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$W_p^{1,2}$ SOLVABILITY FOR THE CAUCHY-DIRICHLET PROBLEM FOR PARABOLIC EQUATIONS WITH VMO COEFFICIENTS

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§0. Introduction

Let L be a linear parabolic operator of the form

$$Lu = u_t - a_{ij}(x)u_{x_i' x_j'} \tag{0.1}$$

where $x = (x', t) = (x_1', ..., x_n', t) \in \mathbf{R}^{n+1}$. The principal part of the operator is symmetric and uniformly elliptic, i.e.

$$a_{ij}(x) = a_{ji}(x) \quad \text{and} \quad \lambda^{-1}|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2 \tag{0.2}$$

for some $\lambda \ge 1$ and for every $\xi \in \mathbf{R}^n$, a.e. $x \in Q_T = \Omega \times (0,T)$ ($\Omega \subset \mathbf{R}^n$ a bounded $C^{1,1}$ domain). We are interested in solutions to the Cauchy-Dirichlet problem

$$\begin{cases} Lu = f & \text{in } Q_T \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x', 0) = 0 & \text{in } \Omega \end{cases} \tag{0.3}$$

with $f \in L^p(Q_T)$, $1 < p < +\infty$.

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When the coefficients $a_{ij}(x)$ are at least uniformly continuous existence and uniqueness results together with a-priori $W_p^{1,2}(Q_T)$ estimates are well known (see e.g. [**14**]), while there is no general theory for operators with discontinuous coefficients.

Here we consider operators with coefficients in the Sarason's class VMO, i.e. the closure in the (parabolic) BMO seminorm of uniformly continuous functions (see §1 for a precise definition). Therefore we allow for some discontinuities in the coefficients. We prove the same results as in the uniformly continuous case, i.e. we first prove a-priori interior and boundary $W_p^{1,2}$ estimates and then, through the usual procedure, we obtain existence and uniqueness for a solution to (0.3). We would like to point out that our results contain those in [**11**], [**12**] and [**2**] as a particular case.

The first step (§1) is to derive explicit representation formulas, both in the interior and at the boundary, for the derivatives of a solution (see thms. 1.4 and 1.5). The formulas we obtain are somewhat different from the usual ones as the singular integrals involved are more closely related to classical Calderòn-Zygmund operators. Moreover they involve "variable kernels". In §2 we prove L^p estimates for the singular integral operators involved in the interior representation formula, by reducing estimates on "variable kernels" to estimates on "constant kernels" (via an expansion in spherical harmonics, a technique dating back to Calderòn and Zygmund in [**4**]). In §3 we derive analogous estimates for the operators involved in the boundary representation formula which need to be handled in a different way.

Finally in §4 we derive interior and boundary estimates in $W_p^{1,2}(Q_T)$ for every $p \in (1, \infty)$, and get existence and uniqueness for the Cauchy-Dirichlet problem (0.3). The constants involved depend on coefficients only through the ellipticity constant λ and the VMO moduli.

The analogous elliptic theory was developped by Chiarenza, Frasca and Longo in [**5**] and [**6**] using a similar general technique.

§1. Fundamental solutions and representation formulas

Throughout this paper we will use x, y, ... to indicate points in $\mathbf{R}^{n+1}$, and x', y', ... for points in $\mathbf{R}^n$ corresponding to the first n coordinates, i.e. we will write $x = (x', t) = (x'_1, ..., x'_n, t)$.

Let's now endow $\mathbf{R}^{n+1}$ with the following parabolic metric, first introduced by Fabes and Rivière in [**9**]

$$d(x, y) = \rho(x - y) \quad \text{where} \quad \rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + 4t^2}}{2}}.$$

The ball of radius ρ and center 0 in this metric is the ellipsoid $E_\rho(0)$ defined by

$$\frac{|x'|^2}{\rho^2} + \frac{t^2}{\rho^4} = 1, \tag{1.1}$$

so that the unit ball coincides with the euclidean unit ball. Moreover the following hold:

i) $\rho(x) < 1 \Longrightarrow |x| \leq \rho(x)$

ii) $\rho(x) > 1 \Longrightarrow |x| \geq \rho(x)$

iii) $\rho(x', 0) = |x'|$

iv) $\rho^2(0, t) = |t|$.

We will use the following parabolic polar change of coordinates:

$$\begin{aligned} x'_1 &= \rho \cos\phi_1 \ldots \cos\phi_n \\ x'_2 &= \rho \cos\phi_1 \ldots \sin\phi_n \\ \ldots \\ x'_n &= \rho \cos\phi_1 \sin\phi_2 \\ t &= \rho^2 \sin\phi_1 \end{aligned} \tag{1.2}$$

and $dx = \rho^{n+1} J(\phi_1, ..., \phi_n) d\rho\, d\phi_1 ... d\phi_n = \rho^{n+1} d\rho\, d\sigma$, where $d\sigma$ is the element of area of $\Sigma_{n+1} \equiv \{x \in \mathbf{R}^{n+1}, |x| = 1\}$. Note that $1 \leq J(\phi_1, ..., \phi_n) \leq M$ and $J(\phi_1, ..., \phi_n) \in C^\infty([0, 2\pi)^{n-1} \times (0, \pi))$.

We define parabolic cubes of center $x = (x', t) \in \mathbf{R}^{n+1}$ and radius r by $I = I_r(x) = \{y = (y', s) \in \mathbf{R}^{n+1} : |x' - y'| < r, |t - s| < r^2\}$. Also by αI we'll mean the cube having the same center as I and radius αr. Note that for given parabolic ball E_ρ there exist two cubes I' and I'' with measures comparable to ρ^{n+2} and with $I' \subset E_\rho \subset I''$.

Let's finally recall the definitions and some properties of parabolic BMO and VMO spaces. We say that $f \in L^1_{loc}$ is in the space $BMO(\mathbf{R}^{n+1})$ if the BMO seminorm

$$\|f\|_* = \sup_I \frac{1}{|I|} \int_I |f(x) - f_I|\, dx$$

is finite, where I ranges over the class of parabolic cubes in $\mathbf{R}^{n+1}$ and $f_I = \frac{1}{|I|} \int_I f(x)\, dx$.

For $f \in BMO$ and $r > 0$ define the VMO modulus of f

$$\eta_f(r) = \sup_{\rho \le r} \frac{1}{|I_\rho|} \int_{I_\rho} |f(x) - f_{I_\rho}| \, dx$$

where I_ρ ranges over the class of parabolic cubes in $\mathbf{R}^{n+1}$ of radius ρ. We say that $f \in BMO$ is in the space $VMO(\mathbf{R}^{n+1})$ if $\eta_f(r) \to 0$ as $r \to 0$.

The following theorem summarizes some of the most useful properties of the VMO space (see [**16**]).

Theorem 1.1 *For $f \in BMO$ the following conditions are equivalent:*

(i) $f \in VMO$;

(ii) f is in the BMO closure of the set of uniformly continuous functions which belong to BMO;

(iii) $\lim_{y \to 0} \|f(x-y) - f(x)\|_* = 0$.

Observe that *(iii)* implies, in particular, that usual mollifiers converge to f in BMO seminorm.

Moreover define the spaces $BMO(Q_T)$ and $VMO(Q_T)$ by taking $I \cap Q_T$ in place of I, in the definitions of $\|f\|_*$ and $\eta_f(r)$. The following theorem is a consequence of results in [**1**].

Theorem 1.2 *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain and $Q_T = \Omega \times (0,T)$. Then for every function $f \in BMO(Q_T)$ there exists $Ef \in BMO(\mathbf{R}^{n+1})$ such that*

$$Ef(x) = f(x), \; \forall x \in Q_T$$

and

$$\|Ef\|_* \le c\|f\|_{BMO(Q_T)}.$$

Moreover if $f \in VMO(Q_T)$ then $Ef \in VMO(\mathbf{R}^{n+1})$ and

$$\eta_{Ef}(r) \le c_1 \eta_{f,Q_T}(c_2 r), \; \forall r > 0,$$

with constants independent of f.

Finally we need to introduce some notation and function spaces. Let

$\mathbf{R}^{n+1}_{+} = \mathbf{R}^{n+1} \cap \{x'_n \geq 0\}$
$\mathbf{R}^{n+1}_{-} = \mathbf{R}^{n+1} \cap \{x'_n \leq 0\}$
$\mathcal{C}_t = \{u \in C_0^\infty(\mathcal{A}) \colon u(x',0) = 0, \text{ with } \mathcal{A} = \mathbf{R}^{n+1} \cap \{t \geq 0\}\}$
$\mathcal{C}_{t,x'} = \{u \in C_0^\infty(\mathcal{B}) : u(x) = 0 \text{ for } t = 0 \vee x'_n = 0, \text{ with } \mathcal{B} = \mathcal{A} \cap \mathbf{R}^{n+1}_{+}\}.$

Note that being $\mathcal{A}$ and $\mathcal{B}$ closed, functions in the above spaces do not need to have derivatives vanishing at the boundary.

Let's now turn to equation (0.1). Throughout the paper the coefficients $a_{ij}(x)$ will satisfy (0.2) a.e. in a smooth cylinder $Q_T \subset \mathbf{R}^{n+1}$ and will belong to $VMO \cap L^\infty(Q_T)$. In view of theorem 1.2 we can assume, without loss of generality, that the coefficients belong to $VMO \cap L^\infty(\mathbf{R}^{n+1})$. Moreover, since functions in VMO are defined a.e., we assume $a_{ij}(x) = \delta_{ij}$ where they were not previously defined.

Now, for fixed $x_0 \in Q_T$, consider the constant coefficients operator

$$L_0 u(x) = u_t(x) - a_{ij}(x_0) u_{x'_i x'_j}(x)$$

obtained by L freezing the coefficients at x_0. Its fundamental solution in $\mathbf{R}^{n+1}$ is

$$\Gamma^0(x) = \Gamma(x_0, x) = \begin{cases} (4\pi t)^{-n/2}\, D_0^{-1/2} \exp\big(-\frac{1}{4t}\, A_0^{ij} x'_i x'_j\big) & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

where D_0 is the determinant of the matrix $\{a_{ij}(x_0)\}$ of the coefficients of L_0 and the A_0^{ij}'s are the elements of the inverse of the same matrix. (See [**14**]). If $u \in C_t$, then

$$u(x) = \int_{\mathbf{R}^{n+1}} \Gamma^0(x-y)\, L_0 u(y)\, dy$$

for all $x \in \mathbf{R}^{n+1}$.

Letting x_0 range over Q_T we regard Γ as a function of two variables and we'll write $\Gamma(x,y)$ where this time x represents the previously frozen variable.

From now on we'll use indexed Γ^0's and Γ's to indicate the derivatives of the kernels as in the following examples:

$$\Gamma_i^0(x) = \frac{\partial \Gamma^0}{\partial x'_i}(x) \quad , \qquad \Gamma_{ij}(x,y) = \frac{\partial^2 \Gamma}{\partial y'_i \partial y'_j}(x,y).$$

Let's recall some properties of $\Gamma^0(x)$ (see e. g. [**14**]):

(*i*) Mixed homogeneity:

$$\begin{aligned} \Gamma^0(rx', r^2 t) &= r^{-n}\,\Gamma^0(x',t); \\ \Gamma^0_i(rx', r^2 t) &= r^{-(n+1)}\,\Gamma^0_i(x',t); \\ \Gamma^0_{ij}(rx', r^2 t) &= r^{-(n+2)}\,\Gamma^0_{ij}(x',t) \qquad \text{for any } r > 0. \end{aligned} \tag{1.3}$$

(*ii*) Regularity:

$$\Gamma^0 \in C^\infty(\mathbf{R}^{n+1} \setminus \{0\})$$

(*iii*) Vanishing property (on hyperplanes):

$$\int_{\mathbf{R}^n} \Gamma^0_{ij}(x',t)\,dx' = 0 \qquad \text{for all } t > 0.$$

(*iv*) Integrability:

$$\Gamma^0,\ \Gamma^0_i \in L^1_{loc}(\mathbf{R}^{n+1}); \qquad \Gamma^0_{ij} \notin L^1_{loc}(\mathbf{R}^{n+1}).$$

(*v*) Boundedness of the derivatives:

$$\sup_{y \in \Sigma_{n+1}} \left| \left(\frac{\partial}{\partial y}\right)^{\beta} \Gamma^0(y) \right| \le C(\beta, \lambda)$$

for every multiindex β and with C independent of x_0.

Moreover the following lemma holds

Lemma 1.3 *Let $K(x)$ be a smooth function satisfying the mixed homogeneity condition*

$$K(rx', r^2 t) = r^{-\alpha}\,K(x',t) \qquad \text{for some } \alpha > 0 \text{ and any } r > 0.$$

Moreover suppose $\sup_{x \in \Sigma_{n+1}} |\nabla K(x)| = C$. *Then*

$$|K_{x'_i}(x)| \le \frac{C}{\rho(x)^{\alpha+1}}; \qquad |K_t(x)| \le \frac{C}{\rho(x)^{\alpha+2}} \qquad \text{for all } x \in \mathbf{R}^{n+1}.$$

Proof. Let $x = (x',t) \in \mathbf{R}^{n+1}$ and $\bar{x} = (\bar{x}', \bar{t}) = \left(\frac{x'}{\rho(x)}, \frac{t}{\rho(x)^2}\right) \in \Sigma_{n+1}$. Then

$$\left| \lim_{h \to 0} \frac{K(x' + he_i, t) - K(x',t)}{h} \right| =$$

$$= \rho(x)^{-\alpha} \cdot \lim_{h\to 0} \frac{\left| K(\bar{x}' + \frac{h}{\rho(x)} e_i, \bar{t}) - K(\bar{x}', \bar{t}) \right|}{|h|} \leq \rho^{-(\alpha+1)} \cdot \sup_{x\in\Sigma_{n+1}} |K_{x'_i}(x)|.$$

The second inequality is proved analogously.

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Lemma 1.3 holds for Γ^0 and for Γ_i^0 (with α equal to n and $n+1$, respectively) because of mixed homogeneity (1.3.i) and boundedness of the derivatives (1.3.v).

We are now ready to prove the interior representation formula.

Theorem 1.4 (Interior Representation Formula). *Let $u \in C_t$. Then, for x in the support of u, the following holds*

$$u_{x'_i x'_j}(x) =$$

$$= \lim_{\epsilon\to 0} \int\limits_{\rho(x-y)>\epsilon} \Gamma_{ij}(x, x-y) \left\{ \sum_{h,k=1}^{n} \left[a_{hk}(y) - a_{hk}(x)\right] u_{y'_h y'_k}(y) + Lu(y) \right\} dy$$

$$+ Lu(x) \int\limits_{\Sigma_{n+1}} \Gamma_j(x,y)\, n_i \, d\sigma(y) \tag{1.4}$$

where n_i is the i-th component of the outer normal to the surface Σ_{n+1}.

Proof. Let $g(x) = L_0 u(x)$.
Then $u(x) = \int_{\mathbf{R}^{n+1}} \Gamma^0(x-y) g(y)\, dy$ and $u_{x'_i}(x) = \int_{\mathbf{R}^{n+1}} \Gamma_i^0(x-y) g(y)\, dy$.
Therefore, for $\epsilon > 0$ fixed

$$u_{x'_i x'_j}(x) = \lim_{h\to 0} \int\limits_{\mathbf{R}^{n+1}} \frac{\Gamma_i^0(x + he_j - y) - \Gamma_i^0(x-y)}{h} \cdot g(y)\, dy =$$

$$= \int\limits_{\rho(x-y)>\epsilon} \Gamma_{ij}^0(x-y) g(y)\, dy +$$

$$+ \lim_{h\to 0} \int\limits_{\rho(x-y)<\epsilon} \frac{\Gamma_i^0(x+he_j-y) - \Gamma_i^0(x-y)}{h} \cdot [g(y) - g(x)]\, dy +$$

$$+ \lim_{h\to 0} g(x) \int\limits_{\rho(x-y)<\epsilon} \frac{\Gamma_i^0(x+he_j-y)-\Gamma_i^0(x-y)}{h}\,dy = A_\epsilon + B_\epsilon + C_\epsilon.$$

Because of Lemma 1.3 and the mean value theorem, we have that $B_\epsilon < C\cdot\epsilon$. Moreover

$$C_\epsilon = \lim_{h\to 0} g(x) \int\limits_{\rho(y)=\epsilon} \frac{\Gamma^0(y+he_j)-\Gamma^0(y)}{h}\, n_i\, d\sigma_\epsilon(y) =$$

$$= g(x) \int\limits_{\rho(y)=\epsilon} \Gamma_j^0(y)\, n_i\, d\sigma_\epsilon(y).$$

Through a change of variable and using the homogeneity of Γ_j^0 it is easy to see that the last integral equals $\int\limits_{\Sigma_{n+1}} \Gamma_j^0(y)\, n_i\, d\sigma(y)$.

Letting now $\epsilon \to 0$ we obtain

$$u_{x_i' x_j'}(x) = \lim_{\epsilon\to 0} \int\limits_{\rho(x-y)>\epsilon} \Gamma_{ij}^0(x-y)\, g(y)\, dy \;+\; g(x) \int\limits_{\Sigma_{n+1}} \Gamma_j^0(y)\, n_i\, d\sigma(y).$$

The above formula holds for any fixed x_0 and x, so in particular for $x_0 = x$. The representation formula is therefore proved writing $g(y) = Lu(y) + (L_0 - L)u(y) = Lu(y) + \sum_{h,k=1}^{n} [a_{hk}(y) - a_{hk}(x)]\, u_{y_h' y_k'}(y)$.

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From now on we'll use the following notation

$$\begin{aligned}
\alpha_{ij}(x) &= \int\limits_{\Sigma_{n+1}} \Gamma_i(x,y)\, n_j\, d\sigma(y) \\
K_{ij}(f)(x) &= \lim_{\epsilon\to 0} \int\limits_{\rho(x-y)>\epsilon} \Gamma_{ij}(x, x-y)\, f(y)\, dy \\
C_{ij}[a,f](x) &= \lim_{\epsilon\to 0} \int\limits_{\rho(x-y)>\epsilon} \Gamma_{ij}(x, x-y)\,[a(y)-a(x)]\, f(y)\, dy =
\end{aligned} \tag{1.5}$$

so that the interior representation formula we proved above may be rewritten as

$$u_{x'_i x'_j}(x) = \sum_{h,k=1}^{n} C_{ij}\left[a_{hk}, u_{y'_h y'_k}\right](x) + K_{ij}(Lu)(x) + \alpha_{ij} \cdot Lu(x). \quad (1.6)$$

In order to write the boundary representation formula we'll need the Green's function $G_0(x,y)$ for the halfspace $\mathbf{R}^{n+1}_+$. It will be constructed as a difference of fundamental solutions via reflection, as in the elliptic case.

Let $u \in C_{t,x'}$; then, for $x'_n > 0$, $u(x) = \int_{\mathbf{R}^{n+1}_+} G_0(x,y)\, L_0 u(y)\, dy$. We claim that

$$G_0(x,y) = \Gamma^0(x-y) - \Gamma^0\big(T(x_0;x) - y\big), \quad (1.7)$$

where, for fixed $x_0 \in \mathbf{R}^{n+1}$, T is the transformation from $\mathbf{R}^{n+1}_+$ to $\mathbf{R}^{n+1}_-$ defined by

$$T(x_0;x) = x - 2x'_n \cdot \frac{a(x_0)}{a_{nn}(x_0)},$$

and $a(x_0)$ is the $(n+1)$-dimensional vector $[a_{1n}(x_0), ..., a_{nn}(x_0), 0]$ constructed with the coefficients of L.

To see this observe that

$$u(x) = \int_{\mathbf{R}^{n+1}_+} \Gamma^0(x-y)\, f(y)\, dy - \int_{\mathbf{R}^{n+1}_+} \Gamma^0\big(T(x_0;x) - y\big)\, f(y)\, dy = \quad (1.8)$$

$$= v(x) - v\big(T(x_0;x)\big),$$

where v is the solution in $\mathbf{R}^{n+1}$ to $L_0 v = \tilde{f}$ and $\tilde{f}(x) = \begin{cases} f(x) & \text{for } x'_n > 0 \\ 0 & \text{for } x'_n < 0 \end{cases}$. A straightforward computation yields $L_0\Big(v\big(T(x_0;x)\big)\Big) = (L_0 v)\big(T(x_0;x)\big)$ and therefore $L_0 u(x) = \tilde{f}(x) - \tilde{f}\big(T(x_0;x)\big) = f(x)$, for $x'_n > 0$.

We are now ready to state

Theorem 1.5 (Boundary representation formula) *Let $u \in C_{t,x'}$. Then, for x in the support of u, the following holds:*

$$u_{x_i'x_j'}(x) =$$

$$= \lim_{\epsilon\to 0} \int_{\substack{\mathbf{R}^{n+1}_+\cap\\ \{\rho(x-y)>\epsilon\}}} \Gamma_{ij}(x, x-y) \left\{ \sum_{h,k=1}^{n} [a_{hk}(y) - a_{hk}(x)]\, u_{y_h' y_k'}(y) + Lu(y) \right\} dy$$

$$+ Lu(x) \int_{\Sigma_{n+1}} \Gamma_j(x,y)\, n_i \, d\sigma(y) \; - \; I_{ij}(x) \tag{1.9}$$

where $I_{ij}(x) =$

$$= \int_{\mathbf{R}^{n+1}_+} \Gamma_{ij}(x, T(x) - y) \left\{ \sum_{h,k=1}^{n} [a_{hk}(y) - a_{hk}(x)]\, u_{y_h' y_k'}(y) \; + \; Lu(y) \right\} dy,$$

for $i, j = 1, ..., n-1;$

$$I_{in}(x) = I_{ni}(x) = \int_{\mathbf{R}^{n+1}_+} \sum_{l=1}^{n} B_l(x)\, \Gamma_{il}(x, T(x) - y) \left\{ \; \ldots . \; \right\} dy,$$

for $i = 1, ... n-1$

and $I_{nn}(x) = \displaystyle\int_{\mathbf{R}^{n+1}_+} \sum_{l,r=1}^{n} B_l(x)\, B_r(x)\, \Gamma_{lr}(x, T(x) - y) \left\{ \; \ldots . \; \right\} dy;$

$T(x) \equiv T(x; x)$, *$B_i(x)$ is the i-th component of the vector $B(x) = T(e_n; x)$ and n_i is the i-th component of the outer normal to the surface Σ_{n+1}. (The expression between graphs is always the same).*

To prove the theorem we take difference quotients in (1.8). The first integral is then handled as in the proof of Theorem 1.4 while in the second one the derivatives can be taken inside the integral sign since the kernel is not singular $(T(x_0; x) \in \mathbf{R}^{n+1}_-$ while the integral is taken over $\mathbf{R}^{n+1}_+)$. For details see [6] where the analogous elliptic case is discussed.

◇

For the operators appearing in the I_{ij} above we'll use the following notation:

$$\begin{aligned} \widetilde{K}_{ij}(f)(x) &= \int_{\mathbf{R}^{n+1}_+} \Gamma_{ij}(x, T(x) - y)\, f(y)\, dy \\ \widetilde{C}_{ij}[a, f](x) &= \int_{\mathbf{R}^{n+1}_+} \Gamma_{ij}(x, T(x) - y)\, [a(y) - a(x)]\, f(y)\, dy. \end{aligned} \tag{1.10}$$

In the next two sections we are going to prove L^p-estimates for operators appearing in representation formulas (1.4) and (1.9) in order to obtain interior and boundary L^p-estimates for the second derivatives of u.

In particular we are going to prove the following estimates for the operators defined in (1.5) and (1.10)

(i) $$\|K_{ij}(f)\|_p \le c\,\|f\|_p$$

(ii) $$\|C_{ij}[a,f]\|_p \le c\,\|a\|_*\,\|f\|_p$$

(iii) $$\left\|\widetilde{K}_{ij}(f)\right\|_p \le c\,\|f\|_p$$

(iv) $$\left\|\widetilde{C}_{ij}[a,f]\right\|_p \le c\,\|a\|_*\,\|f\|_p$$

for all $p \in (1,\infty)$ and with constants depending only on n, p and λ. The last two inequalities involve norms taken over $\mathbf{R}^{n+1}_+$ and the operators they involve are not singular as observed in the proof of Theorem 1.5.

We'll treat the singular case in §2 and the other one in §3.

§2. Singular integrals estimates

Let's start with the following definition that will point out analogies of the kernels we are going to estimate with classical Calderòn-Zygmund kernels.

Definition 2.1 We say that a function K is a *Parabolic Calderòn-Zygmund kernel* (PCZ kernel) on the space $\mathbf{R}^{n+1}$ endowed with the metric ρ introduced in §1, if

i) K is smooth on $\mathbf{R}^{n+1}\setminus\{0\}$;

ii) $K(rx', r^2t) = r^{-(n+2)}\,K(x',t)$ for any $r>0$; (homogeneity condition)

iii) $\displaystyle\int_{\rho(x)=r} K(x)\,d\sigma(x) = 0$ for all $r>0$ (vanishing property on ellipsoids).

For a PCZ kernel let

$$C_1 = \sup_{x\in\Sigma_{n+1}} |\nabla K(x)| \qquad \text{and} \qquad C_2 = \int_{\Sigma_{n+1}} |K(x)|\,d\sigma(x).$$

In what follows constants depending on K will always be of the form $(1 + C_1 + C_2)$ while dependence on n will be left implicit.

Fabes and Rivière in **[9]** show that the Γ^0_{ij}'s are PCZ kernels and that PCZ kernels satisfy the following integral Hörmander condition, analogous to that of classical Calderòn-Zygmund kernels:

$$\int_{S_y} |K(x-y) - K(x)|\, dx \le C(K) \qquad \text{for all } y \in \mathbf{R}^{n+1}$$

where $S_y = \{x \mid \rho(x) \ge 4\rho(y)\}$. They also prove

Theorem 2.2 *For every $p \in (1, +\infty)$ there exists $C = C(p, K)$ such that*

$$\|Kf\|_p \le C\,\|f\|_p.$$

Moreover the following lemma holds

Lemma 2.3 (Pointwise Hörmander condition) *Let K be a PCZ kernel. Then for any parabolic cube I of center x_I*

$$|K(x-y) - K(x_I - y)| \le C(K)\,\frac{\rho(x - x_I)}{\rho(x_I - y)^{n+3}}$$

for $x \in I$, $y \notin 2I$.

Proof. Let $x = (x', t)$, $y = (y', s)$, $x_I = (x'_I, t_I)$. Then

$$|K(x-y) - K(x_I - y)| \le$$

$$|K(x'-y', t-s) - K(x'_I - y', t-s)| + |K(x'_I - y', t-s) - K(x'_I - y', t_I - s)| \le$$

$$\le \sup_{|z'-x'_I| \le \frac{1}{2}|y'-x'_I|} |\nabla_{x'} K(z_I - y', t-s)| \cdot |x' - x'_I| +$$

$$+ \sup_{|\tau - t_I| \le \frac{1}{4}|s - t_I|} |K_t(x'_I - y', \tau - s)| \cdot |t - t_I| \le$$

$$\le \frac{C}{\rho(x'_I - y', t_I - s)^{n+3}} \cdot \rho(x' - x'_I, 0) + \frac{C}{\rho(x'_I - y', t_I - s)^{n+4}} \cdot \rho^2(0, t - t_I) \le$$

$$\le C\,\frac{\rho(x - x_I)}{\rho(x_I - y)^{n+3}},$$

where we used Lemma 1.3 (which holds for K) and properties of the metric ρ.

◇

Definition 2.4 We say that a function $K(x,y)$, $x \in \mathbf{R}^{n+1}$, $y \in \mathbf{R}^{n+1} \setminus \{0\}$ is a *variable PCZ kernel*, if the following conditions hold:

i) $K(x,\cdot)$ is a PCZ kernel for a.e. $x \in \mathbf{R}^{n+1}$,

ii) $\sup_{y \in \Sigma_{n+1}} \left| \left(\frac{\partial}{\partial y}\right)^\beta K(x,y) \right| \le C(\beta)$ independent of x.

In particular the following quantities are finite:

$$\sup_x \sup_{y \in \Sigma_{n+1}} |\nabla_y K(x,y)|$$

$$\sup_x \int_{\Sigma_{n+1}} |K(x,y)| \, d\sigma(y).$$

From now on for a variable PCZ kernel K, constants will depend on K only through the numbers $C(\beta)$ relative to some (fixed) multiindexes β.

Note that by (1.3.v) $\Gamma_{ij}(x,y)$ is a variable PCZ kernel with $C(\beta)$ depending on the ellipticity constant λ.

For a variable PCZ kernel K and an L^p function f define

$$Kf(x) = \lim_{\epsilon \to 0} \int_{\rho(x-y)>\epsilon} K(x, x-y)\, f(y)\, dy;$$

and for $\phi \in L^\infty(\mathbf{R}^{n+1})$ define the *commutator* of K as the function

$$C[\phi, f](x) = K(\phi f)(x) - \phi(x) K f(x). \tag{2.1}$$

The main theorem regarding variable PCZ kernels is the following (see again [**9**])

Theorem 2.5 *Let $K(x,y)$ be a variable PCZ kernel. Then for any $f \in L^p$, $1 < p < \infty$, Kf exists in L^p sense and*

$$\|Kf\|_p \le C\, \|f\|_p$$

for some $C = C(p, K)$.

The technique used to prove the above theorem, consists in eigenfunctions expansion of the kernel to reduce the problem to estimates on operators with constant kernels. Then L^p estimates are proved for these last operators. This is a general technique which was first employed in the elliptic case by Calderòn and Zygmund in [4] in which they use expansion in spherical harmonics to get L^p-estimates for classical kernels. Fabes in [8] uses expansion in Hermite polynomials to prove similar estimates on classic parabolic singular integral operators. Theorem 2.5 is a restatement of this last result. Actually Fabes and Rivière in [9] prove a version of this theorem with K a kernel with a more general mixed homogeneity on a Euclidean space endowed with a suitable metric. They use again expansion in spherical harmonics. Recently Chiarenza, Frasca and Longo in [5] show that it is possible to apply the same technique to extend to the case of variable elliptic kernels estimates on commutator operators of the kind (2.1) for which estimates are proved in [7] when the kernel is constant.

Here we will prove estimates for the commutator operator with (constant) PCZ kernel, and then use spherical harmonics expansion to prove the result for variable kernels.

Theorem 2.6 *Let K be a PCZ (constant) kernel and $a \in L^\infty$. Then for any $f \in L^p$, $1 < p < \infty$, there exists $C = C(p, K)$ such that*

$$\|C[a,f]\|_p \le C\|a\|_*\|f\|_p.$$

When K is a classical Calderòn-Zygmund kernel the result analogous to the above theorem is due to Coifman, Rochberg and Weiss (see [7]).

Before proceeding with the proof, let's recall some definitions and results of real analysis which hold in this "parabolic" context.

For $f \in L^1_{loc}(\mathbf{R}^{n+1})$, define the parabolic maximal function

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)|\, dy;$$

and the parabolic sharp function

$$f^\sharp(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y) - f_I|\, dy,$$

where, in both functions, the sup is taken over all parabolic cubes I in $\mathbf{R}^{n+1}$ containing the point x.

The next three lemmas follow from results stated in [**3**].

Lemma 2.7 (Maximal inequality) *For* $f \in L^p$, $1 < p \le \infty$,

$$\|Mf\|_p \le C(p)\|f\|_p.$$

Lemma 2.8 (John-Nirenberg type lemma) *For* $1 \le p < \infty$, $f \in BMO$ *and* I *a parabolic cube*

$$\left(\frac{1}{|I|}\int_I |f(y) - f_I|^p\, dy\right)^{1/p} \le C(p)\, \|f\|_*.$$

Lemma 2.9 (Sharp inequality) *For every* p, $1 \le p < \infty$, *there exists a constant* $C = C(p)$ *such that if* $f \in L^p$ *then*

$$\|f\|_p \le C\|f^\sharp\|_p.$$

Lemma 2.10 *Let* $f \in BMO$. *Then, for any positive integer* j *and parabolic cube* I

$$|a_{2^j I} - a_I| \le c(n)\cdot j\|a\|_*.$$

Proof. $|a_{2I} - a_I| =$

$$= \left|\tfrac{1}{|I|}\textstyle\int_I \left(a(x) - a_{2I}\right) dx\right| \le \frac{2^{n+2}}{|2I|}\int_{2I} |a(x) - a_{2I}|\, dx \le 2^{n+2}\|a\|_*.$$

Then

$$|a_{2^j I} - a_I| \le |a_{2^j I} - a_{2^{j-1} I}| + \ldots + |a_{4I} - a_{2I}| + |a_{2I} - a_I| \le 2^{n+2}\cdot j\|a\|_*.$$

◇

Remark 2.11. Lemmas 2.7-2.10 still hold if norms are taken on the halfspace $\mathbf{R}^{n+1}_+$. In this case the definitions of maximal function, sharp function, BMO and VMO need to be modified by taking only cubes $I \subseteq \mathbf{R}^{n+1}_+$.

Proof of Theorem 2.6 We are going to adapt an idea of Stromberg (see [**18**] pp.417-418). Let $Tf = C[a, f]$. Observe that it is enough to prove

$$(Tf)^\sharp(\bar{x}) \le C(r, K)\, \|a\|_* \left\{ \left(M(|Kf|^r)(\bar{x})\right)^{1/r} + \left(M(|f|^r)(\bar{x})\right)^{1/r} \right\} \quad (2.2)$$

for all $r > 1$, $\bar{x} \in \mathbf{R}^{n+1}$.

To see this choose $1 < r < p$; then (2.2) combined with Lemmas 2.7 and 2.9 and with the L^p estimate on K implies

$$\|Tf\|_p \le C\|a\|_* \Big\{ \|Kf\|_p + \|f\|_p \Big\} \le C\, \|a\|_* \cdot \|f\|_p.$$

Let's now prove (2.2). For a fixed cube I and $\bar{x},\ x \in I$ write
$Tf(x) = K(af)(x) - a(x)Kf(x) =$

$$= K\big((a-a_I)f\chi_{2I}\big)(x) + K\big((a-a_I)f\chi_{(2I)^c}\big)(x) - \big(a(x)-a_I\big)\cdot Kf(x) =$$

$$= A(x) + B(x) + C(x).$$

Now, for any $r > 1$ and $q \in (1, r)$,

$$\frac{1}{|I|}\int_I |A(x) - A_I|\, dx \le 2\frac{1}{|I|}\int_I \Big| K\big((a-a_I)f\chi_{2I}\big)(x)\Big|\, dx \le$$

$$\le 2\left\{ \frac{1}{|I|}\int_{\mathbf{R}^{n+1}} \Big| K\big((a-a_I)f\chi_{2I}\big)(x)\Big|^q dx \right\}^{1/q} \le$$

$$C(q,K)\left\{ \frac{1}{|I|}\int_{2I} |a(x)-a_I|^q\, |f(x)|^q\, dx \right\}^{1/q} \le$$

$$\le C(q,K)\left\{ \frac{1}{|I|}\left(\int_{2I} |f(x)|^r\, dx\right)^{\frac{q}{r}} \cdot \left(\int_{2I} \Big|a(x)-a_I\Big|^{\frac{qr}{r-q}} dx\right)^{\frac{r-q}{r}} \right\}^{\frac{1}{q}} \tag{2.3}$$

(we used Theorem 2.5). By Lemmas 2.10 and 2.8,
$\int_{2I} \big|a(x)-a_I\big|^{qr/(r-q)}\, dx \le$
$\le C\int_{2I} \big|a(x)-a_{2I}\big|^{qr/(r-q)}\, dx + C\|a\|_*^{qr/(r-q)}|2I| \le C(p,q)\|a\|_*^{qr/(r-q)}|2I|$
and therefore $(2.3) \le$

$$C(r,K)\left(\frac{1}{|I|}\int_{2I}|f(x)|^r\,dx\right)^{1/r}\|a\|_*\le C(r,K)\|a\|_*\cdot\left(M\left(|f|^r\right)(\bar{x})\right)^{1/r} \tag{2.4}$$

To bound $B(x)$ note that $\frac{1}{|I|}\int_I \left|B(x)-B_I\right|dx \le \frac{2}{|I|}\int_I \left|B(x)-B(x_I)\right|dx$. Moreover, for $x\in I$ and letting x_I be the center of I

$$\left|B(x)-B(x_I)\right| = \left|K\left((a-a_I)f\chi_{(2I)^c}\right)(x)-K\left((a-a_I)f\chi_{(2I)^c}\right)(x_I)\right|\le$$

$$\le \int_{(2I)^c}\left|K(x-y)-K(x_I-y)\right|\left|a(y)-a_I\right|\left|f(y)\right|dy\le$$

$$\le C(K)\int_{(2I)^c}\frac{\rho(x-x_I)}{\rho(x_I-y)^{n+3}}\left|a(y)-a_I\right|\left|f(y)\right|dy\le$$

$$\le C(K)\cdot\delta\left(\int_{(2I)^c}\frac{|f(y)|^r}{\rho(x_I-y)^{n+3}}\,dy\right)^{1/r}\cdot\left(\int_{(2I)^c}\frac{|a(y)-a_I|^{r'}}{\rho(x_I-y)^{n+3}}\,dy\right)^{1/r'} \tag{2.5}$$

where δ is the side lenght of I and we used Hörmander condition (Lemma 2.3).

Now we can write the first integral in (2.5) as

$$\sum_{j=2}^{+\infty}\int_{2^jI\setminus 2^{j-1}I}\frac{|f(y)|^r}{\rho(x_I-y)^{n+3}}\,dy\le\sum_{j=2}^{+\infty}\frac{C}{2^j\delta}\,\frac{1}{|2^jI|}\int_{2^jI}|f(y)|^r dy\le$$

$$\le M\left(|f|^r\right)(\bar{x})\cdot\frac{C}{\delta}\sum_{j=2}^{+\infty}\frac{1}{2^j}.$$

Analogously the second integral in (2.5) may be bounded by $\frac{C(r)}{\delta}\,\|a\|_*^{r'}$, using Lemmas 2.10 and 2.8. Therefore we conclude that

$$\frac{1}{|I|}\int\left|B(x)-B_I\right|dx\le C(r,K)\|a\|_*\left(M\left(|f|^r\right)(\bar{x})\right)^{\frac{1}{r}}.$$

Finally to estimate the average of $C(x)$ observe that, by Hölder inequality and John-Nirenberg Lemma 2.8

$$\frac{1}{|I|}\int_I \left|C(x)-C_I\right| dx \le 2\,\frac{1}{|I|}\int_I \left|C(x)\right| dx \le$$
$$\le 2\left(\frac{1}{|I|}\int_I \left|a(x)-a_I\right|^{r'} dx\right)^{1/r'} \cdot \left(\frac{1}{|I|}\int_I \left|Kf(x)\right|^{r} dx\right)^{1/r} \le$$
$$\le C(r)\|a\|_*\Big(M(|Kf|^r)(\bar{x})\Big)^{1/r}.$$

The proof is now complete. ◇

We shall now prove L^p estimates on commutators with variable kernels.

Theorem 2.12 *Let $K(x,y)$ be a variable PCZ kernel and $a \in L^\infty$. Moreover let $C[a,f]$ be the commutator with kernel K. Then for any $1 < p < \infty$ there exists $C = C(p,K)$ such that, for $f \in L^p$,*

$$\|C[a,f]\|_p \le C\|a\|_*\|f\|_p.$$

Before proceeding with the proof of the theorem, let's recall some properties of spherical harmonics.

Since we need expansions in $\mathbf{R}^{n+1}$, we will consider $(n+1)$-dimensional spherical harmonics. An $(n+1)$-dimensional spherical harmonic of degree m is the restriction to the unit sphere Σ_{n+1} in $\mathbf{R}^{n+1}$ of a homogeneous polynomial of degree m which is harmonic in $\mathbf{R}^{n+1}$. For $Y_m(x)$ an $(n+1)$-dimensional spherical harmonic of degree m we have

$$\sup_{x\in\Sigma_{n+1}}\left|\left(\frac{\partial}{\partial x}\right)^{\beta} Y_m(x)\right| \le c\cdot m^{\frac{n-1}{2}+|\beta|} \tag{2.6}$$

Let

$$\left\{Y_{km}(x)\right\}_{k=1}^{g_m} \text{ where } g_m = \binom{m+n}{n} - \binom{m+n-2}{n} \le c(n)\cdot m^{n-1} \tag{2.7}$$

be an orthonormal base for the space of $(n+1)$-dimensional spherical harmonics of degree m. Then $\left\{Y_{km}(x)\right\}_{k,m}$ is a complete orthonormal system of functions for $L^2(\Sigma_{n+1})$. If $f \in C^\infty(\Sigma_{n+1})$ and if $f(x) \sim \sum b_{km}Y_{km}(x)$

is the Fourier series development of $f(x)$ with respect to $\{Y_{km}\}$, where $b_{km} = \int_{\Sigma_{n+1}} f(x)Y_{km}(x)\,d\sigma$, then, for every $r > 1$,

$$|b_{km}| \leq C(r)m^{-2r} \sup_{\substack{|\beta| = 2r \\ x \in \Sigma_{n+1}}} \left| \left(\frac{\partial}{\partial x}\right)^{\beta} f(x) \right|. \tag{2.8}$$

For the proof of the above results see [4].

Proof of Theorem 2.12 For $y \in \Sigma_{n+1}$, $x \in \mathbf{R}^{n+1}$, write

$$K(x,y) = \sum_{k,m} b_{km}(x)Y_{km}(y), \quad \text{with} \quad b_{km}(x) = \int_{\Sigma_{n+1}} K(x,y)Y_{km}(y)\,d\sigma(y).$$

Observe that, from definition 2.4-*ii*) and (2.8), we have for any $r > 1$

$$|b_{km}(x)| \leq C \cdot m^{-2r}, \quad \text{for } x \in \mathbf{R}^{n+1} \tag{2.9}$$

where $C = C(r,K)$

Given $y \in \mathbf{R}^{n+1}$ let $y = \left(\rho(y)\bar{y}', \rho^2(y)\bar{t}\right)$ where $\bar{y} = (\bar{y}', \bar{t}) \in \Sigma_{n+1}$. Because of the homogeneity of K we can then write

$$K(x,y) = \sum_{k,m} b_{km}(x)\frac{Y_{km}(\bar{y})}{\rho(y)^{n+2}}.$$

Let $H_{km} = \dfrac{Y_{km}(\bar{y})}{\rho(y)^{n+2}}$. Then the kernels H_{km} are PCZ (constant) kernels and therefore satisfy the L^p estimate; in particular, by (2.6) and Theorem 2.2,

$$\|H_{km}f\|_p \leq C_{km}\|f\|_p, \quad \text{with } C_{km} = c \cdot m^{\frac{n+1}{2}}.$$

Let now $T_{km}f = C_{km}[a,f] = H_{km}(af) - aH_{km}f$. It follows from Theorem 2.6, (2.9) and (2.7) that

$$\|C[a,f]\|_p \leq$$

$$\leq \sum_{k,m} \|b_{km}\|_\infty \cdot \|T_{km}f\|_p \leq C(r,p,K)\left(\sum_{m=0}^{\infty}\sum_{k=1}^{g_m} m^{-2r}\, m^{\frac{n+1}{2}}\right) \|a\|_* \, \|f\|_p \leq$$

$$\le C(r,p,K)\left(\sum_{m=0}^{\infty} m^{\frac{3n-1}{2}-2r}\right)\|a\|_* \|f\|_p, \tag{2.10}$$

for any $r > 1$. For a suitable choice of r the series in (2.10) converges and the theorem is proved.

◇

We conclude this section with the following corollary, which is a localized version of Theorem 2.6.

Corollary 2.13 *Let K be as in Theorem 2.12 and $a \in VMO \cap L^\infty(\mathbf{R}^{n+1})$. Then for every ϵ there exists r_0 depending only on $\epsilon > 0$ and the VMO modulus η_a of a, such that for every $f \in L^p(B_r)$ with $r \le r_0$*

$$\|C[a,f]\|_p \le C(p,K)\cdot \epsilon \|f\|_p$$

where the norms are taken over B_r.

For the proof see [**5**]. In this proof a main role is played by the properties of VMO functions recalled in Theorem 1.1.

Theorems 2.5 and 2.12 prove (i) and (ii) at the end of §1. In the next section we are going to prove estimates for the "less singular" operators appearing in (iii) and (iv).

§3. Boundary integral operators estimates

In this section L^p norms will always be taken over $\mathbf{R}^{n+1}_+$.

Theorem 3.1 *Let K be a PCZ variable kernel and*

$$\widetilde{K}f(x) = \int_{\mathbf{R}^{n+1}_+} K(x, T(x)-y)\, f(y)\, dy.$$

(Recall T is the transformation introduced in §1, constructing the Green's function). Then for every $p \in (1,+\infty)$ there exists a constant $C = C(p,\lambda,K)$ such that for $f \in L^p(\mathbf{R}^{n+1}_+)$,

$$\left\|\widetilde{K}f\right\|_p \le C\,\|f\|_p\,.$$

Before proceeding with the proof we need the following lemmas. The first one regards the map T.

Lemma 3.2 *For any $x \in \mathbf{R}^{n+1}$, $x = (x_1, ..., x_n, t)$, let $\tilde{x} = (x_1, ..., -x_n, t)$. Then there exists two constants c_1, c_2 depending only on n, λ, such that*

$$c_1\,\rho(\tilde{x} - y) \le \rho(T(x) - y) \le c_1\,\rho(\tilde{x} - y)$$

and

$$c_1\,|\tilde{x}' - y'| \le \left|(T(x))' - y\right| \le c_1\,|\tilde{x}' - y'|$$

for every x, $y \in \mathbf{R}^{n+1}_+$.

Proof It is sufficient to prove the second line inqualities, since $T(x)$ and $\tilde{x}$ have the same t-coordinate. The inequality $c_1\,|\tilde{x}' - y'| \le \left|(T(x))' - y\right|$ is proved in [6], by observing that $\left|(T(x))' - \tilde{x}\right| = 2x_n - \frac{a(x)}{a_{nn}(x)} \le x_n\, c(n, \lambda)$ and $\left|(T(x))' - y\right| \ge x_n$.
To prove the last inequality observe that also $|\tilde{x}' - y'| \ge x_n$. ◇

Let now $Rf(x) = \displaystyle\int_{\mathbf{R}^{n+1}_+} \frac{f(y)}{\rho^{n+2}(\tilde{x} - y)} dy$. The following holds.

Lemma 3.3 *For every $p \in (1, +\infty)$, there exists a constant $C = C(p, \lambda)$, such that for $f \in L^p(\mathbf{R}^{n+1}_+)$,*

$$\|Rf\|_p \le C\,\|f\|_p\,.$$

Proof. For $x \in \mathbf{R}^{n+1}$ let's write $x = (x', t) = (x'', x_n, t)$, with $x'' \in \mathbf{R}^{n-1}$. Then

$$Rf(x) = \int_0^{+\infty} dy_n \int_{\mathbf{R}^n} \frac{f(y'', y_n, \tau)}{\rho^{n+2}(x'' - y'', -x_n - y_n, t - \tau)} dy''\, d\tau.$$

Let $I(x_n) = \displaystyle\int_{\mathbf{R}^n} |Rf(x'', x_n, t)|^p\, dx''\, dt$, so that $\|Rf\|_p^p = \displaystyle\int_0^{+\infty} I(x_n)\, dx_n$.
Then by Minkowsky and Young inequalities, $I(x_n) \le$

$$\leq \left(\int_0^{+\infty} dy_n \left(\int_{\mathbf{R}^n} \left| f(y'', y_n, t) \right|^p dy'' dt \right)^{\frac{1}{p}} \cdot \left(\int_{\mathbf{R}^n} \frac{dy''\, dt}{\rho^{n+2}(y'', x_n + y_n, t)} \right) \right)^p =$$

$$= \left(\int_0^{+\infty} \frac{\phi(y_n)}{x_n + y_n} \, dy_n \right)^p \cdot \left(\int_{\mathbf{R}^n} \frac{dz''\, d\tau}{\rho^{n+2}(z'', 1, \tau)} \right)^p, \tag{3.1}$$

where we set $\phi(y_n) = \left(\int_{\mathbf{R}^n} \left| f(y'', y_n, t) \right|^p dy'' dt \right)^{1/p}$ and changed variables.

Observe that the right term in (3.1) is bounded by a constant depending only n and p, so that

$$\|Rf\|_p^p \leq C(p) \int_0^{+\infty} \left(\int_0^{+\infty} \frac{\phi(\delta\, x_n)}{1+\delta} \, d\delta \right)^p dx_n \leq$$

$$\leq C \left(\int_0^{+\infty} \left(\int_0^{+\infty} \left(\frac{\phi(\delta\, x_n)}{1+\delta} \right) dx_n \right)^{\frac{1}{p}} d\delta \right)^p = C \left(\int_0^{+\infty} \frac{1}{(1+\delta)\delta^{1/p}} \, d\delta \right)^p \|f\|_p^p.$$

And the theorem is proved.

◇

Proof of Theorem 3.1. Let's expand in spherical harmonics the operator $\widetilde{K}$ analogously to the proof of Theorem 2.12. We are then led to consider the operators

$$(\widetilde{H}_{km} f)(x) = \int_{\mathbf{R}^{n+1}_+} H_{km}(x, T(x) - y) \, f(y) \, dy,$$

where H_{km} are the kernels defined in the proof of the same theorem; from (2.6) and Lemma 3.2 we have

$$\left| H_{km}(T(x) - y) \right| \leq c(n, \lambda) \cdot m^{\frac{n-1}{2}} \cdot \frac{1}{\rho^{n+2}(\tilde{x} - y)}.$$

Repeating the argument at the end of the proof of Theorem 2.12, the above estimate combined with Lemma 3.3 proves the theorem.

◇

Finally let's estimate the commutator type boundary operator.

Theorem 3.4 *Let $K(x,y)$ be a PCZ variable kernel, $\widetilde{K}f$ be defined in Theorem 3.1, $a \in L^\infty$ and $\widetilde{C}[a,f](x) = \widetilde{K}(af)(x) - a(x)\widetilde{K}f(x)$. Then for every $p \in (1,+\infty)$, there exists a constant $C = C(p,\lambda,K)$, such that for $f \in L^p(\mathbf{R}^{n+1}_+)$,*

$$\left\|\widetilde{C}[a,f]\right\|_p \leq C\|a\|_* \,\|f\|_p.$$

In the proof of this theorem we cannot use Hörmander type estimates on the kernel $\widetilde{H}_{km}$, since the map T is not continuous. The kernel, though, behaves somewhat "better" in the y variable.

We will prove the following

Theorem 3.5 *Let $K(x)$ PCZ (constant) kernel, define*

$$\widetilde{K}^*f(x) = \int_{\mathbf{R}^{n+1}_+} K(T(y)-x)\,f(y)\,dy.$$

Also for $a \in L^\infty(\mathbf{R}^{n+1}_+)$, let $\widetilde{C}^*[a,f]$ be the commutator associated to $\widetilde{K}^*$. Then for every $p \in (1,+\infty)$ there exists a constant $C = C(p,\lambda,K)$, such that for $f \in L^p(\mathbf{R}^{n+1}_+)$,

$$\left\|\widetilde{C}^*[a,f]\right\|_p \leq C\|a\|_* \,\|f\|_p.$$

We are then going to use the following result from functional analysis.

Theorem 3.6 *Let $H(x,y)$ be a function, $H^*(x,y) = H(y,x)$ and let $a(x)$ be a bounded function. If the maps*

$$f(x) \longmapsto \int |H(x,y)| f(y)\,dy$$

and

$$f(x) \longmapsto C[a,f] = H(af)(x) - a(x)Hf(x)$$

are continuous on L^p, $1 < p < +\infty$, then the maps

$$f(x) \longmapsto \int |H^*(x,y)| f(y)\,dy$$

and

$$f(x) \longmapsto C^*[a,f] = H^*(af)(x) - a(x)H^*f(x)$$

are continuous on $L^{p'}$, with p' conjugate exponent of p. Moreover the $L^{p'}$ norms of the adjoint operators equal the L^p norms of the corresponding operators.

In order to prove Theorem 3.5, we still need the following

Lemma 3.7 (Hörmander condition for adjoint operators) *For a PCZ kernel K, $I \subset \mathbf{R}_+^{n+1}$ a parabolic cube with $x \in I$ and $y \in \mathbf{R}_+^{n+1}$, $y \notin 2I$,*

$$\left|K(x - T(y \qquad {}_I - T(y))\right| \leq C(\lambda, K)(x \frac{\rho(x - x_I)}{\rho(x_I - \tilde{y})^{n+3}}.$$

Proof.

$$\left|K(x - T(y)) - K(x_I - T(y))\right| \leq |x' - x_I'|\, |\nabla_{x'} K(\xi)| + |t - t_I|\, |K_t(\eta)|$$

with $\xi = \bar{x}_1 - T(y)$, $\eta = \bar{x}_2 - T(y)$, for some $\bar{x}_1, \bar{x}_2 \in I$

$$\leq C(K) \left\{ \frac{|x' - x_I'|}{\rho\big(\bar{x}_1 - T(y)\big)^{n+3}} + \frac{|t - t_I|}{\rho\big(\bar{x}_2 - T(y)\big)^{n+4}} \right\}. \tag{3.2}$$

From the hypothesis on y and I, $\tilde{y} \notin 2I$ and $\rho(x_I - \tilde{y}) \geq c_1\, \rho(x_I - \bar{x}_i)$ for $i = 1, 2$ and for some $c_1 > 1$. Then using Lemma 3.2, (3.2) becomes

$$\left|K(x - T(y)) - K(x_I - T(y))\right| \leq$$

$$C(\lambda, K) \frac{1}{\rho(x_I - \tilde{y})^{n+3}} \left\{ \rho(x' - x_I', 0) + \frac{\rho(0, t - t_I)^2}{\rho(x_I - \tilde{y})} \right\} \leq C\, \frac{\rho(x - x_I)}{\rho(x_I - \tilde{y})^{n+3}}.$$

◇

Proof of Theorem 3.5. Let $Sf = \widetilde{C}^*[a, f]$. The proof follows closely that of Theorem 2.6, as we'll prove again that

$$(Sf)^\sharp(\bar{x}) \;\leq\; C(r, \lambda, K)\, \|a\|_* \left\{ \Big(M\big(|\widetilde{K}^* f|^r\big)(\bar{x})\Big)^{1/r} + \big(M\big(|f|^r\big)(\bar{x})\big)^{1/r} \right\}$$

for $\bar{x} \in \mathbf{R}_+^{n+1}$ and $r > 1$, keeping in mind Remark 2.11. Via the analogous decomposition in $A(x) + B(x) + C(x)$ for a cube $I \subset \mathbf{R}_+^{n+1}$, again we obtain

$$\frac{1}{|I|}\int_I |A(x)-A_I|\,dx \le C(r,K)\Big(M(|f|^r(\bar{x}))\Big)^{1/r},$$

$$\frac{1}{|I|}\int_I |C(x)-C_I|\,dx \le C(r)\|a\|_*\left(M(|\widetilde{K}^* f(x)|^r(\bar{x}))\right)^{1/r}$$

and (using the above lemma and observing that $\rho(x_I-\tilde{y})\ge\rho(x_I-y)$)

$$\frac{1}{|I|}\int_I |B(x)-B_I|\,dx \le C(\lambda,K)\rho(x-x_I)\int_{\mathbf{R}^{n+1}_+\setminus 2I}\frac{|a(y)-a(x)|\,|f(x)|}{\rho(x_I-y)^{n+3}}\,dy\,.$$

The end of the proof follows exactly that of Theorem 2.6.

◇

Proof of Theorem 3.4 Theorem 3.5 combined with Theorem 3.6 gives the estimate for the constant kernel operator. Using again spherical harmonics expansion and letting $\widetilde{C}_{km}[a,f]$ be the commutator operators with kernels $H_{km}(x,T(x)-y)$, this implies

$$\left\|\widetilde{C}_{km}[a,f]\right\|_p \le C(\lambda)\cdot m^{\frac{n+1}{2}}\|a\|_*\,\|f\|_p$$

and therefore the thesis of the theorem.

◇

A localized version of Theorem 3.4, analogous to Corollary 2.13, holds:

Corollary 3.8 *Let K be as in Theorem 3.4 and $a\in VMO\cap L^\infty(\mathbf{R}^{n+1}_+)$. Then for every $\epsilon>0$ there exists r_0 depending only on ϵ and the VMO modulus η_a af a, such that for every $f\in L^p(B_r^+)$, $(1<p<\infty,\ B_r^+=B_r\cap\mathbf{R}^{n+1}_+)$, with $r<r_0$*

$$\left\|\widetilde{C}[a,f]\right\|_p \le C(p,\lambda,K)\cdot\epsilon\,\|f\|_p.$$

§4. L^p-estimates, existence and uniqueness for the Cauchy-Dirichlet problem

Let $W^{1,2}_p(Q_T)$ be the Sobolev space of functions u such that u, u_t, $u_{x_i'}$

and $u_{x'_i x'_j} \in L^p$ and let $W_0(Q_T)$ be the closure in the $W_p^{1,2}$ norm of the space $\mathcal{C} = \left\{\phi \in C^\infty(\bar{Q}_T) \; : \; \phi = 0 \text{ for } t = 0 \text{ or } x \in \partial\Omega\right\}$. We are interested in the Cauchy-Dirichlet problem

$$\begin{cases} Lu = f & \text{in } Q_T \\ u \in W_0(Q_T) \end{cases} \tag{4.1}$$

with $f \in L^p(Q_T)$, for some $1 < p < \infty$, and the coefficients of L satisfying (0.2) and belonging to $VMO(Q_T)$. In this section we are going to prove interior and boundary L^p-estimates and existence and uniqueness for solutions to problem (4.1).

Theorem 4.1 (Interior estimates) *For every $p \in (1, +\infty)$ there exist $C(n,p,\lambda)$, and $r_0 = r_0(C, \eta_a)$ such that for $r \leq r_0$, $B_{2r} \subset\subset \Omega \times \mathbf{R}$, and $u \in W_0(Q_T)$ we have*

$$(i) \qquad \|u_{x'_i x'_j}\|_{L^p(B_r^+)} \leq C\left\{\|Lu\|_{L^p(B_{2r}^+)} + r^{-2}\|u\|_{L^p(B_{2r}^+)}\right\}$$

$$(ii) \qquad \|u_t\|_{L^p(B_r^+)} \leq C\left\{\|Lu\|_{L^p(B_{2r}^+)} + r^{-2}\|u\|_{L^p(B_{2r}^+)}\right\},$$

where $B_r^+ = B_r \cap \{t \geq 0\}$.

Proof. Let first $u \in \mathcal{C}_t$ and recall the interior representation formula (Theorem 1.4). By (1.3.v) the functions $\alpha_{ij}(x)$'s defined in (1.5) are bounded. Moreover Theorems 2.5 and 2.12 and Corollary 2.13 hold for $K = \Gamma_{ij}$ and $a = a_{hk}$ (recall that the constant C in (2.9) depends on λ). Therefore we obtain that for any fixed $\epsilon > 0$ there exists r_0 depending on ϵ and the VMO moduli of the coefficients such that

$$\|u_{x'_i x'_j}\|_p \leq C(n,\lambda,p)\left\{\epsilon \cdot \sup_{h,k}\|u_{x'_h x'_k}\|_p + \|Lu\|_p\right\},$$

whenever $sprt\; u \subseteq B_r^+$ for some $r < r_0$.

For a suitable choice of ϵ it follows, for such a u that

$$\|u_{x'_i x'_j}\|_p \leq C(n,\lambda,p)\|Lu\|_p.$$

Using a standard argument involving cutoff functions and interpolation inequalities (see e.g. [10]) we obtain estimate (i) for any function in $\mathcal{C}$ and by density for any function in $W_0(Q_T)$.

To get (ii) it is enough to write

$$u_t = Lu + a_{ij} u_{x'_i x'_j}$$

and apply (i). ◇

Theorem 4.2 (Boundary estimates) *For all $p \in (1, +\infty)$ there exist $C(n, p, \lambda)$, and $r_0 = r_0(C, \eta_a)$ such that for $r \le r_0$, sprt $u \subseteq B_r$, and any $u \in W_0(Q_T)$ we have*

$$(i) \qquad \|u_{x'_i x'_j}\|_{L^p(\bar{B}_r^+)} \le C \left\{ \|Lu\|_{L^p(\bar{B}_{2r}^+)} + \|u\|_{L^p(\bar{B}_{2r}^+)} \right\}$$

$$(ii) \qquad \|u_t\|_{L^p(\bar{B}_r^+)} \le C \left\{ \|Lu\|_{L^p(\bar{B}_{2r}^+)} + \|u\|_{L^p(\bar{B}_{2r}^+)} \right\},$$

where $\bar{B}_r^+ = B_r^+ \cap \mathbf{R}_+^{n+1}$.

Proof. Recall the boundary representation formula (1.9) and observe that the functions B_i in the statement of Theorem 1.5 are bounded because of the ellipticity condition (0.2). (i) and (ii) are then proved (analogously to Theorem 4.1) using Theorem 3.1 and Corollary 3.8. ◇

¿From now on norms will always be taken over Q_T.

Theorem 4.3 (Solvability of the Cauchy-Dirichlet problem) *Let $f \in L^p(Q_T)$, $1 < p < \infty$. Then the Cauchy-Dirichlet problem (4.1) has a unique solution $u \in W_p^{1,2}(Q_T)$. Moreover*

$$\|u\|_{W_p^{1,2}} \le C\|Lu\|_p \tag{4.2}$$

where $C = C(n, \lambda, p, \eta_a, \partial\Omega, |\Omega|, T)$.

Proof. By a standard argument consisting of flattening the boundary and using Theorems 4.1 and 4.2 together with interpolation inequalities to estimate the L^p norm of ∇u, we get that

$$\|u\|_{W_p^{1,2}} \le C(n, \lambda, p, \eta_a, \partial\Omega, |\Omega|, T) \left\{ \|f\|_p + \|u\|_p \right\}. \tag{4.3}$$

The action of a $C^{1,1}$ map on the VMO coefficients in the flattening argument is discussed in [**6**].

In order to prove uniqueness let $u \in \mathcal{C}$ and write $u(x,t) = \displaystyle\int_0^t u_s(x,s)\,ds$:

this implies $\|u\|_p \leq T \cdot \|u_t\|_p$. Then by (4.3) there exists $T_0 > 0$ such that if $T \leq T_0$

$$\|u\|_{W_p^{1,2}} \leq C\|Lu\|_p. \tag{4.4}$$

By density (4.4) holds for $u \in W_0(Q_T)$. Let now u be a solution to (4.1) with $f \equiv 0$ and break up Q_T into a finite number of cylinders $Q_i = \Omega \times [T_i, T_{i+1}]$ with $|T_{i+1} - T_i| \leq T_0$. Apply (4.4) to each Q_i and obtain $u \equiv 0$ in Q_T.

Finally, existence and estimate (4.2) are obtained again through a standard argument from uniqueness and estimate (4.3). (See e.g. [**6**]).

◇

Postscript. In [**15**] the authors note that a result equivalent to our theorem 2.6 can be derived by the results in [**15**] and [**17**].

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