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ERRATUM

Erratum to the paper "Global L^p estimates for degenerate Ornstein–Uhlenbeck operators with variable coefficients"

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Abstract
In this note we point out and correct a mistake in our paper "*Global L^p estimates for degenerate Ornstein–Uhlenbeck operators with variable coefficients*", published in Math. Nachr. **286** (2013), no. 11–12, 1087–1101.

KEYWORDS
global L^p estimates, hypoelliptic operators, nondoubling spaces, Ornstein–Uhlenbeck operators, singular integrals

MSC (2020)
35B45, 35H10, 35K70, 42B20

In this note we want to point out and correct a mistake in our paper [1], where we consider a class of Ornstein–Uhlenbeck operators on $\mathbb{R}^N$

$$\mathcal{A} = \sum_{i,j=1}^{p_0} a_{ij}(x)\partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j},$$

together with the corresponding Kolmogorov–Fokker–Planck operators on $\mathbb{R}^{N+1}$

$$L = \sum_{i,j=1}^{p_0} a_{ij}(z)\partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t$$

(here $z = (x,t)$). We will not recall here the structural assumptions on the matrix $B = \left(b_{ij}\right)_{i,j=1}^N$. The matrix $A_0 = \left(a_{ij}(x)\right)_{i,j=1}^{p_0}$ is a $p_0 \times p_0$ $(p_0 \le N)$ symmetric, bounded and uniformly positive definite matrix:

$$\frac{1}{\Lambda}|\xi|^2 \le \sum_{i,j=1}^{p_0} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \tag{1}$$

for all $\xi \in \mathbb{R}^{p_0}$, $x \in \mathbb{R}^N$ (or $z \in \mathbb{R}^{N+1}$) and for some constant $\Lambda \ge 1$. The entries a_{ij} are assumed to satisfy a continuity condition which will be clarified in a moment. The main result in [1] is the following:

Theorem 1 (See [1, Thm. 1.1]). *For every $p \in (1, \infty)$ there exists a constant $c > 0$, depending on p, N, p_0, the matrix B, the number Λ in (1) and the continuity modulus ω,*

$$\omega(r) = \max_{i,j=1,\dots,p_0} \sup_{\substack{x,y\in\mathbb{R}^N \\ |x-y|\le r}} |a_{ij}(x) - a_{ij}(y)|, \tag{2}$$

such that for every $u \in C_0^\infty(\mathbb{R}^N)$ one has:

$$\sum_{i,j=1}^{p_0} \left\| \partial^2_{x_i x_j} u \right\|_{L^p(\mathbb{R}^N)} \le c\left\{ \|\mathcal{A}u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)} \right\},$$

$$\left\| \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u \right\|_{L^p(\mathbb{R}^N)} \le c\left\{ \|\mathcal{A}u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)} \right\}.$$

This result is derived from an analogous L^p estimate holding for L on a strip $S_T = \mathbb{R}^N \times [-T, T]$:

Theorem 2 (See [1, Thm. 3.1]). *Let L be as above, with uniformly continuous coefficients a_{ij}. For every $p \in (1, \infty)$ there exist constants $c, T > 0$ depending on p, N, p_0, the matrix B, the number Λ in (1), c also depending on the modulus of continuity ω*

$$\omega(r) = \max_{i,j=1,\dots,p_0} \sup_{\substack{z_1,z_2\in\mathbb{R}^{N+1} \\ |z_1-z_2|\le r}} |a_{ij}(z_1) - a_{ij}(z_2)|, \tag{3}$$

such that

$$\sum_{i,j=1}^{p_0} \left\| \partial^2_{x_i x_j} u \right\|_{L^p(S_T)} \le c\left\{ \|Lu\|_{L^p(S_T)} + \|u\|_{L^p(S_T)} \right\}$$

for every $u \in C_0^\infty(S_T)$.

Now, in the statement of the above theorem we assumed the coefficients a_{ij} to be uniformly continuous in S_T *in Euclidean sense*. However, the assumption that we actually use in the proof of Proposition 3.2 (which implies the above theorem), is the global uniform continuity of the coefficients *with respect to the local quasidistance d* in $\mathbb{R}^{N+1}$ which is introduced in the paper. Although the topology induced by d coincides with the Euclidean topology, so that continuity with respect to the two structures is the same thing, global uniform continuity is a different issue. In particular, global uniform continuity in Euclidean sense does not imply global uniform continuity with respect to d. Accordingly, also the assumption in the statement of [1, Thm. 1.1] must be corrected. This means that the coefficients $a_{ij}(x)$, which now are defined in $\mathbb{R}^N$, must be uniformly continuous with respect to d if they are regarded as defined in a strip S_T. Let us make precise the above corrections.

The definition (3) of the continuity modulus $\omega(r)$ (when the coefficients are defined in the strip S_T) must be changed to:

$$\omega_{S_T}(r) = \max_{i,j=1,\dots,p_0} \left\{ \sup\left|a_{ij}(z) - a_{ij}(\zeta)\right| : d(z, \zeta) < r, z, \zeta \in S_T \right\}.$$

The definition (2) of the continuity modulus $\omega(r)$ (when the coefficients are defined in $\mathbb{R}^N$) must be changed to:

$$\omega_{\mathbb{R}^N}(r) = \max_{i,j=1,\dots,p_0} \left\{ \sup\left|a_{ij}(x) - a_{ij}(y)\right| : \inf_{|s|<T, |t|<T} d((x,t),(y,s)) < r, x, y \in \mathbb{R}^N \right\} \tag{4}$$

(where $T > 0$ must be small enough). The global estimates proved in [1, Thm. 1.1, Thm. 3.1] hold, with the same proof, with the constant depending on these moduli.

We end with a remark and an example that should better enlighten the relation between uniform continuity in the two senses.

Remark 3. In the case of time-independent coefficients, let us compare global uniform continuity in Euclidean sense with global uniform continuity w.r.t. $\omega_{\mathbb{R}^N}(r)$ in (4). Since

$$\inf_{|s|<T,|t|<T} d((x,t),(y,s)) \leq d((x,0),(y,0)) = \sum_{i=1}^{N} \left|(x-y)_i\right|^{1/q_i}.$$

(where q_i are the positive integers defined in [1, p. 1091]), for $r < 1$ we have

$$|x-y| < r \Rightarrow \inf_{|s|<T,|t|<T} d((x,t),(y,s)) < N r^{1/q_N}$$

so that

$$\max_{i,j=1,\dots,p_0} \left\{ \sup\left|a_{ij}(x) - a_{ij}(y)\right| \,:\, |x-y| < r \right\} \leq \omega_{\mathbb{R}^N}\left(N r^{1/q_N}\right).$$

This shows that global uniform continuity w.r.t. $\omega_{\mathbb{R}^N}(t)$ implies global uniform continuity in Euclidean sense.

Example 4. Let us consider the simplest example of degenerate KFP operator,

$$\mathcal{L}u = u_{x_1x_1} + x_1 u_{x_2} - u_t.$$

We have (keeping the notation in [1, p. 1089])

$$E(s) = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}$$

and, letting $y = (y_1, y_2)$,

$$E(s)y = \left(y_1, y_2 - s y_1\right).$$

Choosing, for $\varepsilon > 0$,

$$x = \left(\frac{1}{\varepsilon}, 1\right); \;\; y = \left(\frac{1}{\varepsilon}, 2\right); \;\; s = 0; \;\; t = \varepsilon$$

we have $(y,s) - (x,t) = (0, 1, -\varepsilon)$, so that $|(y,s) - (x,t)| \to 1$ as $\varepsilon \to 0$. On the other hand,

$$\begin{aligned} (y,s)^{-1} \circ (x,t) &= (x - E(t-s)y, t-s) = (x - E(\varepsilon)y, \varepsilon) \\ &= \left(\left(\frac{1}{\varepsilon}, 1\right) - \left(\frac{1}{\varepsilon}, 2 - \varepsilon\frac{1}{\varepsilon}\right), \varepsilon \right) = (0,0,\varepsilon) \end{aligned}$$

$$\left\| (y,s)^{-1} \circ (x,t) \right\| = \|(0,0,\varepsilon)\| = \sqrt{\varepsilon} \to 0.$$

This shows that, if x, y are free to move in the whole $\mathbb{R}^N$,

$$\left\| (y,s)^{-1} \circ (x,t) \right\| \to 0 \text{ does not imply } |(y,s) - (x,t)| \to 0.$$

For instance, let

$$a(x_1, x_2, t) = \sin\left(\frac{\pi}{2} x_2\right),$$

then a is uniformly continuous in $\mathbb{R}^{N+1}$, in Euclidean sense. Nevertheless, for (x_1, x_2, t) and (y_1, y_2, s) as above we have

$$d\big((x_1, x_2, t), (y_1, y_2, s)\big) = \sqrt{\varepsilon} \to 0$$

but

$$|a(x_1, x_2, t) - a(y_1, y_2, s)| = \left|\sin\left(\frac{\pi}{2}x_2\right) - \sin\left(\frac{\pi}{2}y_2\right)\right| = \left|\sin\left(\frac{\pi}{2}\right) - \sin\pi\right| = 1,$$

so that the function a is not uniformly continuous in any strip S_T, w.r.t. d.

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REFERENCE

[1] M. Bramanti, G. Cupini, E. Lanconelli, and E. Priola, *Global* L^p *estimates for degenerate Ornstein–Uhlenbeck operators with variable coefficients*, Math. Nachr. **286** (2013), no. 11–12, 1087–1101.