q]$ there exists a constant C depending on n, p, the quantities involved in the assumptions and K such that for every $\lambda \geq 0$ and $u \in W_V^{2,p}(\mathbb{R}^n)$ the following estimate holds:

$$\|u\|_{W_V^{2,p}(\mathbb{R}^n)} \leq C \|Lu + \lambda u\|_{L^p(\mathbb{R}^n)}.$$

Moreover, for every $f \in L^p(\mathbb{R}^n)$ there exists a unique $u \in W_V^{2,p}(\mathbb{R}^n)$ such that $Lu+\lambda u = f$.

To prove the above theorem, we will need the following standard result:

Proposition 24 (Maximum principle, see [9, Thm 1 p. 262]) *Let Ω be a C^2 bounded domain of $\mathbb{R}^n$, let L satisfy assumptions (1, 2), $V \geq 0$, $V \in L^p(\Omega)$ for some $p > 1$. If $u \in W^{2,p}(\Omega)$, $Lu \leq 0$ in Ω and $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in Ω.*

We note that, although in [9] this maximum principle is stated for operators with continuous coefficients, the same proof holds under the VMO assumption, in virtue of Theorem 14 and (24). Also, the same maximum principle holds *a fortiori* for $L + \lambda$ with $\lambda \geq 0$.

We also need the following refinement of Theorem 17:

Proposition 25 *Let L satisfy the assumptions of Theorem 17. Then for any $p_1, p_2 \in (1, q]$, $f \in L^{p_1} \cap L^{p_2}(\mathbb{R}^n)$, $\lambda \geq 0$, the solution u to $Lu + \lambda u = f$ exists in $W_V^{2,p_1}(\mathbb{R}^n) \cap W_V^{2,p_2}(\mathbb{R}^n)$ and is unique in $W_V^{2,p_1}(\mathbb{R}^n) \cup W_V^{2,p_2}(\mathbb{R}^n)$. In particular, if $u \in W_V^{2,p_1}(\mathbb{R}^n)$ solves the equation $Lu + \lambda u = f$ with $f \in L^{p_1} \cap L^{p_2}(\mathbb{R}^n)$, then $u \in W_V^{2,p_2}(\mathbb{R}^n)$.*

Proof Let $p_1, p_2 \in (1, q]$, $f \in L^{p_1} \cap L^{p_2}(\mathbb{R}^n)$, $\lambda \geq 0$; applying the method of continuity as in the proof of Theorem 17, one can easily prove existence and uniqueness of the solution to $Lu + \lambda u = f$ in $W_V^{2,p_1}(\mathbb{R}^n) \cap W_V^{2,p_2}(\mathbb{R}^n)$. It remains to prove uniqueness in $W_V^{2,p_1}(\mathbb{R}^n) \cup W_V^{2,p_2}(\mathbb{R}^n)$. Assume that $u_i \in W_V^{2,p_i}(\mathbb{R}^n)$ (for $i = 1, 2$) solve $Lu = f$ with $f \in L^{p_1} \cap L^{p_2}(\mathbb{R}^n)$ and let us prove that $u_1 = u_2$. Let $w \in W_V^{2,p_1}(\mathbb{R}^n) \cap W_V^{2,p_2}(\mathbb{R}^n)$ be the unique solution to $Lw + \lambda w = f$. Since in $W^{2,p_1}(\mathbb{R}^n)$ we have uniqueness, $w = u_1$; similarly, in $W_V^{2,p_2}(\mathbb{R}^n)$ we have uniqueness, so $w = u_2$, and we are done. □

Proof of Theorem 23 Under the assumption (6), let

$$V_1(x) = \max(V(x), \delta).$$

Clearly, V_1 satisfies (3), with a constant bounded by twice the constant of V. Moreover, V_1 satisfies (5) and $V = V_1 - V_0$ where

$$0 \leq V_0(x) \leq \delta$$
$$V_0 \text{ supported in } B_R(0).$$

Applying Theorem 17 to $A + V_1$ we have, for every $u \in W_V^{2,p}(\mathbb{R}^n)$,

$$\begin{aligned} \|u\|_{W_V^{2,p}(\mathbb{R}^n)} &\leqslant C\, \|Au + V_1 u + \lambda u\|_{L^p(\mathbb{R}^n)} \\ &\leq C \left\{ \|Lu + \lambda u\|_{L^p(\mathbb{R}^n)} + \|V_0 u\|_{L^p(\mathbb{R}^n)} \right\} \\ &\leq C \left\{ \|Lu + \lambda u\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(B_R)} \right\}. \end{aligned} \tag{31}$$

The fact that the L^p norm of u in the right-hand side of (31) is taken over B_R instead of $\mathbb{R}^n$ will be crucial in the following.

We are going to prove that for any operator L satisfying our assumptions, there exists a constant C, only depending on the quantities involved in the assumptions, such that

$$\|u\|_{W_V^{2,p}(\mathbb{R}^n)} \leqslant C\, \|Lu + \lambda u\|_{L^p(\mathbb{R}^n)} \tag{32}$$

for every $u \in W_V^{2,p}(\mathbb{R}^n)$. It is enough to prove this for every $u \in C_0^\infty(\mathbb{R}^n)$. Assume that this is false. This means that for every k there exists $u_k \in C_0^\infty(\mathbb{R}^n)$, coefficients $a_{ij}^{(k)}$ and a potential V_k satisfying (1–3, 6) and $\|V_k\|_{L^1(B_R)} \leq K$ such that

$$\|u_k\|_{W_{V_k}^{2,p}(\mathbb{R}^n)} > k\, \|L_k u_k + \lambda u_k\|_{L^p(\mathbb{R}^n)} \tag{33}$$

and

$$\|u_k\|_{W_{V_k}^{2,p}(\mathbb{R}^n)} \leqslant C \left\{ \|L_k u_k + \lambda u_k\|_{L^p(\mathbb{R}^n)} + \|u_k\|_{L^p(B_R)} \right\}. \tag{34}$$

If $\|u_k\|_{L^p(B_R)} = 0$ for infinitely many k's, then (33) and (34) imply that $\|u_k\|_{W_V^{2,p}(\mathbb{R}^n)} = 0$ for some k, which contradicts (33). Passing to a subsequence, we can therefore, assume that $\|u_k\|_{L^p(B_R)} \neq 0$ for all k and, by normalization, we can assume

$$\|u_k\|_{L^p(B_R)} = 1 \quad \text{for all } k. \tag{35}$$

This gives

$$(k - C) \| L_k u_k + \lambda u_k \|_{L^p(\mathbb{R}^n)} \leq C,$$

hence $L_k u_k + \lambda u_k \to 0$ in $L^p(\mathbb{R}^n)$. Also, this means by (34) that $\|u_k\|_{W^{2,p}(\mathbb{R}^n)}$ is bounded; hence, there exists $u \in W^{2,p}(\mathbb{R}^n)$ and a subsequence, which we keep calling u_k, such that $u_k \to u$ weakly in $W^{2,p}(\mathbb{R}^n)$ and strongly in $L^s(\Omega)$ for $s \in [1, np/(n-2p))$ (if $p < n/2$) or $s \in [1, +\infty)$ (if $p \geq n/2$) and Ω bounded. In particular,

$$\|u\|_{L^p(B_R)} = 1. \tag{36}$$

On the other hand, by an argument in [4, p. 852], the sequence $a_{ij}^{(k)}$ has a subsequence converging a.e. to a_{ij}, satisfying assumptions (1, 2) with the same constants of $a_{ij}^{(k)}$. As to the sequence V_k, by (3) and (6), we can write

$$\left(\frac{1}{|B_{2^h R}|} \int_{B_{2^h R}} V_k^q \mathrm{d}x \right)^{1/q} \leq C \frac{1}{|B_{2^h R}|} \int_{B_{2^h R}} V_k \mathrm{d}x$$

and by the doubling condition on $V_k(x)\,\mathrm{d}x$

$$\leq \frac{C}{|B_{2^h R}|} C^h \int_{B_R} V_k \mathrm{d}x \leq \frac{C}{|B_{2^h R}|} C^h K.$$

This shows that for any bounded domain Ω, the sequence V_k is bounded in $L^q(\Omega)$ and has a subsequence weakly convergent to some V in L^q_{loc}. Let us show that V satisfies assumptions (3) and (6) with the same constants of V_k. Let $\phi \in C_0^\infty(\mathbb{R}^n \setminus B_R)$, $\phi \geq 0$ then

$$\int \delta\phi \leq \int V_k \phi \to \int V\phi$$

which implies (6) for V. Moreover, for any fixed ball B, the weak convergence in $L^q(B)$ gives

$$\|V\|_{L^q(B)} \leq \liminf \|V_k\|_{L^q(B)} \leq C\,|B|^{\frac{1}{q}-1} \liminf \|V_k\|_{L^1(B)} = |B|^{\frac{1}{q}-1} \|V\|_{L^1(B)}.$$

Hence, V satisfies also (3).

Now, we will show that

$$\int (V_k u_k - Vu)\,\phi \to 0$$

for every $\phi \in C_0^\infty(\mathbb{R}^n)$. First of all, observe that since $q > n/2$, by the Sobolev imbedding theorem, u_k and u are in $L^{q'}(\mathbb{R}^n)$. Since $V_k, V \in L^q_{loc}$, this shows that $V_k u_k$ and Vu are locally integrable. Then

$$\int (V_k u_k - Vu)\,\phi = \int V_k (u_k - u)\,\phi + \int u\,(V_k - V)\,\phi \equiv A_I + A_{II}.$$

Now,

$$|A_I| \leq \|V_k\|_{L^q(B)} \|(u_k - u)\,\phi\|_{L^{q'}(B)} \to 0$$

because $\|V_k\|_{L^q(B)}$ is bounded while $\|(u_k - u)\,\phi\|_{L^{q'}(B)} \to 0$ by the strong convergence of u_k on bounded sets.

Next, we observe that $A_{II} \to 0$ because $V_k \to V$ weakly in L^q and $u\phi \in L^{q'}$.

Let now $L_k w = -a_{ij}^{(k)} w_{x_i x_j} + V_k w$. We want to show that for any $\phi \in C_0^\infty(\mathbb{R}^n)$

$$\int (L_k u_k + \lambda u_k)\,\phi \to \int (Lu + \lambda u)\,\phi, \tag{37}$$

which will imply $Lu + \lambda u = 0$ a.e. in $\mathbb{R}^n$.

First of all,

$$\begin{aligned}&\int \left[a_{ij}^{(k)} (u_k)_{x_i x_j} - a_{ij} u_{x_i x_j}\right]\phi \\ &= \int \left[a_{ij}^{(k)} - a_{ij}\right](u_k)_{x_i x_j}\,\phi + \int a_{ij}\left[(u_k)_{x_i x_j} - u_{x_i x_j}\right]\phi \equiv B_I + B_{II}.\end{aligned}$$

Now:

$$|B_I| \le \left\|(u_k)_{x_i x_j}\right\|_{L^p} \left\|\left[a_{ij}^{(k)} - a_{ij}\right]\phi\right\|_{L^{p'}} \to 0$$

because $\left\|(u_k)_{x_i x_j}\right\|_{L^p}$ is bounded and $\left\|\left[a_{ij}^{(k)} - a_{ij}\right]\phi\right\|_{L^{p'}} \to 0$ by Lebesgue's theorem, since $\left|\left[a_{ij}^{(k)} - a_{ij}\right]\phi\right| \le C\,|\phi|$ and $\left[a_{ij}^{(k)} - a_{ij}\right] \to 0$ a.e. On the other hand,

$$B_{II} \to 0$$

because $a_{ij}\phi \in L^{p'}(\mathbb{R}^n)$ and $(u_k)_{x_i x_j} \to u_{x_i x_j}$ weakly in L^p.

This completes the proof of (37) and allows to conclude that

$$Lu + \lambda u = 0 \text{ a.e.}$$

We can then apply a bootstrap argument and (31) to show that $u \in W^{2,p}(\mathbb{R}^n)$ with $p > \frac{n}{2}$. Namely: let us rewrite our equation as

$$Au + V_1 u + \lambda u = V_0 u.$$

Since V_0 is bounded and $u \in W^{2,p}(\mathbb{R}^n) \subset L^{p_2^*}(\mathbb{R}^n)$ $(p_2^* = np/(n-2p))$ by Proposition 25 $u \in W^{2,p_2^*}(\mathbb{R}^n)$. If $p_2^* \ge q$ we have finished; otherwise we can iterate the same reasoning, finding after a finite number of steps that $u \in W^{2,p}(\mathbb{R}^n)$ with $p > \frac{n}{2}$. Therefore, by the Sobolev imbedding theorem, u is continuous and

$$\lim_{|x| \to +\infty} u(x) = 0$$

(the last assertion depending on the fact that, being $W^{2,p}(\mathbb{R}^n) = W_0^{2,p}(\mathbb{R}^n)$ continuously embedded in $L^\infty(\mathbb{R}^n)$, u is a uniform limit of $C_0^\infty(\mathbb{R}^n)$ functions).

Let now $m = \sup_{\mathbb{R}^n} u(x)$ and suppose, by way of contradiction, the $m > 0$. Let R_0 sufficiently large so that $|u(x)| \leqslant \frac{m}{2}$ for $|x| > \frac{R_0}{2}$. Let $w(x) = u(x) - \frac{m}{2}$, then $Lw + \lambda w = -\frac{m}{2}(V + \lambda) \leqslant 0$. Applying Proposition 24 to the function w in B_{R_0} shows that $w(x) \leqslant 0$; hence $u(x) \leqslant \frac{m}{2}$ for $x \in B_{R_0}$ and therefore, in the whole $\mathbb{R}^n$, which is a contradiction. It follows that $m = 0$. The same argument applied to $-u$ shows that $u \equiv 0$. This contradicts (36) and so finishes the proof of (32). As in the proof of Theorem 17, this a priori bound and the method of continuity give the desired result. □

Acknowledgments This research was carried out while the authors were visiting the Department of Mathematics of Universidad Autonoma of Madrid, Spain. We wish to thank Josè Luis Torrea for his kind invitation and his warm hospitality.

References

1. Bramanti, M.: Commutators of integral operators with positive kernels. Matematiche (Catania) **49** (1994)(1), 149–168 (1995)
2. Chen, Y.-Z., Wu, L.-C.: Second order elliptic equations and elliptic systems. Transl. Math. Monogr. **174**, A.M.S. (1991)
3. Chiarenza, F., Frasca, M., Longo, P.: Interior $W^{2,p}$-estimates for nondivergence elliptic equations with discontinuous coefficients. Ricerche di Mat. XL, 149–168 (1991)
4. Chiarenza, F., Frasca, M., Longo, P.: $W^{2,p}$-solvability of the Dirichlet problem for non divergence elliptic equations with VMO coefficients. Trans. Am. Math. Soc. **336**(1), 841–853 (1993)
5. Dziubanski, J.: Note on H^1 spaces related to degenerate Schrödinger operators. Ill. J. Math. **49**(4), 1271–1297 (2005)
6. Gehring, F.W.: The L^p-integrability of the partial derivatives of a quasiconformal mapping. Acta Math. **130**, 265–277 (1973)
7. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer Series "Classics in Mathematics" (1976)
8. Guo, Z., Li, P., Peng, L.: L^p boundedness of commutators of Riesz transforms associated to Schrödinger operator. J. Math. Anal. Appl. **341**, 421–432 (2008)
9. Krylov, N.V.: Lectures on Elliptic and Parabolic Equations in Sobolev Spaces. Graduate Studies in Mathematics, A.M.S. (2008)
10. Kurtz, D.S., Wheeden, R.L.: Results on weighted norm inequalities for multipliers. Trans. Am. Math. Soc. **255**, 343–362 (1979)
11. Lorente, M., Martell, J.M., Riveros, M.S., de la Torre, A.: Generalized Hörmander's conditions, commutators and weights. J. Math. Anal. Appl. **342**, 1399–1425 (2008)
12. Martell, J.M., Pérez, C., Trujillo-González, R.: Lack of natural weighted estimates for some singular integral operators. Trans. Amer. Math. Soc. **357**(1), 385–396 (2005)
13. Rubio de Francia, J.L., Ruiz, F.J., Torrea, J.L.: Calderón-Zygmund theory for vector-valued functions. Adv. Math. **62**, 7–48 (1986)
14. Sarason, D.: Functions of vanishing mean oscillation. Trans. Amer. Math. Soc. **207**, 391–405 (1975)
15. Segovia, C., Torrea, J.L.: Vector-valued commutators and applications. Indiana Univ. Math. J. **38**(4), 959–971 (1989)
16. Shen, Z.: L^p estimates for Schrodinger's operators with certain potentials. Ann. Inst. Fourier t. **45**(2), 513–546 (1995)
17. Shen, Z.: On the Neumann problem for Schrödinger operators in Lipschitz domains. Indiana Univ. Math. J. **43**(1), 143–176 (1994)
18. Stein, E.M.: Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals. Princeton Univ. Press, Princeton (1993)
19. Thangavelu, S.: Riesz transforms and the wave equation for the Hermite operator. Comm. Partial Differ. Equ. **15**(8), 1199–1215 (1990)
20. Vitanza, C.: A new contribution to the $W^{2,p}$ regularity for a class of elliptic second order equations with discontinuous coefficients. Le Matematiche (Catania) **48**(1993)(2), 287–296 (1994)