Fundamental solutions and local solvability for nonsmooth Hörmander's operators*

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Abstract

We consider operators of the form $L = \sum_{i=1}^{n} X_i^2 + X_0$ in a bounded domain of $\mathbb{R}^p$ where $X_0, X_1, \ldots, X_n$ are *nonsmooth* Hörmander's vector fields of step r such that the highest order commutators are only Hölder continuous. Applying Levi's parametrix method we construct a local fundamental solution γ for L and provide growth estimates for γ and its first derivatives with respect to the vector fields. Requiring the existence of one more derivative of the coefficients we prove that γ also possesses second derivatives, and we deduce the local solvability of L, constructing, by means of γ, a solution to $Lu = f$ with Hölder continuous f. We also prove $C_{X,loc}^{2,\alpha}$ estimates on this solution.

Contents

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1 Introduction

Object and main results of the paper
In the study of elliptic-parabolic degenerate partial differential operators, an important class is represented by Hörmander's operators

$$L=\sum_{i=1}^{n}X_i^2+X_0 \tag{1.1}$$

built on real smooth vector fields

$$X_i=\sum_{j=1}^{p}b_{ij}\left(x\right)\partial_{x_j} \tag{1.2}$$

which are defined in some domain $\Omega\subset\mathbb{R}^p$. A famous theorem by Hörmander [17] states that if the Lie algebra generated by the X_i's ($i=0,1,2,...,n$) coincides with the whole $\mathbb{R}^p$ at any point of Ω, then L is hypoelliptic in Ω, that is any distributional solution to the equation $Lu=f\in C^\infty\left(\Omega\right)$ belongs to $C^\infty\left(\Omega\right)$. Over the years, a number of deep properties of Hörmander's operators and systems of Hörmander's vector fields have been established. Some of them are related to the metric induced by Hörmander's vector fields (connectivity property, doubling property for metric balls, see [31]), or to the "gradient" associated to Hörmander's vector fields (Poincaré's inequality, see [18]); other properties are related to second order Hörmander's operators (properties of fundamental solutions, see [13], [31], [34], or a priori estimates on the second order derivatives with respect to the vector fields, see [13], [33]).

One can note that, apart from Hörmander's hypoellipticity theorem, which intrinsically requires C^∞ regularity of the vector fields, most of the important existing results in this area are expressed by statements which are meaningful, and hopefully still hold, under much less regularity of the vector fields. So a natural question consists in asking how much of the classical theory of Hörmander's vector fields and Hörmander's operators still holds if we consider a family of vector fields whose coefficients possess just the right number of derivatives which are enough to check that Hörmander's condition at some step r holds (see section 2 for the definition). However, this generalization is far from being obvious, since if one tries to repeat the classical proofs just paying attention to the minimal regularity required, one finds that some arguments need the existence of a very high number of derivatives (for instance, the double of the step r), while others simply cannot be repeated. Experience shows that proving relevant results about nonsmooth vector fields under reasonably weak assumptions is almost always a hard task. Nevertheless, this is a natural problem if one hopes to settle the basis for applications to nonlinear equations which involve vector fields depending on the solution itself (such as Levi-type equations that we will discuss later in this introduction).

This paper is the third step in a larger project started by three of us in [5] and [6], and devoted to this issue. Our framework is the following. Let $X_0,X_1,...,X_n$ be a system of real vector fields, defined in a bounded domain $\Omega\subset\mathbb{R}^p$. We assume that for some integer $r\geqslant 2$ and some $\alpha\in(0,1]$ the coefficients of the vector fields $X_1,X_2,...,X_n$ belong to $C^{r-1,\alpha}\left(\Omega\right)$, while the coefficients of X_0 belong to $C^{r-2,\alpha}\left(\Omega\right)$. If $r=2$, we assume $\alpha=1$. Here and in the following, $C^{k,\alpha}$ stands for the classical space of functions which are differentiable up to order k, with α-Hölder continuous derivatives of order k. Moreover, we assume that $X_0, X_1, ..., X_n$ satisfy Hörmander's condition of *weighted step* r in Ω: if we assign weight 1 to $X_1, X_2, ..., X_n$ and weight 2 to X_0, then the commutators of the vector fields X_i, up to weight r, span $\mathbb{R}^p$ at any point of Ω (more precise definitions will be given later).

An extension to this nonsmooth context of some basic properties of the distance induced by the vector fields, Chow's connectivity theorem, the estimate on the volume of metric balls, the doubling condition, and Poincaré's inequality has been given in [5]. These results also imply a Sobolev embedding and the validity of Moser's iteration technique to handle operators of the kind

$$\sum_{i,j=1}^{n} X_i^* \left(a_{ij}\left(x\right) X_j u\right).$$

In [6] the same authors have extended to the nonsmooth context the lifting and approximation theory developed in the smooth case by Rothschild-Stein [33] and some related results, such as the comparison between volumes of balls in the lifted and original space. Starting with the paper [33], this technique has been used, in the smooth case, to reduce the study of general Hörmander's operators (1.1) to that of left invariant homogeneous operators on homogeneous groups, for which Folland's theory developed in [13] applies, granting the existence of a homogeneous left invariant fundamental solution, which is a good starting point to prove a-priori estimates of several types.

Following this idea, in the present paper we use tools and results from [5] and [6] to study Hörmander's operators (1.1) built with nonsmooth vector fields or, briefly, *nonsmooth Hörmander's operators*. Namely, we are able to adapt to this situation the classical Levi's parametrix method, in order to build a fundamental solution $\gamma(x, y)$ for L (in the small), possessing some good properties. More precisely, under the above assumptions we prove (see Thm. 4.8) that for any $x_0 \in \Omega$ there exists a neighborhood $U(x_0)$ and a function $\gamma(x, y)$, defined and continuous in the joint variables for $x, y \in U(x_0)$, $x \neq y$, satisfying

$$\int \gamma\left(x, y\right) L^* \omega\left(x\right) dx = -\omega\left(y\right) \tag{1.3}$$

for any $\omega \in C_0^\infty(U(x_0))$; moreover, γ satisfies the bounds

$$\left|\gamma\left(x, y\right)\right| \leqslant c \frac{d\left(x, y\right)^2}{\left|B\left(x, d\left(x, y\right)\right)\right|}; \tag{1.4}$$

$$\left|X_i \gamma\left(x, y\right)\right| \leqslant c \frac{d\left(x, y\right)}{\left|B\left(x, d\left(x, y\right)\right)\right|}, \qquad i = 1, 2, ..., n, \tag{1.5}$$

where, here and in the following, $X_i \gamma(x, y)$ denotes the X_i derivative with respect to the *first* variable, x, the distance d is the one induced by the vector fields X_i, and $B(x, r)$ are the corresponding balls.

Under the stronger assumption that the coefficients of the X_i's $(i = 1, 2, ..., n)$ belong to $C^{r,\alpha}(\Omega)$ and the coefficients of X_0 belong to $C^{r-1,\alpha}(\Omega)$, we are able to prove that γ also possesses second derivatives with respect to the vector fields, satisfying the bounds

$$
\begin{aligned}
|X_j X_i \gamma(x,y)| &\leqslant \frac{c}{|B(x,d(x,y))|}, \qquad i,j=1,2,...,n, \\
|X_0 \gamma(x,y)| &\leqslant \frac{c}{|B(x,d(x,y))|},
\end{aligned} \tag{1.6}
$$

and that $\gamma(\cdot,y)$ is a classical solution to the equation $L\gamma(x,y)=0$ for $x\neq y$ (see Thm. 5.9). Exploiting these results we prove (see Thm. 5.18) the following local solvability result for L: for every $x_0\in\Omega$ there exists a neighborhood U such that for any $\beta>0$ and $f\in C_X^{\beta}(U)$ (i.e., β-Hölder continuous with respect to the distance d) there exists a classical solution u to the equation $Lu=f$ in U. Pushing even forward our analysis, we show that the functions $X_iX_j\gamma$ satisfy the following local Hölder estimate: for every $x_1,x_2,y\in U$ such that $d(x_1,y)\geqslant 2d(x_1,x_2)$,

$$
|X_iX_j\gamma(x_1,y)-X_iX_j\gamma(x_2,y)|\leqslant c_{\varepsilon}\left(\frac{d(x_1,x_2)}{d(x_1,y)}\right)^{\alpha-\varepsilon}\frac{1}{|B(x_1,d(x_1,y))|} \tag{1.7}
$$

for any $\varepsilon\in(0,\alpha)$ and $i,j=1,2,...,n$, with c_{ε} depending on ε (Thm. 5.17). As a consequence, we eventually show that the local solution w to $Lw=f$ that we have built for $f\in C_X^{\beta}(U)$, with $\beta<\alpha$, actually belongs to $C_{X,loc}^{2,\beta}(U)$ (see Thm. 5.20).

Comparison with the existent literature

The study of nonsmooth Hörmander's vector fields has been carried out by several authors; we refer to the introduction of [5] for a detailed discussion of the related bibliography. Here we just point out that the peculiarity of the research project consisting in the present paper and [5], [6] is that of considering nonsmooth Hörmander's vector fields of completely general form. Indeed, with the notable exception of the papers [28], [29], [30] by Montanari-Morbidelli and some papers by Karmanova-Vodopyanov (see [20], [35] and the references therein), all the other previous results about nonsmooth vector fields either hold only for vector fields with a particular structure, or assume axiomatically some important properties of the metric induced by the vector fields themselves. Another characteristic feature of the present research is to take explicitly into account the possibility that one of the vector fields X_0 ("the drift") could have weight two, as in the case of Hörmander's operators (1.1). This is relevant for instance in view of the possible application of the present theory to operators of Kolmogorov-Fokker-Planck type with nonsmooth drift. While the literature devoted to the geometry of nonsmooth vector fields is quite large, the one about *Hörmander's operators* built on nonsmooth vector fields is much narrower. Particular classes of operators of this kind have been studied in the framework of regularity results for nonlinear equations of Levi type by Citti, Lanconelli, Montanari, starting with the paper [8] and continuing with [10], [9], [25] (see also references therein). A somewhat related field of research is that about the Levi-Monge-Ampère equation, see [26], [27], which also motivates the study of nonvariational operators modeled on (possibly nonsmooth) Hörmander's vector fields. Another application of this circle of ideas to a nonlinear regularization problem has been given by Citti, Pascucci, Polidoro in [11]. However, the present paper seems to be the first one where Hörmander's operators built with nonsmooth vector fields of general structure are studied.

Let us come to some remarks about the techniques used. The parametrix method was originally developed more than a century ago by E. E. Levi to study uniformly elliptic equations of order $2n$ (see [22]), and later extended to uniformly parabolic operators (see e.g. [14]). For more details about this method in the elliptic case we refer to [24, § 19], [16, Part IV, Chap.3] and [19]. In particular, the last reference contains a rich account of the previous literature on this subject and a careful discussion of the assumptions made by different authors to implement the method. The parametrix method was first adapted to hypoelliptic ultraparabolic operators of Kolmogorov-Fokker-Planck type by Polidoro in [32], exploiting the knowledge of an explicit expression for the fundamental solution of the "frozen" operator, which had been constructed in [21]. It was later adapted by Bonfiglioli, Lanconelli, Uguzzoni in [1] to a general class of operators structured on homogeneous left invariant (smooth) vector fields on Carnot groups, for which no explicit fundamental solution is known in general, and by Bramanti, Brandolini, Lanconelli, Uguzzoni in [4] to the more general context of arbitrary (smooth) Hörmander's vector fields. Finally, in the nonsmooth context, the parametrix method has been exploited by Manfredini in [23] to deal with sum of squares of $C^{1,\alpha}$-intrinsic vector fields of step 2, with a particular structure.

In order to evaluate our assumptions about the regularity of vector fields, one can draw a comparison with the assumptions made in the elliptic case, as reported in [19]. Rewriting our operator in the form

$$L=\sum_{j,k=1}^{p}a_{jk}\left(x\right)\partial_{x_jx_k}^{2}+\sum_{k=1}^{p}b_k\left(x\right)\partial_{x_k}+c\left(x\right)$$

one can see that our stronger assumptions (see Assumptions B in § 5) imply in the simplest degenerate case $r=2$

$$a_{jk}\in C^{2,1}\left(\Omega\right),b_k\in C^{1,1}\left(\Omega\right),c\in C^{1,1}\left(\Omega\right)$$

while in the elliptic case [19, Thm.3] it is essentially required that

$$a_{jk}\in C^{2}\left(\Omega\right)\cap C^{0,\alpha}\left(\Omega\right),b_j\in C^{1}\left(\Omega\right)\cap C^{0,\alpha}\left(\Omega\right),c\in C^{0,\alpha}\left(\Omega\right).$$

Strategy and plan of the paper

The technique of "lifting and approximation" developed by Rothschild-Stein in [33] and extended to nonsmooth vector fields in [6], coupled with the results by Folland [13] suggests that, in order to study the (nonsmooth) operator (1.1), natural steps consist in lifting L, in a neighborhood of a point $x_0\in\mathbb{R}^p$, to a new (nonsmooth) operator

$$\widetilde{L}=\sum_{i=1}^{n}\widetilde{X}_i^2+\widetilde{X}_0$$

defined in a neighborhood $\mathcal{U}$ of $(x_0,0)\in\mathbb{R}^{p+m}$, and then approximate $\widetilde{L}$ with a (smooth) left invariant homogeneous operator

$$\mathcal{L}=\sum_{i=1}^{n}Y_i^2+Y_0$$

which possesses a homogeneous left invariant fundamental solution $\Gamma\left(v^{-1}\circ u\right)$, with respect to a structure of homogeneous group in $\mathbb{R}^{p+m}$. Then, a natural parametrix of $\widetilde{L}$ can be defined by

$$P_0\left(\xi,\eta\right)=\Gamma\left(\Theta_\eta\left(\xi\right)\right),$$

where the map $\Theta_\eta\left(\xi\right)$ (a nonsmooth version, introduced in [6], of the function defined by Rothschild-Stein in [33]) is, for any fixed $\eta\in\mathcal{U}$, a smooth diffeomorphism which allows to approximate $\widetilde{L}$ with $\mathcal{L}$ near η, and $\Theta_\eta\left(\xi\right)$ depends on η in a Hölder continuous way. Hence $P_0\left(\xi,\eta\right)$ is smooth in ξ but just Hölder continuous in η (or $C^{1,\alpha}$ in η, if the coefficients of the X_i's are $C^{r,\alpha}$ and the coefficients of X_0 are $C^{r-1,\alpha}$, see Proposition 5.4). This rough asymmetry in the properties of P_0 with respect to the two variables prevents us from repeating Rothschild-Stein's technique to prove L^p or C^α estimates for second order derivatives with respect to the vector fields, for a solution to $Lu=f$. Instead, one can think to adapt to this case the classical Levi's parametrix method, which is compatible with a different degree of regularity of P_0 in the two variables. Now, if we applied the parametrix method directly to the kernel P_0 we would build a local fundamental solution for $\widetilde{L}$. Starting from this object, however, there is no obvious way to produce a local fundamental solution for L. Instead, we have to define directly a parametrix for L, shaped on P_0 saturating the lifted variables by integration, in the following way:

$$P\left(x,y\right)=\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\Gamma\left(\Theta_{(y,k)}\left(x,h\right)\right)\varphi\left(h\right)dh\right)\varphi\left(k\right)dk,\quad\text{for } x,y\in U,\qquad(1.8)$$

where $\varphi\in C_0^\infty\left(\mathbb{R}^m\right)$ is a cutoff function fixed once and for all, equal to one in a neighborhood of the origin. This P turns out to be a good parametrix for L, and starting with it we can actually construct a local fundamental solution for L, satisfying natural growth estimates and regularity properties. However, performing this construction (see §4) is a hard task, since we are forced to work in a metric measure space where the measure of balls does not behave like a fixed power of the radius, in particular there is not a homogeneous dimension. Therefore a good deal of preliminary work (see §3) has to be done to craft the geometric and real analysis tools necessary to make the Levi method work. In particular, it turns out that the right function to measure the size of a kernel $k\left(x,y\right)$ is

$$\phi_\beta\left(x,y\right)=\int_{d(x,y)}^R\frac{r^{\beta-1}}{\left|B\left(x,r\right)\right|}dr$$

which for $\beta\in(0,p)$, is bounded by (but not equivalent to)

$$c\frac{d\left(x,y\right)^\beta}{\left|B\left(x,d\left(x,y\right)\right)\right|}\qquad(1.9)$$

and satisfies a key property which is very useful in iterative computations (see Theorem 3.5), and could not be proved for (1.9).

The Levi method is then implemented as follows. We look for a fundamental solution for L of the form

$$\gamma\left(x,y\right)=P\left(x,y\right)+J\left(x,y\right)$$

where P is as in (1.8) and

$$J(x,y) = \int_U P(x,z)\,\Phi(z,y)\,dz.$$

In turn, we will find Φ as the series

$$\Phi(z,y) = \sum_{j=1}^{\infty} Z_j(z,y) \text{ for } z \neq y$$

where the Z_j's are defined inductively by

$$\begin{aligned} Z_1(x,y) &= L(P(\cdot,y))(x) \\ Z_{j+1}(x,y) &= \int_U Z_1(x,z)\,Z_j(z,y)\,dz \qquad \text{for } x \neq y. \end{aligned}$$

In §4, exploiting the results of §3 and some results proved in [5], [6] and recalled in §2, we prove the basic properties and upper bounds satisfied by the functions Z_1, Z_j, Φ, J, and we deduce the existence of a local fundamental solution γ satisfying (1.3), (1.4), (1.5).

The next step, in §5, is then to compute the second derivatives of γ, that is

$$X_iX_j\gamma(x,y) = X_iX_jP(x,y) + X_i\int_U X_jP(x,z)\,\Phi(z,y)\,dz$$

(all the X_i derivatives being taken with respect to the x variable). In order to do that one has to exploit, in particular, Hölder continuity (with respect to d) of $z \mapsto \Phi(z,y)$, to allow differentiation under the integral sign. Proving Hölder continuity of Φ and the existence of $X_iX_j\gamma$ forces us to deepen the analysis of the properties of the map $\Theta_\eta(\xi)$ and to strengthen our assumptions on the vector fields, requiring from now on $X_i \in C^{r,\alpha}$ and $X_0 \in C^{r-1,\alpha}$. Once the existence of $X_iX_j\gamma$ and the upper bound (1.6) are proved, we can show that for any $\beta > 0$ and $f \in C_X^{\beta}(U)$, the function

$$w(x) = -\int_U \gamma(x,y)\,f(y)\,dy \tag{1.10}$$

is a classical solution to the equation $Lw = f$ in U. In particular, we establish an "explicit" representation formula for X_iX_jw (see Corollary 5.19), containing singular integrals, fractional integrals, and multiplicative terms. This formula, although rather involved, is designed in view of the subsequent proof of Hölder continuity. The point is that, for technical reasons related to the starting definition of the parametrix $P(x,y)$, which is assigned by an integral with respect to the "lifted variables", the singular part of

$$X_i\int_U X_j\gamma(x,y)\,f(y)\,dy$$

cannot be easily written in a form like

$$\lim_{\varepsilon\to 0}\int_{d(x,y)>\varepsilon} X_iX_j\gamma(x,y)\,f(y)\,dy,$$

which should allow to apply directly some abstract theory of singular integrals. Instead, we have to rewrite properly the integral, to transform the singular part into something like

$$\int k(x,y)[f(y)-f(x)]\,dy \tag{1.11}$$

with k singular near the diagonal.

In §5 we also prove Hölder estimates on $X_iX_j\gamma$, the difficult part of the estimate being that on X_iX_jJ. We then pass to prove that the solution (1.10) to $Lu=f$ possesses locally Hölder continuous derivatives X_iX_jw. This amounts to proving Hölder continuity of each term of the representation formula for X_iX_jw previously established. While for the fractional integrals it is fairly enough to exploit Hölder continuity of X_iX_jJ, the singular integral term also requires the proof of a cancellation property of the kind

$$\left|\int_{r_1<d(x,y)<r_2} k(x,y)\,dy\right| \leqslant c \quad \text{for any } r_1<r_2.$$

In order to prove C_X^α continuity of singular and fractional integrals we both apply some abstract results proved in [7] for locally homogeneous spaces and revise some techniques used in [3].

Finally, in Appendix we give some examples of nonsmooth Hörmander's operators satisfying assumption A in §2 or assumption B in §5.

2 Some known results about nonsmooth Hörmander's vector fields

In this section we fix precisely our notation and assumptions, and recall a number of known facts which will be used throughout the paper. In some cases, we do not recall the complete definitions given in [5, 6], but only the properties that are needed for our current purposes.

Let $X_0, X_1, ..., X_n$ be a system of real vector fields

$$X_i = \sum_{j=1}^{p} b_{ij}(x)\,\partial_{x_j},$$

defined in a bounded, arcwise connected open set $\Omega \subset \mathbb{R}^p$. Let us assign to each X_i a *weight* p_i, saying that

$$p_0 = 2 \text{ and } p_i = 1 \text{ for } i = 1, 2, \ldots, n.$$

For any multiindex

$$I = (i_1, i_2, \ldots, i_k)$$

we define the *weight* of I as

$$|I| = \sum_{j=1}^{k} p_{i_j}$$

and we set

$$X_I = X_{i_1}X_{i_2}...X_{i_k}$$

and

$$X_{[I]} = \left[X_{i_1}, \left[X_{i_2}, ... \left[X_{i_{k-1}}, X_{i_k}\right] ...\right]\right],$$

where $[X, Y]$ is the usual Lie bracket of vector fields. If $I = (i_1)$, then

$$X_{[I]} = X_{i_1} = X_I.$$

As usual, $X_{[I]}$ can be seen either as a differential operator or as a vector field. We will write $X_{[I]}f$ to denote the differential operator $X_{[I]}$ acting on a function f, and $\left(X_{[I]}\right)_x$ to denote the vector field $X_{[I]}$ evaluated at the point x.

For a positive integer k and $\alpha \in (0, 1]$ we define the (classical) Hölder space $C^{k,\alpha}(\Omega)$ of functions k times differentiable (in classical sense), with derivatives of order k belonging to the Hölder (or Lipschitz) space $C^{\alpha}(\Omega)$, defined by the finiteness of the norm

$$\|f\|_{C^{\alpha}(\Omega)} = \sup_{x\in\Omega} |f(x)| + |f|_{C^{\alpha}(\Omega)},$$

with

$$|f|_{C^{\alpha}(\Omega)} = \sup_{x,y\in\Omega, x\neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}.$$

Assumptions A. We assume that for some integer $r \geqslant 2$ and some $\alpha \in (0, 1]$, the coefficients of the vector fields $X_1, X_2, ..., X_n$ belong to $C^{r-1,\alpha}(\Omega)$, while the coefficients of X_0 belong to $C^{r-2,\alpha}(\Omega)$. If $r = 2$, we assume $\alpha = 1$. Moreover, we assume that $X_0, X_1, ..., X_n$ satisfy Hörmander's condition of step r in Ω, i.e. the vectors

$$\left\{\left(X_{[I]}\right)_x\right\}_{|I|\leqslant r}$$

span $\mathbb{R}^p$ for any $x \in \Omega$. (For examples of systems of vector fields satisfying the assumptions, see the Appendix).

We note that under our assumptions, for any $1 \leqslant k \leqslant r$, the differential operators $\{X_I\}_{|I|\leqslant k}$ and the vector fields $\left\{X_{[I]}\right\}_{|I|\leqslant k}$ are well defined, and have $C^{r-k,\alpha}$ coefficients.

We will sometimes need the *transpose operator*

$$L^* = \sum_{i=1}^{n} (X_i^*)^2 + X_0^* \tag{2.1}$$

defined by the transpose operators X_i^* of the vector fields, which act on smooth functions as

$$X_i^* u(x) = -\sum_{j=1}^{p} \partial_{x_j} (b_{ij}(x) u(x)).$$

Note that, in order for L^*u to be well defined, at least as an L^{∞} function, we need the b_{ij}'s to be at least $C^{1,1}$ for $i = 1, 2, ..., p$, and $C^{0,1}$ for $i = 0$. This is one of the reasons why we need $\alpha = 1$ if $r = 2$. We will also use this in the proof of Theorem 2.10.

The subelliptic metric, analogous to that introduced by Nagel-Stein-Wainger in [31], is defined as follows:

Definition 2.1 *For any* $\delta > 0$, *let* $C(\delta)$ *be the class of absolutely continuous mappings* $\varphi : [0,1] \longrightarrow \Omega$ *which satisfy*

$$\varphi'(t) = \sum_{|I| \leqslant r} a_I(t) \left(X_{[I]}\right)_{\varphi(t)} \quad a.e.$$

with $a_I : [0,1] \to \mathbb{R}$ *measurable functions,*

$$|a_I(t)| \leqslant \delta^{|I|}.$$

Then define

$$d(x,y) = \inf \{\delta > 0 : \exists \varphi \in C(\delta) \text{ with } \varphi(0) = x, \varphi(1) = y\}$$

and denote $B(x,\rho)$ *the associated ball of center* x *and radius* ρ.

The finiteness of d for any couple of points of Ω, as well as the basic properties of this distance in the nonsmooth context have been established in [5]. In particular, we will use the following facts:

Proposition 2.2 (Relation with the Euclidean distance) *There exist a positive constant* c_1 *depending on* Ω *and the* X_i*'s and, for every* $\Omega' \Subset \Omega$, *a positive constant* c_2 *depending on* Ω' *and the* X_i*'s, such that*

$$c_1 |x-y| \leqslant d(x,y) \leqslant c_2 |x-y|^{1/r} \text{ for any } x,y \in \Omega'. \tag{2.2}$$

In particular, the distance d *induces Euclidean topology.*

Theorem 2.3 (Doubling condition) *Under the previous assumptions, for any domain* $\Omega' \Subset \Omega$, *there exist positive constants* c, ρ_0, *depending on* Ω, Ω' *and the* X_i*'s, such that*

$$|B(x,2\rho)| \leqslant c\, |B(x,\rho)|$$

for any $x \in \Omega'$, $\rho < \rho_0$.

Theorem 2.4 (Volume of metric balls) *For any family* $\mathcal{I}$ *of* p *multiindices* $I_1, I_2, ..., I_p$ *with* $|I_j| \leqslant r$, *let* $|\mathcal{I}| = \sum_{j=1}^{p} |I_j|$ *and* $\lambda_{\mathcal{I}}(x)$ *be the determinant of the* $p \times p$ *matrix with rows* $\left\{ \left(X_{[I_j]}\right)_x \right\}_{I_j \in \mathcal{I}}$. *For any* $\Omega' \Subset \Omega$ *there exist positive constants* c_1, c_2, ρ_0 *depending on* Ω, Ω' *and the* X_i*'s, such that*

$$c_1 \sum_{\mathcal{I}} |\lambda_{\mathcal{I}}(x)| \rho^{|\mathcal{I}|} \leqslant |B(x,\rho)| \leqslant c_2 \sum_{\mathcal{I}} |\lambda_{\mathcal{I}}(x)| \rho^{|\mathcal{I}|} \tag{2.3}$$

for any $\rho < \rho_0$, $x \in \Omega'$, *where the sum is taken over any family* $\mathcal{I}$ *with the above properties.*

Definition 2.5 (Hölder spaces) *For any* $U \Subset \Omega$ *we can introduce Hölder spaces* $C_X^\alpha(U)$ *with respect to the distance* d, *letting for* $\alpha > 0$,

$$\|f\|_{C_X^\alpha(U)} = \sup_{x \in U} |f(x)| + |f|_{C_X^\alpha(U)},$$

with

$$|f|_{C_X^\alpha(U)} = \sup_{x,y \in U, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}.$$

Also, we let

$$C_X^{2,\alpha}(U) = \{f : U \to \mathbb{R} |\ \|f\|_{C_X^{2,\alpha}(U)} < \infty\}$$

where

$$\|f\|_{C_X^{2,\alpha}(U)} = \|f\|_{C_X^{\alpha}(U)} + \sum_{|I|\leqslant 2} \|X_I f\|_{C_X^{\beta}(U)}.$$

By (2.2) the following hold:

$$f \in C^{\alpha}(\Omega') \Rightarrow f \in C_X^{\alpha}(\Omega')$$
$$f \in C_X^{\alpha}(\Omega') \Rightarrow f \in C^{\alpha/r}(\Omega').$$

Note, in particular, that saying "$f \in C^{\beta}(\Omega')$ for some $\beta > 0$" is the same as "$f \in C_X^{\beta}(\Omega')$ for some $\beta > 0$".

We will also need the following property, which is similar to that proved in [5, Thm. 5.11]. For convenience of the reader, we recall here its short proof.

Proposition 2.6 *Let $\overline{x} \in \Omega$ and let $B(\overline{x}, R) \subset \Omega$. For any $f \in C^1(B(\overline{x}, R))$, one has*

$$|f(x) - f(\overline{x})| \leqslant d(x, \overline{x}) \left(\sum_{i=1}^{n} \sup_{B(\overline{x},R)} |X_i f| + d(x, \overline{x}) \sup_{B(\overline{x},R)} |X_0 f| \right)$$

for any $x \in B(\overline{x}, R)$.

Proof. Let $x \in B(\overline{x}, R)$, hence by Definition 2.1 there exists a curve $\varphi(t)$, such that $\varphi(0) = \overline{x}, \varphi(1) = x$, and

$$\varphi'(t) = \sum_{i=0}^{n} \lambda_i(t) (X_i)_{\varphi(t)}$$

with $|\lambda_0(t)| \leqslant d(x, \overline{x})^2$ and $|\lambda_i(t)| \leqslant d(x, \overline{x})$ for $i = 1, \ldots, n$. Moreover, every point $\gamma(t)$ for $t \in (0, 1)$ belongs to $B(\overline{x}, R)$. Then we can write:

$$\begin{aligned} |f(x) - f(\overline{x})| &= \left| \int_0^1 \frac{d}{dt} f(\varphi(t))\, dt \right| = \left| \int_0^1 \sum_{i=0}^{n} \lambda_i(t) (X_i f)_{\varphi(t)}\, dt \right| \\ &\leqslant d(x, \overline{x}) \sum_{i=1}^{n} \sup_{B(\overline{x},R)} |X_i f| + d(x, \overline{x})^2 \sup_{B(\overline{x},R)} |X_0 f|, \end{aligned}$$

as desired. ■

In [6] an extension to nonsmooth vector fields of some known results by Rothschild-Stein [33] are proved. The first one is:

Theorem 2.7 (Lifting theorem) *For every $x_0 \in \Omega$, there exist a neighborhood $U(x_0)$, an integer m and vector fields of the form*

$$\widetilde{X}_k = X_k + \sum_{j=1}^{m} u_{kj}(x, h_1, h_2, ..., h_{j-1}) \frac{\partial}{\partial h_j} \tag{2.4}$$

($k = 0, 1, ..., n$), where the u_{kj}'s are polynomials of degree at most $r-1$, such that the $\widetilde{X}_k$'s are free up to step r and such that $\left\{ \left(\widetilde{X}_{[I]} \right)_{(x,h)} \right\}_{|I|\leqslant r}$ span $\mathbb{R}^{p+m} \equiv \mathbb{R}^N$ for every $(x, h) \in U(x_0) \times I$, where I is a neighborhood of $0 \in \mathbb{R}^m$.

We do not repeat here the exact definition of *free* vector fields, in our weighted situations, because we will never use it explicitly.

An easy consequence of the structure (2.4) of the lifted vector fields is that for any differentiable function $f(x,h)$ and any smooth cutoff function $\varphi(h)$ we have

$$\int_{\mathbb{R}^m} \widetilde{X}_k \left[f(x,h) \varphi(h) \right] dh = \int_{\mathbb{R}^m} X_k f(x,h) \varphi(h) \, dh \tag{2.5}$$

since the integrals $\int_{\mathbb{R}^m} \frac{\partial}{\partial h_j}(...) \, dh$ vanish.

We will denote by $\widetilde{d}$ the distance induced in $U(x_0) \times I$ by the lifted vector fields $\widetilde{X}_i$ $(i = 0, 1, 2, ..., n)$, as in Definition 2.1, and by $\widetilde{B}(\eta, r)$ the corresponding metric ball of center η and radius r. We will also set

$$\widetilde{L} = \sum_{i=1}^{n} \widetilde{X}_i^2 + \widetilde{X}_0.$$

Let us recall that a structure of *homogeneous group* $\mathbb{G}$ on $\mathbb{R}^N$ consists in a Lie group operation $\circ$ (which we think of as *translation*) such that the origin is the unit in the group and the Euclidean opposite is the inverse in the group, and a one-parameter family $\{D(\lambda)\}_{\lambda>0}$ of group automorphisms (which we think of as *dilations*), acting as follows:

$$D(\lambda)(u_1, u_2, ..., u_N) = (\lambda^{\alpha_1} u_1, \lambda^{\alpha_2} u_2, ..., \lambda^{\alpha_N} u_N), \tag{2.6}$$

for some positive integers $\alpha_1, \alpha_2, ..., \alpha_N$. The sum of these integers is called the *homogeneous dimension* Q of $\mathbb{G}$.

A homogeneous norm on $\mathbb{G}$ is any function $\|\cdot\| : \mathbb{G} \to [0, \infty)$ such that
$\|u\| = 0 \Longleftrightarrow u = 0$, $\|D(\lambda) u\| = \lambda \|u\|$ for any $\lambda > 0$,
$\|u_1 \circ u_2\| \leqslant c(\|u_1\| + \|u_2\|)$, $\left\|u^{-1}\right\| \leqslant c\|u\|$ for any $u, u_1, u_2 \in \mathbb{G}$.

Such a homogeneous norm naturally induces a distance $\left\|u_1^{-1} \circ u_2\right\|$ in $\mathbb{G}$; the (Lebesgue) measure of the corresponding ball in $\mathbb{G}$ is translation invariant, and multiple of r^Q. In the following we will use a fixed homogeneous norm on $\mathbb{G}$.

Definition 2.8 *(See [6]) We say that a vector field*

$$R = \sum_{j=1}^{N} c_j(u) \, \partial_{u_j}$$

on the group $\mathbb{G}$ *has* weight $\geqslant \beta$, *for some* $\beta \in \mathbb{R}$, *if*

$$|c_j(u)| \leqslant c \|u\|^{\alpha_j + \beta}$$

for u *in a neighborhood of* 0.

The second basic result proved in [6] is:

Theorem 2.9 (Approximation by left invariant Hörmander's operator) *Let x_0, $U(x_0)$, and I be as in the lifting theorem. There exist a structure of homogeneous group $\mathbb{G}$ on $\mathbb{R}^N$, $N = p + m$, a family of homogeneous left invariant Hörmander's vector fields $Y_0, Y_1, Y_2, ..., Y_n$ on $\mathbb{G}$ and an open set $V \subset U(x_0) \times I$, such that for any $\eta \in V$ there exists a smooth diffeomorphism Θ_η from a neighborhood of η containing V onto a neighborhood of the origin in $\mathbb{G}$ such that*

$\Theta_\eta(\xi)$ and its first order derivatives with respect to ξ depend on η in a C^α continuous way, locally uniformly in ξ, and for any smooth function $f:\mathbb{G}\to\mathbb{R}$,

$$\widetilde{X}_i\left(f\circ\Theta_\eta\right)(\xi)=\left(Y_if+R_i^\eta f\right)\left(\Theta_\eta(\xi)\right)\ \forall\xi,\eta\in V \tag{2.7}$$

($i=0,1,...,n$) where R_i^η are $C^{r-p_i,\alpha}$ vector fields of weight $\geqslant\alpha-p_i$. Moreover:

1. *The following equivalences hold:*

$$c_1\left|\Theta_\eta(\xi)\right|\leqslant c_2\widetilde{d}(\eta,\xi)\leqslant\left\|\Theta_\eta(\xi)\right\|\leqslant c_3\widetilde{d}(\eta,\xi)\quad\leqslant c_4\left|\Theta_\eta(\xi)\right|^{1/r} \tag{2.8}$$

   for any $\xi,\eta\in V$. Also,

$$c_1\rho^Q\leqslant\left|\widetilde{B}(\xi,\rho)\right|\leqslant c_2\rho^Q \text{ for any } \xi\in V,\rho\leqslant\rho_0 \tag{2.9}$$

   where Q is the homogeneous dimension of the group $\mathbb{G}$ and c_i, ρ_0 are suitable positive constants.

2. *The modulus of the Jacobian determinant of $\xi\mapsto\Theta_\eta(\xi)$ has the form*

$$d\xi=c(\eta)\left(1+O\left(\|u\|\right)\right)du, \tag{2.10}$$

   where

$$c(\eta)=\left|\det\left(\left(\widetilde{X}_{[I]}\right)_{I\in B}\right)_\eta\right|$$

   is a C^α function, bounded and bounded away from zero. (Here B is the set of multiindices giving the basis $\{\widetilde{X}_{[I]}\}_{I\in B}$ involved in the definition of the map Θ_η.) More explicitly, (2.10) means that

$$d\xi=c(\eta)\left[1+\omega(\eta,u)\right]du$$

   with $|\omega(\eta,u)|\leqslant c\|u\|$, ω smooth in u and C^α with respect to η, uniformly in u.

The diffeomorphism $\Theta_\eta(\cdot)$ is defined as the inverse of the exponential function

$$u\mapsto E(u,\eta)=\exp\left(\sum_{I\in B}u_IS_{[I],\eta}\right)(\eta)$$

where the vector fields $S_{[I],\eta}$ are smooth vector fields depending on η in a C^α way (see [6, §3] for the details).

In the next theorem we will show that both $E(\cdot,\eta)$ and $\Theta_\eta(\cdot)$ have derivatives that depend on η in a C^α way. As a consequence we will prove some properties of the coefficients of the vector fields R_i^η.

Theorem 2.10

1. *For every multi-index β the derivatives $\frac{\partial^{|\beta|}E}{\partial u^\beta}(u,\eta)$ and $\frac{\partial^{|\beta|}\Theta_\eta}{\partial\xi^\beta}(\xi)$ depend on η in a C^α way.*

2. *If $R_i^\eta=\sum_{k=1}^N c_{ik}^\eta(u)\,\partial_{u_k}$ then:*

i. the functions $c_{ik}^{\eta}(u)$ (for $0 \leqslant i \leqslant n$) and $\frac{\partial c_{ik}^{\eta}}{\partial u_j}(u)$ (for $1 \leqslant i \leqslant n$) depend on η in a C^{α} way, locally uniformly with respect to u;

ii. the vector fields $\sum_{k=1}^{N} \frac{\partial c_{ik}^{\eta}}{\partial u_j}(u)\,\partial_{u_k}$ (for $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant N$) have weight $\geqslant \alpha - 2$.

Proof. We start with $\frac{\partial^{|\beta|}E}{\partial u^{\beta}}$. We know that

$$E(u,\eta) = \gamma(1,u,\eta)$$

where γ solves the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\gamma(t,u,\eta) = \displaystyle\sum_{I\in B} u_I \left(S_{[I],\eta}\right)_{\gamma(t,u,\eta)} \\ \gamma(0,u,\eta) = \eta. \end{cases}$$

For a fixed η the solution $\gamma(\cdot,\cdot,\eta)$ is smooth; moreover γ depends on η in a C^{α} way. Therefore

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial \gamma}{\partial u_J}(t,u,\eta) = \displaystyle\sum_{I\in B} u_I \frac{\partial S_{[I],\eta}}{\partial \xi}(\gamma(t,u,\eta))\frac{\partial \gamma}{\partial u_J}(t,u,\eta) + \left(S_{[J],\eta}\right)_{\gamma(t,u,\eta)} \\ \dfrac{\partial \gamma}{\partial u_J}(0,u,\eta) = 0. \end{cases}$$

Let now

$$\begin{aligned} \omega(t,u,\eta) &= \frac{\partial \gamma}{\partial u_J}(t,u,\eta), \\ A(t,u,\eta) &= \sum_{I\in B} u_I \frac{\partial S_{[I],\eta}}{\partial \xi}(\gamma(t,u,\eta)), \\ B_J(t,u,\eta) &= \left(S_{[J],\eta}\right)_{\gamma(t,u,\eta)}. \end{aligned}$$

Since $\left(S_{[J],\eta}\right)_{\xi}$ and $\frac{\partial S_{[I],\eta}}{\partial \xi}(\xi)$ are smooth in the ξ variable and C^{α} in the η variable, the functions $A(t,u,\eta)$ and $B_J(t,u,\eta)$ are smooth in (t,u) and C^{α} in η. With the above notation,

$$\begin{cases} \frac{d}{dt}\omega(t,u,\eta) = A(t,u,\eta)\ \omega(t,u,\eta) + B_J(t,u,\eta) \\ \omega(0,u,\eta) = 0 \end{cases}$$

whence we readily see that ω is C^{α} in η. This shows that $\frac{\partial E}{\partial u_J}(u,\eta) = \omega(1,u,\eta)$ has the same property. An iteration of this argument shows that also $\frac{\partial^{|\beta|}E}{\partial u^{\beta}}$ is C^{α} with respect to η.

To prove the analogous result for $\frac{\partial^{|\beta|}\Theta_{\eta}}{\partial \xi^{\beta}}(\xi)$ we differentiate with respect to ξ the identity

$$\xi = E(\Theta_{\eta}(\xi),\eta)$$

finding the matrix identity

$$I = \frac{\partial E}{\partial u}(\Theta_{\eta}(\xi),\eta)\frac{\partial \Theta_{\eta}}{\partial \xi}(\xi)$$

and then

$$\frac{\partial \Theta_\eta}{\partial \xi}(\xi) = \left[\frac{\partial E}{\partial u}(\Theta_\eta(\xi), \eta)\right]^{-1}.$$

Since $\frac{\partial E}{\partial u}(\xi, \eta)$ is smooth in ξ and C^α in η and $\Theta_\eta(\xi)$ is C^α in η, we get the desired result. An iteration of this argument shows that also $\frac{\partial^{|\beta|}\Theta_\eta}{\partial \xi^\beta}(\xi)$ is C^α in η.

To prove 2.i, let $f(u) = u_k$ and $g_\eta(\xi) = f(\Theta_\eta(\xi)) = (\Theta_\eta(\xi))_k$. Then, by (2.7), we have

$$\widetilde{X}_i g_\eta(\xi) = (Y_i u_k)(\Theta_\eta(\xi)) + c_{ik}^\eta(\Theta_\eta(\xi)),$$

so that

$$c_{ik}^\eta(u) = \widetilde{X}_i g_\eta\left(\Theta_\eta^{-1}(u)\right) - Y_i u_k. \tag{2.11}$$

Since $Y_i u_k$ is independent of η it is enough to consider the term $\widetilde{X}_i g_\eta\left(\Theta_\eta^{-1}(u)\right)$. Let us write

$$\begin{aligned}
&\left|\widetilde{X}_i g_{\eta_1}\left(\Theta_{\eta_1}^{-1}(u)\right) - \widetilde{X}_i g_{\eta_2}\left(\Theta_{\eta_2}^{-1}(u)\right)\right| \\
&\leqslant \left|\widetilde{X}_i g_{\eta_1}\left(\Theta_{\eta_1}^{-1}(u)\right) - \widetilde{X}_i g_{\eta_1}\left(\Theta_{\eta_2}^{-1}(u)\right)\right| + \left|\widetilde{X}_i g_{\eta_1}\left(\Theta_{\eta_2}^{-1}(u)\right) - \widetilde{X}_i g_{\eta_2}\left(\Theta_{\eta_2}^{-1}(u)\right)\right|
\end{aligned}$$

and

$$\widetilde{X}_i g_\eta(\xi) = \sum \widetilde{b}_{ij}(\xi) \frac{\partial g_\eta}{\partial \xi_j}(\xi).$$

By Assumption A the coefficients $\widetilde{b}_{ij}$ are at least Lipschitz. Since $\frac{\partial g_\eta}{\partial \xi_j}(\xi)$ are smooth in ξ we have

$$\begin{aligned}
\left|\widetilde{X}_i g_{\eta_1}\left(\Theta_{\eta_1}^{-1}(u)\right) - \widetilde{X}_i g_{\eta_1}\left(\Theta_{\eta_2}^{-1}(u)\right)\right| &\leqslant c\left|\Theta_{\eta_1}^{-1}(u) - \Theta_{\eta_2}^{-1}(u)\right| \\
&\leqslant c\left|\eta_1 - \eta_2\right|^\alpha.
\end{aligned}$$

Also, since $\frac{\partial g_\eta}{\partial \xi_j}(\xi)$ depends on η in a C^α way, we have

$$\left|\widetilde{X}_i g_{\eta_1}\left(\Theta_{\eta_2}^{-1}(u)\right) - \widetilde{X}_i g_{\eta_2}\left(\Theta_{\eta_2}^{-1}(u)\right)\right| \leqslant c\left|\eta_1 - \eta_2\right|^\alpha.$$

By (2.11) this shows that $\eta \mapsto c_{ik}^\eta(u)$ is C^α. Let us consider now

$$\frac{\partial c_{ik}^\eta}{\partial u_j}(u) = \nabla_\xi\left(\widetilde{X}_i g_\eta\right)\left(\Theta_\eta^{-1}(u)\right) \cdot \frac{\partial \Theta_\eta^{-1}(u)}{\partial u_j} - \frac{\partial}{\partial u_j}(Y_i u_k).$$

Since $\frac{\partial \Theta_\eta^{-1}(u)}{\partial u_j}$ depends on η in a C^α way it is enough to study $\nabla_\xi\left(\widetilde{X}_i g_\eta\right)\left(\Theta_\eta^{-1}(u)\right)$. We have

$$\frac{\partial}{\partial \xi_\ell}\widetilde{X}_i g_\eta(\xi) = \sum \frac{\partial \widetilde{b}_{ij}}{\partial \xi_\ell}(\xi) \frac{\partial g_\eta}{\partial \xi_j}(\xi) + \sum \widetilde{b}_{ij}(\xi) \frac{\partial^2 g_\eta}{\partial \xi_\ell \partial \xi_j}(\xi).$$

By Assumption A, for $i \neq 0$, $\widetilde{b}_{ij} \in C^{r-1,\alpha}$, so that $\frac{\partial \widetilde{b}_{ij}}{\partial \xi_\ell} \in C^{r-2,\alpha}$. Since for $r = 2$ we have $\alpha = 1$, $\frac{\partial \widetilde{b}_{ij}}{\partial \xi_\ell}$ is at least Lipschitz therefore $\frac{\partial}{\partial \xi_\ell}\widetilde{X}_i g_\eta(\xi)$ is Lipschitz with respect to ξ and C^α with respect to η. The proof now follows as in the previous case.

To show 2.ii, we first note that, from the proof of [6, Prop. 3.5], one reads that

$$\left|c_{ik}^{\eta}(u)\right| \leqslant c\,|u|^{r-1+\alpha}. \tag{2.12}$$

On the other hand, we know that $c_{ik}^{\eta}(\cdot) \in C^{r-1,\alpha}$, hence the Taylor expansion of $c_{ik}^{\eta}(\cdot)$ and the bound (2.12) imply

$$\left|\frac{\partial c_{ik}^{\eta}}{\partial u_j}(u)\right| \leqslant c\,|u|^{r-2+\alpha} \leqslant c\,\|u\|^{\alpha_k-2+\alpha}.$$

This implies 2.ii. ■

The assertions on the "weight" of the remainders R_i^{η} in point 2.ii of the previous theorem in particular mean that, whenever $f : \mathbb{G} \to \mathbb{R}$ is homogeneous of degree $-k$ (with respect to the dilations $D(\lambda)$), then near the origin

$$\left|R_i^{\eta} f(u)\right| \leqslant \frac{c}{\|u\|^{k+p_i-\alpha}} \text{ for } i = 0, 1, ..., n. \tag{2.13}$$

Moreover, the statements 2.i and 2.ii in the above theorem immediately imply:

Corollary 2.11 *All the differential operators D_{ij}^{η} defined by the compositions*

$$Y_j R_i^{\eta},\ R_i^{\eta} Y_j,\ R_i^{\eta} R_j^{\eta}\ \ (i, j = 1, 2, ..., n)$$

satisfy the bound

$$\left|D_{ij}^{\eta} f(u)\right| \leqslant \frac{c}{\|u\|^{k+2-\alpha}} \tag{2.14}$$

for u in a neighborhood of the origin, whenever $f : \mathbb{G} \to \mathbb{R}$ is $D(\lambda)$-homogeneous of degree $-k$. Also, the coefficients of D_{ij}^{η} depend on η in a C^{α} way.

Next, we have to point out some properties related to the volume of metric balls.

Remark 2.12 *In contrast with (2.9), if we apply the estimates (2.3) for x in the neighborhood $U(x_0)$ where the lifting theorem applies, we find the following useful inequalities*

$$c_1\left(\frac{r_1}{r_2}\right)^{p} \leqslant \frac{|B(x, r_1)|}{|B(x, r_2)|} \leqslant c_2\left(\frac{r_1}{r_2}\right)^{Q} \tag{2.15}$$

for any r_1, r_2 with $\rho_0 > r_1 > r_2 > 0$. This follows from the inequalities $p \leqslant |\mathcal{I}| \leqslant Q$, holding for each $\mathcal{I}$ in the sums appearing in (2.3).

The following nonsmooth version of a well-known result by Sánchez-Calle [34] and Nagel, Stein, Wainger [31], has been proved in [5], and allows one to compare the volume of balls in the lifted and in the original variables.

Theorem 2.13 *Let x_0, $U(x_0)$, and I be as in the lifting theorem. Then, up to possibly shrinking the set $U(x_0)$, there exist positive constants c_1, c_2, ρ_0, and $\delta \in (0, 1)$ such that for any $(x, h) \in U(x_0) \times I$, any $y \in B(x, \delta\rho)$, $0 < \rho < \rho_0$, we have*

$$c_1\frac{\left|\widetilde{B}((x, h), \rho)\right|}{|B(x, \rho)|} \leqslant \int_{\mathbb{R}^m} \chi_{\widetilde{B}((x,h),\rho)}(y, s)\, ds \leqslant c_2\frac{\left|\widetilde{B}((x, h), \rho)\right|}{|B(x, \rho)|}. \tag{2.16}$$

Actually the second inequality holds for every $y \in U(x_0)$. Also, the projection of $\widetilde{B}((x,h),\rho)$ on $\mathbb{R}^p$ is exactly $B(x,\rho)$.

Remark 2.14 *Actually (2.16) is stated in [5] when X_0 is lacking; however, the proof given in [5] relies on the analog result which holds for smooth Hörmander's vector fields. In turn, the result for smooth Hörmander's vector fields has been proved in [34] when X_0 is lacking, while just one of the two inequalities in (2.16) has been proved in [31] also in presence of X_0; however, as shown in [18], the same argument used in [31] allows one to prove also the other inequality. Hence (2.16) holds in the smooth case also in presence of X_0, and the same is true for nonsmooth Hörmander's vector fields.*

Notation. Throughout the paper we will handle four types of vector fields, which will be regarded as differential operators acting on different variables. The vector fields

$$Y_i \text{ and } R_i^\eta$$

act on the variable u in the group $\mathbb{G}$ (that is, they are written in the coordinates u), and we will often have $u = \Theta_\eta(\xi)$; moreover, the coefficients of the R_i^η's depend on the variable η as a parameter. The vector fields

$$X_i \text{ and } \widetilde{X}_i$$

act on $\mathbb{R}^p$, $\mathbb{R}^N$, respectively; they are often applied on a function of two variables, and in this case, they will *always* be seen as acting on the *first* variable, which in $\mathbb{R}^p$ is called x and in $\mathbb{R}^N$ is called $\xi = (x,h)$. For instance,

$$X_i f(x,y) = X_i[f(\cdot,y)](x);$$
$$\widetilde{X}_i f(\xi,\eta) = \widetilde{X}_i[f(\cdot,\eta)](\xi).$$

These conventions will be applied consistently throughout the paper.

3 Geometric estimates

In this section we establish some estimates which relate the growth of some kernels defined in the lifted space with that of kernels defined in the original space $\mathbb{R}^p$. The fact that the volume of metric balls in $\mathbb{R}^p$ does not behave like a fixed power of the radius makes these estimates delicate to be proved. These results will be fundamental throughout the following.

Let $\Omega \subset \mathbb{R}^p$ be a domain where our assumptions are satisfied, $\Omega' \Subset \Omega, x_0 \in \Omega'$, $U(x_0) = B(x_0,r_0) \Subset \Omega$ a neighborhood of x_0 where the lifting and approximation theorem is applicable, R a number small enough so that $B(x,2R) \Subset \Omega$ for any $x \in U(x_0)$. Let $\varphi,\psi \in C_0^\infty(\mathbb{R}^m)$ be supported in the neighborhood I of the origin which appears in Theorem 2.7. Shrinking if necessary $U(x_0)$ and the supports of φ,ψ, we can assume that $4r_0 \leqslant R$ and

$$\widetilde{d}((x,h),(y,k)) < R$$

for $x,y \in U(x_0)$ and h,k in the supports of φ,ψ, respectively. With this notation, we have the following:

Lemma 3.1 *For every $\beta \in \mathbb{R}$ there exists $c > 0$ such that*

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{\psi(h)}{\left\|\Theta_{(y,k)}(x,h)\right\|^{Q-\beta}} dh\, \varphi(k)\, dk \leqslant c \int_{d(x,y)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} dr$$

for any $x, y \in U(x_0)$.

This is just [31, Thm. 5], in our nonsmooth context. It can be proved at the same way using Theorem 2.13.

It is convenient to give a name to the function which appears in the previous Lemma, since it will be a central object throughout the following.

Definition 3.2 *For $x, y \in U(x_0), x \neq y$ and $\beta \in \mathbb{R}$, let*

$$\phi_\beta(x,y) = \int_{d(x,y)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} dr. \tag{3.1}$$

The estimate in the previous lemma is made more readable by the next:

Lemma 3.3 *For $x, y \in U(x_0), x \neq y$, the following inequalities hold:*

$$\phi_\beta(x,y) \leqslant \begin{cases} c\dfrac{d(x,y)^\beta}{|B(x,d(x,y))|} & \text{for } \beta < p \\[2ex] c\dfrac{d(x,y)^p}{|B(x,d(x,y))|} \log \dfrac{R}{d(x,y)} & \text{for } \beta = p \\[2ex] c\dfrac{d(x,y)^p}{|B(x,d(x,y))|} R^{\beta-p} & \text{for } \beta > p \end{cases}$$

(recall that p is the Euclidean dimension of the space of variables x, y).

Proof. By (2.15) we have:

$$|B(x,r)| \geqslant c\,|B(x,d(x,y))| \left(\frac{r}{d(x,y)}\right)^p \text{ for } d(x,y) < r < R.$$

Hence, for $\beta < p$,

$$\begin{aligned} \int_{d(x,y)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} dr &\leqslant c\frac{d(x,y)^p}{|B(x,d(x,y))|} \int_{d(x,y)}^{R} \frac{r^{\beta-1}}{r^p} dr \\ &= c\frac{d(x,y)^p}{|B(x,d(x,y))|} \left[\frac{d(x,y)^{\beta-p} - R^{\beta-p}}{p-\beta}\right] \\ &\leqslant c\frac{d(x,y)^p}{|B(x,d(x,y))|} d(x,y)^{\beta-p} = c\frac{d(x,y)^\beta}{|B(x,d(x,y))|}. \end{aligned}$$

The proof in other cases is analogous. ■

By a standard computation the previous lemma immediately implies

Corollary 3.4 *For any $\beta > 0$ the following bounds hold:*

$$\Psi_\beta(x,r) \equiv \int_{d(x,y)<r} \phi_\beta(x,y)\,dy \leqslant \begin{cases} cr^\beta & \text{if } \beta < p \\ c_\varepsilon r^{\beta-\varepsilon} & \text{if } \beta = p \ (\text{any } \varepsilon > 0) \\ cr^p & \text{if } \beta > p \end{cases}$$

where in case $\beta < p$ the constant c is independent of R. In any case, $\Psi_\beta(x,r) \to 0$ as $r \to 0$, uniformly in x.

Theorem 3.5 *We have the following:*

1) there exists $c > 0$ such that for every $\beta, \gamma > 0$:

$$\int_{U(x_0)} \phi_\beta(x,y)\,\phi_\gamma(y,z)\,dy \leqslant c\left(\frac{1}{\beta} + \frac{1}{\gamma}\right)\phi_{\beta+\gamma}(x,z)$$

for every $x, z \in U(x_0)$.

2) there exists $c > 0$ such that for every $\gamma > Q$

$$\phi_\gamma(x,y) \leqslant cR^{\gamma-Q}$$

for every $x, y \in U(x_0)$. (Recall that Q is the homogeneous dimension of the group in the lifted space).

Remark 3.6 *Comparing point 2) in the statement of the above theorem with the case $\beta > p$ in the statement of Lemma 3.3, one can see why in our context it is necessary to work with the functions ϕ_β instead of the simpler functions*

$$\psi_\beta(x,y) = \frac{d(x,y)^\beta}{|B(x,d(x,y))|}.$$

The point is that the functions ϕ_β are bounded for β large enough, so that an iterative construction involving integrals of the kind

$$\int_{U(x_0)} \phi_\beta(x,y)\,\phi_\gamma(y,z)\,dy$$

ends with a bounded function. On the other hand, if one tries to prove an analog of the previous theorem for the ψ_β's, the best upper bound one can find is

$$\frac{d(x,y)^p}{|B(x,d(x,y))|}$$

which is generally unbounded, because $|B(x,d(x,y))| \geqslant cd(x,y)^Q$ with $Q > p$. This "dimensional gap" occurs in our general context since the measure of a ball does not behave like a fixed power of the radius.

Proof. We start by noting that

$$\phi_\beta(x,y) \leqslant c\phi_\beta(y,x).$$

Indeed,

$$\phi_\beta(x,y) = \int_{d(x,y)}^R \frac{r^{\beta-1}}{|B(x,r)|}dr \leqslant c\int_{d(x,y)}^R \frac{r^{\beta-1}}{|B(y,r)|}dr = c\phi_\beta(y,x)$$

since for $d(x,y) < r$ we have $B(y,r) \subset B(x,2r)$; for $x \in U(x_0)$ and $r \leqslant R$ the doubling condition is applicable and gives

$$|B(y,r)| \leqslant |B(x,2r)| \leqslant c\,|B(x,r)|.$$

Also, since $R \geqslant 4r_0 \geqslant 2d(x,y)$ for any $x, y \in U(x_0)$, we have

$$\begin{aligned}
\int_{\frac{1}{2}d(x,y)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} dr &= \int_{\frac{1}{2}d(x,y)}^{R/2} \frac{r^{\beta-1}}{|B(x,r)|} dr + \int_{R/2}^{R} \frac{r^{\beta-1}}{|B(x,r)|} dr \\
&\leqslant \frac{c}{2^{\beta}} \int_{d(x,y)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} dr + \int_{d(x,y)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} dr \\
&\leqslant c \int_{d(x,y)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} dr \qquad (3.2)
\end{aligned}$$

where c is independent of β. Now,

$$\begin{aligned}
&\int_{U(x_0)} \phi_{\beta}(x,y)\,\phi_{\gamma}(y,z)\,dy \\
&= \int_{d(x,y)<\frac{1}{2}d(x,z)} (\ldots)\,dy + \int_{d(z,y)<\frac{1}{2}d(x,z)} (\ldots)\,dy + \int_{\substack{d(x,y)\geqslant\frac{1}{2}d(x,z)\\ d(z,y)\geqslant\frac{1}{2}d(x,z)}} (\ldots)\,dy \\
&\equiv I + II + III.
\end{aligned}$$

To bound I we note that $\frac{1}{2}d(y,z) \leqslant d(x,z) \leqslant 2d(y,z)$, hence

$$\begin{aligned}
I &= \int_{d(x,y)<\frac{1}{2}d(x,z)} \left(\int_{d(x,y)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} dr \int_{d(y,z)}^{R} \frac{s^{\gamma-1}}{|B(z,s)|} ds \right) dy \\
&\leqslant c \int_{\frac{1}{2}d(x,z)}^{R} \frac{s^{\gamma-1}}{|B(z,s)|} ds \int_{d(x,y)<\frac{1}{2}d(x,z)} \left(\int_{d(x,y)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} dr \right) dy
\end{aligned}$$

and, applying Fubini's theorem in the integral in $drdy$,

$$d(x,y) < \frac{1}{2}d(x,z),\, d(x,y) < r < R \Longrightarrow 0 < r < R,\, d(x,y) < \min\left(\frac{1}{2}d(x,z), r\right),$$

we have that

$$\begin{aligned}
I &\leqslant c \int_{\frac{1}{2}d(x,z)}^{R} \frac{s^{\gamma-1}}{|B(z,s)|} ds \int_{0}^{R} \frac{r^{\beta-1}}{|B(x,r)|} \left(\int_{d(x,y)<\frac{1}{2}d(x,z)\wedge r} dy \right) dr \\
&= c \int_{\frac{1}{2}d(x,z)}^{R} \frac{s^{\gamma-1}}{|B(z,s)|} ds \left\{ \int_{0}^{\frac{1}{2}d(x,z)} \frac{r^{\beta-1}}{|B(x,r)|} \left(\int_{d(x,y)<r} dy \right) dr \right. \\
&\qquad \left. + \int_{\frac{1}{2}d(x,z)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} \left(\int_{d(x,y)<\frac{1}{2}d(x,z)} dy \right) dr \right\} \\
&\leqslant c \int_{\frac{1}{2}d(x,z)}^{R} \frac{s^{\gamma-1}}{|B(z,s)|} ds \left\{ \int_{0}^{\frac{1}{2}d(x,z)} r^{\beta-1} dr + \int_{\frac{1}{2}d(x,z)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} |B(x, d(x,z))|\, dr \right\} \\
&\equiv I_A + I_B.
\end{aligned}$$

In turn,

$$I_A = \frac{c}{\beta}\left(\frac{1}{2}d\left(x,z\right)\right)^{\beta}\int_{\frac{1}{2}d(x,z)}^{R}\frac{s^{\gamma-1}}{\left|B\left(z,s\right)\right|}ds \leqslant \frac{c}{\beta}\int_{d(x,z)}^{R}\frac{s^{\beta+\gamma-1}}{\left|B\left(z,s\right)\right|}ds$$

and, using the notation $B(x;z) = B\left(x, d\left(x,z\right)\right)$ and applying (3.2),

$$\begin{aligned} I_B &\leqslant c\left|B\left(x;z\right)\right|\int_{d(x,z)}^{R}\frac{s^{\gamma-1}}{\left|B\left(z,s\right)\right|}ds\int_{d(x,z)}^{R}\frac{r^{\beta-1}}{\left|B\left(x,r\right)\right|}dr \\ &= c\left|B\left(x;z\right)\right|\int_{d(x,z)}^{R}\frac{s^{\gamma-1}}{\left|B\left(z,s\right)\right|}\left(\int_{d(x,z)}^{s}\frac{r^{\beta-1}}{\left|B\left(x,r\right)\right|}dr + \int_{s}^{R}\frac{r^{\beta-1}}{\left|B\left(x,r\right)\right|}dr\right)ds \\ &\equiv I_{B_1} + I_{B_2}, \end{aligned}$$

where, since in I_{B_2} we have $d\left(x,z\right) < s < r$, then $\left|B\left(x;z\right)\right| \leqslant \left|B\left(z,s\right)\right|$ and therefore

$$I_{B_2} \leqslant c\int_{d(x,z)}^{R}s^{\gamma-1}\left(\int_{s}^{R}\frac{r^{\beta-1}}{\left|B\left(x,r\right)\right|}dr\right)ds \tag{3.3}$$

applying Fubini's theorem:
$d\left(x,z\right) < s < R, s < r < R \Longrightarrow d\left(x,z\right) < r < R, d\left(x,z\right) < s < r$

$$\begin{aligned} &= c\int_{d(x,z)}^{R}\frac{r^{\beta-1}}{\left|B\left(x,r\right)\right|}\left(\int_{d(x,z)}^{r}s^{\gamma-1}ds\right)dr \\ &\leqslant c\int_{d(x,z)}^{R}\frac{r^{\beta-1}}{\left|B\left(x,r\right)\right|}\left(\int_{0}^{r}s^{\gamma-1}ds\right)dr \\ &= \frac{c}{\gamma}\int_{d(x,z)}^{R}\frac{r^{\beta+\gamma-1}}{\left|B\left(x,r\right)\right|}dr. \end{aligned}$$

As to I_{B_1}, applying once more Fubini's theorem,

$$d\left(x,z\right) < s < R, d\left(x,z\right) < r < s \Longrightarrow d\left(x,z\right) < r < R, r < s < R,$$

we have

$$I_{B_1} = c\left|B\left(x;z\right)\right|\int_{d(x,z)}^{R}\frac{r^{\beta-1}}{\left|B\left(x,r\right)\right|}\left(\int_{r}^{R}\frac{s^{\gamma-1}}{\left|B\left(z,s\right)\right|}ds\right)dr$$

since $d\left(x,z\right) < r$ implies $\left|B\left(x;z\right)\right| \leqslant \left|B\left(x,r\right)\right|$,

$$\leqslant c\int_{d(x,z)}^{R}r^{\beta-1}\left(\int_{r}^{R}\frac{s^{\gamma-1}}{\left|B\left(z,s\right)\right|}ds\right)dr$$

and this can be handled as I_{B_2} (see (3.3)).

We have therefore proved that I satisfies the desired bound. The term II can be handled analogously (by symmetry).

Let us come to the bound on III. Since

$$d\left(x,y\right) \geqslant \frac{1}{2}d\left(x,z\right),\ d\left(z,y\right) \geqslant \frac{1}{2}d\left(x,z\right) \text{ and } d\left(x,y\right) < r < R, d\left(y,z\right) < s < R$$

imply

$$\frac{1}{2}d(x,z)<r<R, \frac{1}{2}d(x,z)<s<R$$

and

$$\frac{1}{2}d(x,z)<d(x,y)<r, \frac{1}{2}d(x,z)<d(y,z)<s,$$

applying Fubini's theorem in the triple integral gives

$$\begin{aligned}
III &= \int_{\frac{1}{2}d(x,z)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} \int_{\frac{1}{2}d(x,z)}^{R} \frac{s^{\gamma-1}}{|B(z,s)|} \left(\int_{\substack{\frac{1}{2}d(x,z)<d(x,y)<r \\ \frac{1}{2}d(x,z)<d(y,z)<s}} dy \right) ds dr \\
&\leqslant \int_{\frac{1}{2}d(x,z)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} \left(\int_{\frac{1}{2}d(x,z)}^{R} \frac{s^{\gamma-1}}{|B(z,s)|} |B(x,r)\cap B(z,s)|\, ds \right) dr \\
&= \int_{\frac{1}{2}d(x,z)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} \left(\int_{\frac{1}{2}d(x,z)}^{r} + \int_{r}^{R} \right) \left(\frac{s^{\gamma-1}}{|B(z,s)|} |B(x,r)\cap B(z,s)|\, ds \right) dr \\
&\equiv III_A + III_B.
\end{aligned}$$

Now,

$$\begin{aligned}
III_A &\leqslant \int_{\frac{1}{2}d(x,z)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} \left(\int_{\frac{1}{2}d(x,z)}^{r} s^{\gamma-1} ds \right) dr \\
&\leqslant \int_{\frac{1}{2}d(x,z)}^{R} \frac{r^{\beta-1}}{|B(x,r)|} \left(\int_{0}^{r} s^{\gamma-1} ds \right) dr \\
&= \frac{1}{\gamma} \int_{\frac{1}{2}d(x,z)}^{R} \frac{r^{\beta+\gamma-1}}{|B(x,r)|} dr \\
&\leqslant \frac{c}{\gamma} \int_{d(x,z)}^{R} \frac{r^{\beta+\gamma-1}}{|B(x,r)|} dr
\end{aligned}$$

by (3.2). As to III_B, since

$$\frac{1}{2}d(x,z)<r<R, r<s<R \Longrightarrow \frac{1}{2}d(x,z)<s<R, \frac{1}{2}d(x,z)<r<s,$$

by Fubini's theorem,

$$\begin{aligned}
III_B &\leqslant \int_{\frac{1}{2}d(x,z)}^{R} r^{\beta-1} \left(\int_{r}^{R} \frac{s^{\gamma-1}}{|B(z,s)|} ds \right) dr \\
&= \int_{\frac{1}{2}d(x,z)}^{R} \frac{s^{\gamma-1}}{|B(z,s)|} \left(\int_{\frac{1}{2}d(x,z)}^{s} r^{\beta-1} dr \right) ds \\
&\leqslant \int_{\frac{1}{2}d(x,z)}^{R} \frac{s^{\gamma-1}}{|B(z,s)|} \left(\int_{0}^{s} r^{\beta-1} dr \right) ds \\
&= \frac{1}{\beta} \int_{\frac{1}{2}d(x,z)}^{R} \frac{s^{\beta+\gamma-1}}{|B(z,s)|} ds \\
&\leqslant \frac{c}{\beta} \int_{d(x,z)}^{R} \frac{s^{\beta+\gamma-1}}{|B(z,s)|} ds.
\end{aligned}$$

This shows that also III satisfies the desired bound, and point 1 of the theorem is proved.

As to point 2, the volume estimate (2.15) gives, for any $r < R$,

$$|B(x,r)| \geqslant c\left(\frac{r}{R}\right)^{Q}|B(x,R)| \geqslant cr^{Q}$$

since

$$\inf_{x\in\Omega'}|B(x,R)| \geqslant c > 0$$

as easily follows by the doubling condition. Then, for any $\gamma > Q$,

$$\int_{d(x,y)}^{R}\frac{r^{\gamma-1}}{|B(x,r)|}dr \leqslant \int_{d(x,y)}^{R}\frac{r^{\gamma-1}}{cr^{Q}}dr \leqslant c\int_{0}^{R}r^{\gamma-1-Q}dr = cR^{\gamma-Q}.$$

■

In order to deal with continuity matters of the next sections, we will need the following

Proposition 3.7 *Let $T \subset U(x_0)$ be an open set.*

(i) Let $f(x,y), g(x,y)$ be two functions defined in $T\times T$ satisfying

$$|f(x,y)| \leqslant c\phi_{\beta}(x,y);$$
$$|g(x,y)| \leqslant c\phi_{\gamma}(x,y),$$

for some $\beta,\gamma > 0$ and any $x,y \in T, x \neq y$. Assume that both f and g are continuous in the joint variables (x,y) for $x \neq y$. Then the function

$$h(x,y) = \int_{T} f(x,z)\,g(z,y)\,dz$$

is jointly continuous in $T\times T$ for $x \neq y$.

(ii) Let $f(x,y)$ be a function defined in $T\times T$ satisfying

$$|f(x,y)| \leqslant c\frac{d(x,y)^{\beta}}{|B(x,d(x,y))|}$$

for some $\beta > 0$, $f(x,y)$ measurable with respect to y for every x, and continuous with respect to x at any $x \neq y$, for a.e. y. Then the function

$$m(x) = \int_{T} f(x,y)\,dy$$

is continuous in T.

Proof. (i) Let $\varphi_{\varepsilon} : [0,\infty) \to [0,1]$ be a continuous function such that $\varphi_{\varepsilon}(t) = 0$ for $t \leqslant \varepsilon/2$, $\varphi_{\varepsilon}(t) = 1$ for $t \geqslant \varepsilon$, and define

$$f_{\varepsilon}(x,y) = f(x,y)\,\varphi_{\varepsilon}(d(x,y));$$
$$g_{\varepsilon}(x,y) = g(x,y)\,\varphi_{\varepsilon}(d(x,y));$$
$$h_{\varepsilon}(x,y) = \int_{T} f_{\varepsilon}(x,z)\,g_{\varepsilon}(z,y)\,dz.$$

For any fixed $\varepsilon > 0$ the function

$$f_\varepsilon(x,z)\,g_\varepsilon(z,y)$$

is measurable with respect to z for every (x,y) and, for any $z \in T$, continuous in the joint variables (x,y). Moreover by our assumption on f, g and Lemma 3.3,

$$|f_\varepsilon(x,z)\,g_\varepsilon(z,y)| \leqslant c\frac{1}{|B(x,\varepsilon)|}\frac{1}{|B(y,\varepsilon)|} \leqslant c(\varepsilon).$$

Then, by Lebesgue theorem, h_ε is continuous in $T\times T$, since T has finite measure. Let us show that $h_\varepsilon(x,y) \to h(x,y)$ locally uniformly for $x \neq y$, which will imply the continuity of h. To see this, let us write

$$\begin{aligned} h_\varepsilon(x,y) - h(x,y) &= \int_T [f_\varepsilon(x,z) - f(x,z)]\,g_\varepsilon(z,y)\,dz \\ &+ \int_T f(x,z)\,[g_\varepsilon(z,y) - g(z,y)]\,dz \end{aligned}$$

and

$$\begin{aligned} |h_\varepsilon(x,y) - h(x,y)| &\leqslant c\int_{d(x,z)<\varepsilon} \phi_\beta(x,z)\,\phi_\gamma(z,y)\,dz \\ &+ c\int_{d(z,y)<\varepsilon} \phi_\beta(x,z)\,\phi_\gamma(z,y)\,dz = I + II. \end{aligned}$$

Now, for $d(x,y) \geqslant \delta > 0$ and $\varepsilon < \delta/2$, $d(x,z) < \varepsilon$ implies $d(z,y) > \delta/2$, hence by Lemma 3.3 $\phi_\gamma(z,y) \leqslant c(\delta)$ and

$$I \leqslant c(\delta)\int_{d(x,z)<\varepsilon} \phi_\beta(x,z)\,dz = c(\delta)\,\Psi_\beta(x,\varepsilon) \to 0$$

as $\varepsilon \to 0$, uniformly for $d(x,y) \geqslant \delta > 0$ (see Corollary 3.4). Analogously

$$II \leqslant c(\delta)\int_{d(z,y)<\varepsilon} \phi_\gamma(z,y)\,dz = c(\delta)\,\Psi_\gamma(y,\varepsilon) \to 0$$

as $\varepsilon \to 0$, uniformly for $d(x,y) \geqslant \delta > 0$. Hence (i) is proved. The proof of (ii) is similar but easier. ■

4 The parametrix method

Let x_0, $U(x_0)$, and I be as in the previous sections. To shorten notation, in the following we will write U instead of $U(x_0)$. We will denote by ξ, η lifted variables ranging in the small domain

$$V \subset U \times I \subset \mathbb{R}^{p+m},$$

as in the approximation theorem. By known results of Folland [13], the operator

$$\mathcal{L} = \sum_{i=1}^{p} Y_i^2 + Y_0$$

possesses a fundamental solution Γ on $\mathbb{G}$, left invariant and homogeneous of degree $2-Q$. (Recall that, in order for Folland's theory to be applicable, the homogeneous dimension Q of $\mathbb{G}$ must be $\geqslant 3$. However, this restriction only rules out uniformly elliptic operators in two variables).

In particular, this means that for some positive constant c we have

$$\begin{aligned}
\left|\Gamma\left(\Theta_\eta\left(\xi\right)\right)\right| &\leqslant \frac{c}{\left\|\Theta_\eta\left(\xi\right)\right\|^{Q-2}};\\
\left|\left(Y_i\Gamma\right)\left(\Theta_\eta\left(\xi\right)\right)\right|, \left|\widetilde{X}_i\left[\Gamma\left(\Theta_\eta\left(\xi\right)\right)\right]\right| &\leqslant \frac{c}{\left\|\Theta_\eta\left(\xi\right)\right\|^{Q-1}};\\
\left|\left(Y_iY_j\Gamma\right)\left(\Theta_\eta\left(\xi\right)\right)\right|, \left|\widetilde{X}_i\widetilde{X}_j\left[\Gamma\left(\Theta_\eta\left(\xi\right)\right)\right]\right| &\leqslant \frac{c}{\left\|\Theta_\eta\left(\xi\right)\right\|^{Q}};\\
\left|\left(Y_0\Gamma\right)\left(\Theta_\eta\left(\xi\right)\right)\right|, \left|\widetilde{X}_0\left[\Gamma\left(\Theta_\eta\left(\xi\right)\right)\right]\right| &\leqslant \frac{c}{\left\|\Theta_\eta\left(\xi\right)\right\|^{Q}},
\end{aligned} \tag{4.1}$$

for every $\eta,\xi\in V$, $\eta\neq\xi$, where the $\widetilde{X}$-derivatives act on the ξ variable. Recall that, according to the Notation stated at the end of § 2, we will always assume that differential operators act on the ξ variable of $\Gamma\left(\Theta_\eta\left(\xi\right)\right)$. Also, recall that by (2.8) $\left\|\Theta_\eta\left(\xi\right)\right\|$ is equivalent to $\widetilde{d}\left(\eta,\xi\right)$.

Let us define the following (local) *parametrix for the operator* L. For $x,y\in U$, we set

$$P\left(x,y\right)=\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\Gamma\left(\Theta_{(y,k)}\left(x,h\right)\right)\varphi\left(h\right)dh\right)\varphi\left(k\right)dk, \tag{4.2}$$

where $\varphi\in C_0^\infty\left(\mathbb{R}^m\right)$ is a cutoff function fixed once and for all, equal to one in a neighborhood of the origin and supported in I. It is worth telling that the alternative definition

$$\int_{\mathbb{R}^m}\Gamma\left(\Theta_{(y,0)}\left(x,h\right)\right)\varphi\left(h\right)dh$$

of the parametrix (as in [31, eq.(20)]) would be fit for the purposes of this section, but not for those of section 5. Let us also note that, in case our vector fields X_i were free up to step s, the lifting procedure would be unnecessary, we would simply have $\widetilde{X}_i=X_i$ and:

$$P\left(x,y\right)=\Gamma\left(\Theta_y\left(x\right)\right).$$

As already sketched in the introduction, the strategy is then the following. We look for a fundamental solution for L of the form

$$\gamma\left(x,y\right)=P\left(x,y\right)+J\left(x,y\right)$$

where

$$J\left(x,y\right)=\int_U P\left(x,z\right)\Phi\left(z,y\right)dz.$$

In turn, we will find Φ as the series

$$\Phi\left(z,y\right)=\sum_{j=1}^{\infty}Z_j\left(z,y\right)\ \text{ for } z\neq y \tag{4.3}$$

where the Z_j's are defined inductively by

$$Z_1(x,y) = LP(x,y) \tag{4.4}$$
$$Z_{j+1}(x,y) = \int_U Z_1(x,z)\, Z_j(z,y)\, dz \quad \text{for } x \neq y.$$

More precisely, we will eventually find that the above identities need to be slightly modified multiplying some of the involved functions by a suitable coefficient $c_0(x)$; the necessity of this will be clear in the following.

Before carrying out this plan step by step, let us clarify the way how our constants will depend on the vector fields:

Dependence of the constants. All the constants in the upper bounds proved in this section will depend on the vector fields X_i's only through the following quantities:

(i) the norms $C^{r-1,\alpha}(\Omega)$ of the coefficients of X_i $(i = 1,2,...,n)$ and the norms $C^{r-2,\alpha}(\Omega)$ of the coefficients of X_0;

(ii) a positive constant c_0 such that the following bound holds:

$$\inf_{x\in\Omega} \max_{|I_1|,|I_2|,...,|I_p|\leqslant r} \left|\det\left(\left(X_{[I_1]}\right)_x, \left(X_{[I_2]}\right)_x, ..., \left(X_{[I_p]}\right)_x\right)\right| \geqslant c_0,$$

where "det" denotes the determinant of the $p\times p$ matrix having the vectors $\left(X_{[I_i]}\right)_x$ as rows.

Proposition 4.1 (Properties of P) *Under the above assumptions and with the above notation, we have, for any $x,y\in U$:*

$$P(\cdot,y) \in C^{\infty}(U\setminus\{y\}); \tag{4.5}$$
$$P(x,\cdot) \in C^{\alpha}_{loc}(U\setminus\{x\}); \tag{4.6}$$
$$P \in C(U\times U\setminus\Delta); \tag{4.7}$$
$$X_iP,\ X_jX_iP,\ X_0P \in C(U\times U\setminus\Delta) \tag{4.8}$$

for $i,j = 1,2,...,n$, where $\Delta = \{(x,x) : x\in U\}$ and the exponent $\alpha\in(0,1]$ is the one appearing in the assumptions on the coefficients of the vector fields X_i's. Moreover:

$$|P(x,y)| \leqslant c\phi_2(x,y); \tag{4.9}$$
$$|X_iP(x,y)| \leqslant c\phi_1(x,y) \text{ for } i = 1,2,...,n; \tag{4.10}$$
$$|X_jX_iP(x,y)|, |X_0P(x,y)| \leqslant c\phi_0(x,y) \text{ for } i,j = 1,2,...,n. \tag{4.11}$$

(For the meaning of the symbol $X_iP(x,y)$, recall the Notation fixed at the end of § 2). Note that, regardless the infinite differentiability of $P(\cdot,y)$, only r derivatives of $P(\cdot,y)$ *with respect to the vector fields X_i* exist (since the vector fields themselves are nonsmooth). In particular, recalling that $r\geqslant 2$, we have that $X_iX_jP(x,y)$ is well defined for any $x\neq y$.

Proof. From (4.2) we read that for any $x\neq y$ the integral defining P is absolutely convergent, and P can be differentiated under the integral sign. Since Γ is smooth outside the origin, by the properties of the map Θ stated in Theorem 2.9, condition (4.5) immediately follows. To prove (4.6) and (4.7) we will show that for $x \neq y$ we have a locally uniform (in x) control on the C^{α}_{loc} modulus of continuity in y for $P(x,\cdot)$. Namely, since Γ is smooth outside the origin, we can write

$$|\Gamma(u_1) - \Gamma(u_2)| \leqslant c(\delta)\,|u_1 - u_2|$$

if $|u_1| \geqslant \delta$ and $|u_1 - u_2| \leqslant \delta/2$. Also, we know that, by Theorem 2.9

$$d(x,y) \leqslant \widetilde{d}((x,h),(y,k)) \leqslant c\left\|\Theta_{(y,k)}(x,h)\right\| \leqslant c\left|\Theta_{(y,k)}(x,h)\right|^{1/r}$$

and

$$\left|\Theta_{(y_1,k)}(x,h) - \Theta_{(y_2,k)}(x,h)\right| \leqslant c\,|y_1 - y_2|^{\alpha},$$

hence there exist constants c_1, c_2 such that for any fixed $\delta > 0$, if $d(x,y_1) \geqslant c_1\delta^{1/r}$ and $|y_1 - y_2| \leqslant c_2\delta^{1/\alpha}$ then

$$\begin{aligned}|P(x,y_1) - P(x,y_2)| &\leqslant \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\left|\Gamma\left(\Theta_{(y_1,k)}(x,h)\right) - \Gamma\left(\Theta_{(y_2,k)}(x,h)\right)\right|\varphi(h)\,dh\right)\varphi(k)\,dk \\ &\leqslant c(\delta)\,|y_1 - y_2|^{\alpha}\end{aligned}$$

which means that $P(x,\cdot)$ is C^{α} locally uniformly for $x \neq y$.

Lemma 3.1 with $\beta = 2$ together with (2.8), (4.1) and (4.2) implies (4.9).

Moreover, by (2.5) and (2.7),

$$\begin{aligned}X_iP(x,y) &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\widetilde{X}_i\left[\Gamma\left(\Theta_{(y,k)}(x,h)\right)\varphi(h)\right]dh\,\varphi(k)\,dk \\ &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\left\{\left[(Y_i\Gamma)\left(\Theta_{(y,k)}(x,h)\right) + \left(R_i^{(y,k)}\Gamma\right)\left(\Theta_{(y,k)}(x,h)\right)\right]\varphi(h)\right. \\ &\quad \left. + \,\Gamma\left(\Theta_{(y,k)}(x,h)\right)\widetilde{X}_i\varphi(h)\right\}dh\,\varphi(k)\,dk \\ &\equiv \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}(Y_i\Gamma)\left(\Theta_{(y,k)}(x,h)\right)\varphi(h)\,dh\,\varphi(k)\,dk \\ &\quad + \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\sum_{l=1}^{2}Q_l(y,k;\;x,h)\,\varphi_l(h)\,dh\,\varphi(k)\,dk\end{aligned}$$

where $\varphi_l \in C_0^{\infty}(\mathbb{R}^m)$ and, by (2.13) and (4.1),

$$\begin{aligned}|Q_l(y,k;\;x,h)| &\leqslant \frac{c}{\left\|\Theta_{(y,k)}(x,h)\right\|^{Q-1+\alpha}}, \\ \left|(Y_i\Gamma)\left(\Theta_{(y,k)}(x,h)\right)\right| &\leqslant \frac{c}{\left\|\Theta_{(y,k)}(x,h)\right\|^{Q-1}},\end{aligned}$$

so that Lemma 3.1 implies (4.10).

The proof of (4.11) is an iteration of the previous argument:

$$
\begin{aligned}
X_j X_i P(x,y) &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \widetilde{X}_j \widetilde{X}_i \left[\Gamma\left(\Theta_{(y,k)}(x,h)\right)\varphi(h)\right] dh\, \varphi(k) dk \\
&= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \left\{ (Y_j Y_i \Gamma)\left(\Theta_{(y,k)}(x,h)\right)\varphi(h) \right. \\
&\quad + \left(Y_j R_i^{(y,k)}\Gamma + R_j^{(y,k)} Y_i \Gamma + R_j^{(y,k)} R_i^{(y,k)}\Gamma\right)\left(\Theta_{(y,k)}(x,h)\right)\varphi(h) \\
&\quad + \left[(Y_j\Gamma)\left(\Theta_{(y,k)}(x,h)\right) + \left(R_j^{(y,k)}\Gamma\right)\left(\Theta_{(y,k)}(x,h)\right)\right]\widetilde{X}_i\varphi(h) \\
&\quad + \left[(Y_i\Gamma)\left(\Theta_{(y,k)}(x,h)\right) + \left(R_i^{(y,k)}\Gamma\right)\left(\Theta_{(y,k)}(x,h)\right)\right]\widetilde{X}_j\varphi(h) \\
&\quad \left. + \Gamma\left(\Theta_{(y,k)}(x,h)\right)\widetilde{X}_j\widetilde{X}_i\varphi(h)\right\} dh\, \varphi(k) dk \\
&= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} (Y_j Y_i \Gamma)\left(\Theta_{(y,k)}(x,h)\right)\varphi(h)\, dh \varphi(k) dk \\
&\quad + \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \sum_s Q_s(y,k;\ x,h)\,\varphi_s(h)\, dh\, \varphi(k)\, dk
\end{aligned}
$$

with $\varphi_s \in C_0^\infty(\mathbb{R}^m)$ and, as above, exploiting now also Corollary 2.11,

$$
\begin{aligned}
|Q_s(y,k;\ x,h)| &\leqslant \frac{c}{\left\|\Theta_{(y,k)}(x,h)\right\|^{Q-\alpha}}, \\
\left|(Y_j Y_i \Gamma)\left(\Theta_{(y,k)}(x,h)\right)\right| &\leqslant \frac{c}{\left\|\Theta_{(y,k)}(x,h)\right\|^{Q}}
\end{aligned}
$$

which by Lemma 3.1 implies (4.11) for $X_j X_i P(x,y)$. The proof of the bound on $X_0 P(x,y)$ is similar.

Finally, the explicit expression of the derivatives $X_i P, X_j X_i P, X_0 P$ allows us to repeat the argument used to prove (4.7), showing that also (4.8) holds. ■

Proposition 4.2 (Properties of Z_1) *Let L be as in (1.1) and, for $x,y \in U$, $x \neq y$, let*

$$
Z_1(x,y) = LP(x,y). \tag{4.12}
$$

Under the above assumptions and with the above notation, we have:

$$
Z_1(\cdot,y) \in C_{loc}^{r-2,\alpha}(U \setminus \{y\}); \tag{4.13}
$$

$$
Z_1(x,\cdot) \in C_{loc}^{\alpha}(U \setminus \{x\}); \tag{4.14}
$$

$$
Z_1 \in C(U \times U \setminus \Delta). \tag{4.15}
$$

Moreover:

$$
|Z_1(x,y)| \leqslant c_1 \phi_\alpha(x,y). \tag{4.16}
$$

Proof. Let us first prove (4.16). The computation is similar to that of the previous proof. However we have to write it explicitly because we will need it in the following. By (2.5) and Theorem 2.9 we have

$$Z_1(x,y)=\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\widetilde{L}\left[\Gamma\left(\Theta_{(y,k)}(x,h)\right)\varphi(h)\right]dh\,\varphi(k)dk$$

$$=\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\Big\{(\mathcal{L}\Gamma)\left(\Theta_{(y,k)}(x,h)\right)\varphi(h)+$$

$$+\left(\sum_j\left(Y_jR_j^{(y,k)}\Gamma+R_j^{(y,k)}Y_j\Gamma+R_j^{(y,k)}R_j^{(y,k)}\Gamma\right)+R_0^{(y,k)}\Gamma\right)\left(\Theta_{(y,k)}(x,h)\right)\varphi(h)$$

$$+2\sum_j\left[(Y_j\Gamma)\left(\Theta_{(y,k)}(x,h)\right)+R_j^{(y,k)}\Gamma\left(\Theta_{(y,k)}(x,h)\right)\right]\widetilde{X}_j\varphi(h)$$

$$+\Gamma\left(\Theta_{(y,k)}(x,h)\right)\widetilde{L}\varphi(h)\Big\}\,dh\,\varphi(k)dk.$$

Since $(\mathcal{L}\Gamma)(u)=0$ for $u\neq 0$, then $(\mathcal{L}\Gamma)\left(\Theta_{(y,k)}(x,h)\right)=0$ for $(x,h)\neq(y,k)$, so that, for $x\neq y$,

$$Z_1(x,y)=\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\sum_{i=1}^{3}Q_i(y,k,\ x,h)\,\varphi_i(h)\,dh\,\varphi(k)dk \tag{4.17}$$

where $\varphi_i\in C_0^\infty(\mathbb{R}^m)$ and, by Corollary 2.11,

$$|Q_i(y,k,\ x,h)|\leqslant\frac{c}{\left\|\Theta_{(y,k)}(x,h)\right\|^{Q-\alpha}}.$$

It follows that

$$|Z_1(x,y)|\leqslant c\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\frac{\psi(h)}{\left\|\Theta_{(y,k)}(x,h)\right\|^{Q-\alpha}}dh\right)\varphi(k)dk$$

for some $\psi\in C_0^\infty(\mathbb{R}^m)$. By Lemma 3.1, (4.16) follows.

As to the regularity of Z_1, let us inspect for instance the term

$$\sum_j\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\left(R_j^{(y,k)}R_j^{(y,k)}\Gamma\right)\left(\Theta_{(y,k)}(x,h)\right)\varphi(h)\,dh\,\varphi(k)\,dk$$

(all the others being more regular). By Corollary 2.11,

$$u\mapsto R_j^{(y,k)}R_j^{(y,k)}\Gamma(u)$$

is a $C_{loc}^{r-2,\alpha}$ function outside the origin. Since $\xi\mapsto\Theta_{(y,k)}(\xi)$ is smooth,

$$(x,h)\mapsto R_j^{(y,k)}R_j^{(y,k)}\Gamma\left(\Theta_{(y,k)}(x,h)\right)$$

is at least $C_{loc}^{r-2,\alpha}$ for $(x,h)\neq(y,k)$, and $Z_1(\cdot,y)\in C_{loc}^{r-2,\alpha}(U\setminus\{y\})$.

To deal with the regularity of $Z_1(x,\cdot)$ note that by Corollary 2.11, $R_j^\eta R_j^\eta\Gamma(u)$ depends on η in a C^α continuous way locally uniformly in $u\neq 0$. It follows that

$$y\mapsto\left(R_j^{(y,k)}R_j^{(y,k)}\Gamma\right)\left(\Theta_{(y,k)}(x,h)\right)$$

is $C^{\alpha}_{loc}(U \setminus \{x\})$ and the same is true for $Z_1(x,\cdot)$, by an argument similar to that used in the proof of Proposition 4.1 to deal with $P(x,\cdot)$. Joint continuity of Z_1 in (x,y), outside the diagonal, also follows from these facts by a local uniformity argument. ■

Next, we can prove:

Proposition 4.3 (Properties of Φ) *Let, for $j = 1,2,3,...$,*

$$Z_{j+1}(x,y) = \int_U Z_1(x,z)\, Z_j(z,y)\, dz \quad \text{for } x,y \in U, x \neq y. \tag{4.18}$$

Then the functions $Z_j(x,y)$ are well defined for $x,y \in U, x \neq y$. Moreover, shrinking U if necessary, the series

$$\Phi(x,y) = \sum_{j=1}^{\infty} Z_j(x,y) \tag{4.19}$$

converges for $x,y \in U, x \neq y$ and the function Φ satisfies the bound

$$|\Phi(x,y)| \leqslant c\phi_{\alpha}(x,y) \tag{4.20}$$

and the integral equation

$$\Phi(x,y) = Z_1(x,y) + \int_U Z_1(x,z)\,\Phi(z,y)\,dz \quad \text{for } x,y \in U, x \neq y. \tag{4.21}$$

Finally,

$$Z_j, \Phi \in C(U \times U \setminus \Delta).$$

Proof. By definition of Z_j, the bound (4.16) and Theorem 3.5 we have, recursively:

$$\begin{aligned}
|Z_2(x,y)| &\leqslant c_1^2 c \frac{2}{\alpha} \phi_{2\alpha}(x,y);\\
|Z_3(x,y)| &\leqslant c_1^3 \left(c\frac{2}{\alpha}\right) c \left(\frac{1}{\alpha} + \frac{1}{2\alpha}\right) \phi_{3\alpha}(x,y) \leqslant c_1^3 \left(c\frac{2}{\alpha}\right)^2 \phi_{3\alpha}(x,y);\\
&\dots\\
|Z_{j_0}(x,y)| &\leqslant c_1^{j_0} \left(c\frac{2}{\alpha}\right)^{j_0-1} \phi_{j_0\alpha}(x,y) \leqslant CR^{j_0\alpha - Q} \leqslant CR^{\alpha},
\end{aligned}$$

where j_0 is the least integer such that $j_0 > Q/\alpha$. Then:

$$|Z_{j_0+k}(x,y)| \leqslant CR^{\alpha}(cc_1R^{\alpha})^k \text{ for any } k \geqslant 0.$$

We now choose U small enough in order to get

$$\delta \equiv cc_1 R^{\alpha} < 1.$$

Then

$$|Z_{j_0+k}(x,y)| \leqslant C\delta^k$$

so that the series

$$\sum_{j=j_0}^{\infty} Z_j(x,y)$$

totally converges and the upper bound (4.20) holds. Moreover, we can write, for any $x, y \in U, x \neq y$:

$$\begin{aligned}
& Z_1(x,y) + \int_U Z_1(x,z)\,\Phi(z,y)\,dz \\
& = Z_1(x,y) + \int_U Z_1(x,z) \sum_{j=1}^{\infty} Z_j(z,y)\,dz \\
& = Z_1(x,y) + \sum_{j=1}^{\infty} \int_U Z_1(x,z)\,Z_j(z,y)\,dz \\
& = Z_1(x,y) + \sum_{j=1}^{\infty} Z_{j+1}(x,y) \\
& = \sum_{j=1}^{\infty} Z_j(x,y) = \Phi(x,y)
\end{aligned}$$

so that (4.21) holds. Let us come to the continuity properties of Z_j, Φ. By (4.15) and Lemma 3.7, the definition (4.18) recursively implies that

$$Z_j \in C(U \times U \setminus \Delta) \text{ for } j = 2, 3, ...$$

Since, by the above proof, the series in (4.3) totally converges, this also implies that

$$\Phi \in C(U \times U \setminus \Delta).$$

■

Proposition 4.4 (Properties of J) *Let U be as in the previous proposition. For $x, y \in U, x \neq y$, let*

$$J(x,y) = \int_U P(x,z)\,\Phi(z,y)\,dz. \tag{4.22}$$

Then: J and $X_i J$ $(i = 1, 2, ..., n)$ are well defined for any $x, y \in U, x \neq y$;

$$J,\ X_i J \in C(U \times U \setminus \Delta); \tag{4.23}$$

moreover, the following estimates hold $(i = 1, 2, ..., n)$:

$$|J(x,y)| \leqslant c\phi_{2+\alpha}(x,y); \tag{4.24}$$

$$|X_i J(x,y)| \leqslant c\phi_{1+\alpha}(x,y). \tag{4.25}$$

Proof. By (4.9), (4.20) and Theorem 3.5, we have

$$|J(x,y)| \leqslant c \int_U \phi_2(x,z)\,\phi_\alpha(z,y)\,dz \leqslant c\phi_{2+\alpha}(x,y).$$

Also, $X_i J$ is well defined, indeed

$$\begin{aligned}
X_i J(x,y) &= X_i \int_U P(x,z)\,\Phi(z,y)\,dz \\
&= \int_U X_i P(x,z)\,\Phi(z,y)\,dz
\end{aligned}$$

and, by Propositions 4.1, 4.3 and Theorem 3.5,

$$|X_i J(x,z)| \leqslant c \int_U \phi_1(x,z)\,\phi_\alpha(z,y)\,dz \leqslant c\phi_{1+\alpha}(x,y)\,.$$

As to the continuity properties: by Proposition 4.1 and Proposition 4.3 we know that $P(x,y), X_i P(x,y), \Phi(x,y)$ are continuous in the joint variables for $x \neq y$ and satisfy the bounds

$$\begin{aligned} |P(x,y)| &\leqslant c\phi_2(x,y)\,; \\ |X_i P(x,y)| &\leqslant c\phi_1(x,y) \quad (i = 1,2,...,n)\,; \\ |\Phi(x,y)| &\leqslant c\phi_\alpha(x,y)\,. \end{aligned}$$

Hence Proposition 3.7 implies (4.23). ■

Proposition 4.5 *The following identity and upper bound hold in weak sense:*

$$LJ(x,y) = \int_U Z_1(x,z)\,\Phi(z,y)\,dz - c_0(x)\,\Phi(x,y)\,, \tag{4.26}$$

$$|LJ(x,y)| \leqslant c\phi_\alpha(x,y) \tag{4.27}$$

where

$$c_0(x) = \int_{\mathbb{R}^m} c(x,k)\,\varphi^2(k)dk$$

and $c(x,k)$ is defined in (2.10).

Explicitly, denoting by $G(x,y)$ the right hand side of (4.26), we have

$$\int_U J(x,y)\,L^*\psi(x)\,dx = \int_U G(x,y)\,\psi(x)\,dx \tag{4.28}$$

for any $\psi \in C_0^\infty(U)$ and $y \in U$, where L^* is the transposed operator of L (see (2.1)), and

$$|G(x,y)| \leqslant c\phi_\alpha(x,y)\,.$$

For the proof of the above proposition we need the following lemma.

Lemma 4.6 *Let ω be a smooth function on $\mathbb{G}$ such that $\omega(u) = 0$ for $\|u\| < \frac{1}{2}$ and $\omega(u) = 1$ for $\|u\| > 1$ and let $\omega_\varepsilon(u) = \omega\left(D\left(\varepsilon^{-1}\right)u\right)$. Let R_1 and R_2 be vector fields on $\mathbb{G}$ given by*

$$R_1 = \sum_{j=1}^N a_j(u)\,\partial_{u_j}, R_2 = \sum_{j=1}^N b_j(u)\,\partial_{u_j}$$

and assume that, for a couple of $s_1, s_2 \in \mathbb{R}$ and some constant $c > 0$, every $j,k = 1,2,...,N$,

$$\begin{aligned} |a_j(u)| &\leqslant c\,\|u\|^{s_1+\alpha_j}\,; \\ |b_j(u)| &\leqslant c\,\|u\|^{s_2+\alpha_j}\,; \\ |\partial_{u_k} b_j(u)| &\leqslant c\,\|u\|^{s_2+\alpha_j-\alpha_k} \end{aligned}$$

where the α_j's are as in (2.6). Then there exists $c' > 0$ such that for every $\varepsilon > 0$ and $u \in \mathbb{G}$

$$\begin{aligned} |R_1\omega_\varepsilon(u)| &\leqslant c'\,\|u\|^{s_1} \\ |R_1R_2\omega_\varepsilon(u)| &\leqslant c'\,\|u\|^{s_1+s_2}\,. \end{aligned}$$

Proof. We have

$$\begin{aligned}|R_1\omega_\varepsilon(u)| &\leqslant \sum |a_j(u)| \left|\frac{\partial \omega_\varepsilon}{\partial u_j}(u)\right| \\ &\leqslant \sum \|u\|^{s_1+\alpha_j} \frac{1}{\varepsilon^{\alpha_j}} \left|\frac{\partial \omega}{\partial u_j}\left(D\left(\varepsilon^{-1}\right)u\right)\right| \\ &\leqslant c\,\|u\|^{s_1}\end{aligned}$$

since on the support of $\frac{\partial\omega}{\partial u_j}\left(D\left(\varepsilon^{-1}\right)u\right)$ we have $\|u\| \leqslant \varepsilon$. Similarly,

$$\begin{aligned}&|R_1R_2\omega_\varepsilon(u)| = \\ &= \left|\left(\sum_{k=1}^N a_k(u)\,\partial_{u_k}\sum_{j=1}^N b_j(u)\,\partial_{u_j}\right)\omega_\varepsilon(u)\right| \\ &\leqslant c\sum_{k=1}^N \|u\|^{s_1+\alpha_k}\sum_{j=1}^N\left|\partial_{u_k}b_j(u)\,\partial_{u_j}\omega_\varepsilon(u)+b_j(u)\,\partial^2_{u_ku_j}\omega_\varepsilon\right| \\ &\leqslant c\sum_{k=1}^N \|u\|^{s_1+\alpha_k}\sum_{j=1}^N\left(\frac{\|u\|^{s_2+\alpha_j-\alpha_k}}{\varepsilon^{\alpha_j}}\left|\partial_{u_j}\omega\left(D\left(\frac{1}{\varepsilon}\right)u\right)\right| + \frac{\|u\|^{s_2+\alpha_j}}{\varepsilon^{\alpha_k+\alpha_j}}\left|\partial^2_{u_ku_j}\omega\left(D\left(\frac{1}{\varepsilon}\right)u\right)\right|\right) \\ &\leqslant c\sum_{k=1}^N \|u\|^{s_1+\alpha_k}\,\|u\|^{s_2-\alpha_k} = c\,\|u\|^{s_1+s_2}.\end{aligned}$$

■

Proof of Proposition 4.5. To prove (4.26) we use a distributional argument. Let ω_ε be as in the previous Lemma, let $\Gamma_\varepsilon = \omega_\varepsilon\Gamma$ and define

$$P_\varepsilon(x,y) = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\Gamma_\varepsilon\left(\Theta_{(y,k)}(x,h)\right)\varphi(h)\,dh\right)\varphi(k)\,dk$$

and

$$J_\varepsilon(x,y) = \int_U P_\varepsilon(x,z)\,\Phi(z,y)\,dz.$$

We have

$$J_\varepsilon(x,y) = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\int_U \Gamma_\varepsilon\left(\Theta_{(z,k)}(x,h)\right)\Phi(z,y)\,dz\;\varphi(h)\,dh\right)\varphi(k)\,dk$$

and

$$\begin{aligned}&LJ_\varepsilon(x,y) \\ &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_U \widetilde{L}\left[\Gamma_\varepsilon\left(\Theta_{(z,k)}(x,h)\right)\varphi(h)\right]\Phi(z,y)\,\varphi(k)\,dzdhdk\end{aligned}$$

$$= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_U \mathcal{L}\Gamma_\varepsilon\left(\Theta_{(z,k)}(x,h)\right)\varphi(h)\Phi(z,y)\varphi(k)\,dzdhdk$$

$$+\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_U \left(\sum_i \left(Y_i R_i^{(z,k)} + R_i^{(z,k)}Y_i + R_i^{(z,k)}R_i^{(z,k)} + R_0^{(z,k)}\right)\Gamma_\varepsilon\left(\Theta_{(z,k)}(x,h)\right)\right)\times$$

$$\times\varphi(h)\Phi(z,y)\varphi(k)\,dzdhdk$$

$$+\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_U 2\sum_i Y_i\Gamma_\varepsilon\left(\Theta_{(z,k)}(x,h)\right)\widetilde{X}_i\varphi(h)\Phi(z,y)\varphi(k)\,dzdhdk$$

$$+\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_U \Gamma_\varepsilon\left(\Theta_{(z,k)}(x,h)\right)\widetilde{L}^{(x,h)}\varphi(h)\Phi(z,y)\varphi(k)\,dzdhdk.$$

To bound

$$\sum_i \left(Y_i R_i^{(z,k)} + R_i^{(z,k)}Y_i + R_i^{(z,k)}R_i^{(z,k)} + R_0^{(z,k)}\right)\Gamma_\varepsilon(u)$$

we now recall that, by Theorem 2.9, the vector fields $R_i^{(z,k)}, Y_i, R_0^{(z,k)}$ satisfy the assumptions of Lemma 4.6 with s_1 or s_2 equal to $\alpha-1, -1, \alpha-2$, respectively. A simple computation shows that

$$\left|\sum_i \left(Y_i R_i^{(z,k)} + R_i^{(z,k)}Y_i + R_i^{(z,k)}R_i^{(z,k)} + R_0^{(z,k)}\right)\Gamma_\varepsilon(u)\right| \leqslant \frac{c}{\|u\|^{Q-\alpha}}.$$

Hence for suitable $\varphi_j \in C_0^\infty(\mathbb{R}^m)$ and $Q_{\varepsilon,j}$ satisfying

$$\left|Q_{\varepsilon,j}(z,k;\ x,h)\right| \leqslant \frac{c}{\left\|\Theta_{(z,k)}(x,h)\right\|^{Q-\alpha}}$$

we have

$$LJ_\varepsilon(x,y) = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_U \mathcal{L}\Gamma_\varepsilon\left(\Theta_{(z,k)}(x,h)\right)\varphi(h)\Phi(z,y)\varphi(k)\,dzdhdk$$
$$+\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_U \sum_{j=1}^3 Q_{\varepsilon,j}(z,k;\ x,h)\varphi_j(h)\Phi(z,y)\varphi(k)\,dzdhdk.$$

Let now $\psi \in C_0^\infty(U)$ be any test function. Then

$$\int_{\mathbb{R}^p} LJ_\varepsilon(x,y)\psi(x)\,dx =$$
$$= \int_U\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^p} \mathcal{L}\Gamma_\varepsilon\left(\Theta_{(z,k)}(x,h)\right)\psi(x)\varphi(h)\varphi(k)\,dxdhdk\Phi(z,y)\,dz$$
$$+\int_U\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^p} \sum_{j=1}^3 Q_{\varepsilon,j}(z,k;\ x,h)\psi(x)\varphi_j(h)\varphi(k)\,dxdhdk\Phi(z,y)\,dz.$$

Let now change variable in the first integral setting $u = \Theta_{(z,k)}(x,h)$. Then, by (2.10),

$$dxdh = c(z,k)\left(1 + O(\|u\|)\right)du,$$

and setting

$$\widehat{\varphi}_{(z,k)}(u) = \varphi(h)\psi(x)\Big|_{u=\Theta_{(z,k)}(x,h)}$$

we have

$$\int_{\mathbb{R}^p} LJ_\varepsilon(x,y)\,\psi(x)\,dx = \int_U \int_{\mathbb{R}^m} \int_{\mathbb{G}} \mathcal{L}\Gamma_\varepsilon(u)\,\widehat{\varphi}_{(z,k)}(u)\,du\; c(z,k)\varphi(k)dk\Phi(z,y)\,dz$$
$$+ \int_U \int_{\mathbb{R}^m} \int_{\mathbb{G}} \mathcal{L}\Gamma_\varepsilon(u)\,O(\|u\|)\,\widehat{\varphi}_{(z,k)}(u)\,du\; c(z,k)\varphi(k)dk\Phi(z,y)\,dz$$
$$+ \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^p} \sum_{j=1}^{3} Q_{\varepsilon,j}(z,k;\; x,h)\,\varphi_j(h)\,\psi(x)\,dx\varphi(k)dhdk\Phi(z,y)\,dz.$$

Since $\mathcal{L}\Gamma_\varepsilon(u) = 0$ for $\|u\| > \varepsilon$, letting $\varepsilon \to 0$ the second integral in the right hand side vanishes, by Lebesgue's theorem, and integrating by part in the first integral we get:

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{\mathbb{R}^p} LJ_\varepsilon(x,y)\,\psi(x)\,dx &= \\ &= \int_U \int_{\mathbb{R}^m} \int_{\mathbb{G}} \Gamma(u)\,\mathcal{L}^*\widehat{\varphi}_{(z,k)}(u)\,du\; c(z,k)\varphi(k)dk\Phi(z,y)\,dz \\ &+ \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^p} \sum_{j=1}^{3} Q_j(z,k;\; x,h)\,\varphi_j(h)\,\psi(x)\,\varphi(k)dxdhdk\Phi(z,y)\,dz\end{aligned}$$

where Q_j are as in (4.17) and

$$\mathcal{L}^* = \sum_{i=1}^{n} Y_i^2 - Y_0$$

is the adjoint operator of $\mathcal{L}$, so that

$$\int_{\mathbb{G}} \Gamma(u)\,\mathcal{L}^*\widehat{\varphi}_{(z,k)}(u)\,du = -\widehat{\varphi}_{(z,k)}(0) = -\varphi(k)\,\psi(z)$$

and

$$\begin{aligned}&\lim_{\varepsilon\to 0} \int_{\mathbb{R}^p} LJ_\varepsilon(x,y)\,\psi(x)\,dx \\ &= -\int_U \psi(z)\,c_0(z)\,\Phi(z,y)\,dz + \int_U \int_{\mathbb{R}^p} Z_1(x,z)\,\Phi(z,y)\,\psi(x)\,dxdz,\end{aligned}$$

having set

$$c_0(z) = \int_{\mathbb{R}^m} c(z,k)\,\varphi^2(k)dk. \tag{4.29}$$

On the other hand,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^p} LJ_\varepsilon(x,y)\,\psi(x)\,dx = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^p} J_\varepsilon(x,y)\,L^*\psi(x)\,dx = \int_{\mathbb{R}^p} J(x,y)\,L^*\psi(x)\,dx,$$

which easily follows by Lebesgue's dominated convergence theorem and the bound (4.24) on J, J_ε. Therefore

$$LJ(x,y) = -c_0(x)\,\Phi(x,y) + \int_U Z_1(x,y)\,\Phi(z,y)\,dz,$$

which is (4.26). This also implies, by (4.20), (4.16) and Theorem 3.5:

$$\begin{aligned}|LJ(x,y)| &\leqslant c\phi_\alpha(x,y)+c\int_U \phi_\alpha(x,z)\,\phi_\alpha(z,y)\,dz\\ &\leqslant c\phi_\alpha(x,y)+c\phi_{2\alpha}(x,y)\leqslant c\phi_\alpha(x,y),\end{aligned}$$

which is (4.27). ■

In view of the presence of the term $c_0(x)$ in the identity (4.26) we now modify our previous construction as follows:

$$\begin{aligned}Z_1'(x,y) &= \frac{1}{c_0(x)}Z_1(x,y);\\ Z_{k+1}'(x,y) &= \int_U Z_1'(x,z)\,Z_k'(z,y)\,dz;\\ \Phi'(x,y) &= \sum_{k=1}^{\infty} Z_k'(x,y);\\ J'(x,y) &= \int_U P(x,z)\,\Phi'(z,y)\,dz.\end{aligned}$$

With these definitions, the following hold:

$$\Phi'(x,y)=Z_1'(x,y)+\int_U Z_1'(x,z)\,\Phi'(z,y)\,dz; \tag{4.30}$$

$$c_0(x)\,\Phi'(x,y)=Z_1(x,y)+\int_U Z_1(x,z)\,\Phi'(z,y)\,dz; \tag{4.31}$$

$$LJ'(x,y)=\int_U Z_1(x,z)\,\Phi'(z,y)\,dz-c_0(x)\,\Phi'(x,y). \tag{4.32}$$

Remark 4.7 *Recalling that*

$$0<c_1\leqslant c_0(x)\leqslant c_2$$

for any $x\in U$, and that $c_0\in C^\alpha(U)$ (since by Theorem 2.9 the function c is Hölder continuous), it is immediate to check that the functions Z_1', Z_k', Φ', J' satisfy the same upper bounds (with different constants) and continuity properties proved in Propositions 4.2, 4.3, 4.5 for Z_1, Z_k, Φ, J, respectively.

We have, at last:

Theorem 4.8 (Existence of fundamental solution) *Let*

$$\gamma(x,y)=\frac{1}{c_0(y)}\left[P(x,y)+J'(x,y)\right].$$

Then $\gamma(x,y)$ and $X_i\gamma(x,y)$ $(i=1,2,...,n)$ are well defined and continuous in the joint variables $x,y\in U, x\neq y$, and satisfy the following bounds:

$$|\gamma(x,y)|\leqslant c\phi_2(x,y); \tag{4.33}$$

$$|X_i\gamma(x,y)|\leqslant c\phi_1(x,y). \tag{4.34}$$

Moreover, $\gamma(\cdot,y)$ is a weak solution to $L\gamma(\cdot,y) = -\delta_y$, that is:

$$\int_U \gamma(x,y)\, L^*\psi(x)\, dx = -\psi(y) \tag{4.35}$$

for any $\psi \in C_0^\infty(U), y \in U$. Finally, if $X_0 \equiv 0$, then there exists $\varepsilon > 0$ such that

$$\gamma(x,y) > 0 \text{ for } d(x,y) < \varepsilon. \tag{4.36}$$

Remark 4.9 *When X_0 does not vanish the fundamental solution Γ of the homogeneous operator can be proved to be only non-negative, as the example of the heat operator suggests. As a consequence nothing can be said in this case about the sign of γ near the pole.*

Proof. By (4.7), (4.8), (4.23) and Remark 4.7 the functions $\gamma(x,y)$ and $X_i\gamma(x,y)$ are continuous in the joint variables $x, y \in U, x \neq y$.

The bounds (4.33), (4.34) follow from Proposition 4.4 and Proposition 4.1. As to (4.36),

$$|c_0(y)\gamma(x,y) - P(x,y)| = |J'(x,y)| \leqslant c\phi_{2+\alpha}(x,y) \leqslant c\frac{d(x,y)^{2+\alpha}}{|B(x,d(x,y))|}.$$

If $X_0 \equiv 0$, then also $Y_0 \equiv 0$ and by [1, Prop. 5.3.13, p.243] the function Γ is strictly positive, hence

$$\Gamma(u) \geqslant \frac{c}{\|u\|^{Q-2}}$$

and, reasoning like in Lemma 3.1 one can check that

$$\begin{aligned} P(x,y) &\geqslant c\int_{|k|\leqslant\varepsilon}\int_{|h|\leqslant\varepsilon} \frac{dhdk}{\left\|\Theta_{(y,k)}(x,h)\right\|^{Q-2}} \\ &\geqslant c\int_{d(x,y)}^{R} \frac{r}{|B(x,r)|} dr \geqslant c\frac{d(x,y)^2}{|B(x,d(x,y))|} \end{aligned}$$

and (4.36) follows.

To prove (4.35), we have to show that for any test function ψ

$$-\psi(y)\, c_0(y) = \int P(x,y)\, L^*\psi(x)\, dx + \int J'(x,y)\, L^*\psi(x)\, dx \equiv A + B.$$

As to A, exploiting the same computation performed in the proof of Proposition 4.5,

$$\begin{aligned} A &= \lim_{\varepsilon\to 0} \int P_\varepsilon(x,y)\, L^*\psi(x)\, dx = \lim_{\varepsilon\to 0} \int LP_\varepsilon(x,y)\, \psi(x)\, dx \\ &= -\psi(y)\, c_0(y) + \int_{\mathbb{R}^p} Z_1(x,y)\, \psi(x)\, dx. \end{aligned}$$

On the other hand, by (4.32),

$$B = \int_{\mathbb{R}^p} \psi(x) \left\{ \int Z_1(x,z)\, \Phi'(z,y)\, dz - c_0(x)\, \Phi'(x,y) \right\} dx.$$

By (4.31),

$$\begin{aligned}A+B &= -\psi(y)\,c_0(y)+\\ &+\int_{\mathbb{R}^p}\psi(x)\left\{Z_1(x,y)+\int Z_1(x,z)\,\Phi'(z,y)\,dz-c_0(x)\,\Phi'(x,y)\right\}dx\\ &= -\psi(y)\,c_0(y)\end{aligned}$$

and we are done. ■

5 Further regularity of the fundamental solution and local solvability of L

In this section, under a stronger regularity assumption on the coefficients of the vector fields, we will show that the fundamental solution $\gamma(\cdot,y)$ actually possesses second order derivatives with respect to the vector fields, satisfying natural growth bounds, and $\gamma(\cdot,y)$ satisfies the equation $Lu=0$ (outside the pole) in classical sense. As a consequence, we can establish a local solvability result for the operator L.

Assumptions B. In this section we assume that for some integer $r\geqslant 2$ and some $\alpha\in(0,1]$, the coefficients of the vector fields $X_1,X_2,...,X_n$ belong to $C^{r,\alpha}(\Omega)$, while the coefficients of X_0 belong to $C^{r-1,\alpha}(\Omega)$. If $r=2$ we assume $\alpha=1$. Moreover, we still assume that $X_0,X_1,...,X_n$ satisfy Hörmander's condition of step r in Ω: the vectors

$$\left\{\left(X_{[I]}\right)_x\right\}_{|I|\leqslant r}$$

span $\mathbb{R}^p$ for any $x\in\Omega$. (For examples of systems of vector fields satisfying the assumptions, see the Appendix).

Throughout this section we keep using the notation introduced in §4; in particular, U stands for a fixed neighborhood of a point $x_0\in\Omega$ where all the previous construction can be performed. Accordingly to Assumptions B, from now on the constants appearing in our estimates will have the following dependence on the vector fields:

Dependence of the constants. All the constants appearing in the upper bounds proved in this section will depend on the vector fields only through the following quantities:

(i) the norms $C^{r,\alpha}(\Omega)$ of the coefficients of X_i $(i=1,2,...,n)$ and the norms $C^{r-1,\alpha}(\Omega)$ of the coefficients of X_0;

(ii) a positive constant c_0 such that the following bound holds:

$$\inf_{x\in\Omega}\max_{|I_1|,|I_2|,...,|I_p|\leqslant r}\left|\det\left(\left(X_{[I_1]}\right)_x,\left(X_{[I_2]}\right)_x,...,\left(X_{[I_p]}\right)_x\right)\right|\geqslant c_0.$$

Before proceeding we need to define precisely our functional framework and the notion of solution.

Definition 5.1 *If u is a function, not necessarily smooth, defined in an open set $D \subseteq \Omega$, then:*

we say that $X_i u$ exists in D if the classical X_i-directional derivative of u exists in D;

we say that $u \in C^1_X(D)$ if for $i = 1, 2, ..., n$, the derivatives $X_i u$ exist and are continuous in D;

we say that $u \in C^2_X(D)$ if $u \in C^1_X(D)$ and for $i, j = 1, 2, ..., n$, the derivatives $X_i X_j u$ and $X_0 u$ exist and are continuous in D.

Let f be a continuous function in D. We say that u is a (classical) solution to

$$Lu = f \text{ in } D$$

if $u \in C^2_X(D)$ and $Lu(x) = f(x)$ for every $x \in D$.

We say that the operator L is locally solvable *in Ω if for every $x_0 \in \Omega$ there exists a neighborhood $U(x_0)$ such that for every $\beta > 0$ and $f \in C^\beta(U(x_0))$ the equation $Lw = f$ has at least a $C^2_X(U(x_0))$ solution.*

Note that, by Proposition 2.6, any $C^2_X(D)$ function is necessarily continuous; if $X_0 \equiv 0$ the same conclusion holds for $C^1_X(D)$ functions.

Remark 5.2 *We recall that, even for the classical Laplacian, under the mere assumption of continuity of f in D, a $C^2(D)$ solution to $\Delta w = f$ may not exist. A counterexample is given for instance in [15, exercise 4.9, p.71]. Therefore the condition $f \in C^\beta(D)$ in the definition of solvability is a natural requirement.*

The existence of $X_i u$ will be sometimes established by the following:

Lemma 5.3 *Let $D \subset \mathbb{R}^p$ be an open set and let X be a $C^1(D)$ vector field. Let w be a $C(D)$ function and let $w_\varepsilon \in C^1(D)$ be such that for $x \in D$, $w_\varepsilon(x) \to w(x)$ as $\varepsilon \to 0$ and $Xw_\varepsilon \to g$ uniformly on D. Then w is differentiable along X and $Xw = g$.*

Proof. Let $x \in D$ and let $\upsilon(t)$ be an integral curve of X such that $\upsilon(0) = x$ and let $h_\varepsilon(t) = w_\varepsilon(\upsilon(t))$. Since $h'_\varepsilon(t)$ converges uniformly we have

$$\begin{aligned} g(\upsilon(t)) &= \lim_{\varepsilon \to 0} Xw_\varepsilon(\upsilon(t)) = \lim_{\varepsilon \to 0} h'_\varepsilon(t) \\ &= \frac{d}{dt}\Big(\lim_{\varepsilon \to 0} h_\varepsilon(t)\Big) = \frac{d}{dt}\Big(\lim_{\varepsilon \to 0} w_\varepsilon(\upsilon(t))\Big) = Xw(\upsilon(t)), \end{aligned}$$

so that

$$g(x) = Xw(x).$$

■

5.1 Preliminary results

We now need to sharpen the analysis of the map $\Theta_\eta(\xi)$ performed in [6] showing that, under the above (stronger) Assumptions B, this function possesses reasonable properties also with respect to the "bad" variable η. Namely, the following holds:

Proposition 5.4 *Under Assumptions B:*

i) the vector fields R_i^η appearing in (2.7) are $C^{r+1-p_i,\alpha}$ vector fields of weight $\geqslant \alpha - p_i$, depending on η in a C^α way.

ii) the coefficients of the differential operators D_i^η defined by the compositions

$$Y_i R_j^\eta R_k^\eta,\ R_i^\eta R_j^\eta R_k^\eta,\ Y_i Y_j R_k^\eta,\ R_i^\eta Y_j^\eta R_k^\eta,\ Y_i R_j^\eta Y_k,\ R_i^\eta R_j^\eta Y_k,\ Y_i R_0^\eta,\ R_i^\eta R_0^\eta$$

$(i = 0, 1, 2, ..., n; j, k = 1, 2, ..., n)$ satisfy the bound

$$|D_i^\eta f(u)| \leqslant \frac{c}{\|u\|^{\mu+2+p_i-\alpha}}$$

for u in a neighborhood of the origin, whenever $f : \mathbb{G} \to \mathbb{R}$ is $D(\lambda)$-homogeneous of degree $-\mu$. Also, the coefficients of D_i^η depend on η in a C^α way.

iii) the change of variables $\eta \mapsto u = \Theta_\eta(\xi)$ is a $C^{1,\alpha}$ diffeomorphism in a neighborhood of the origin and its inverse $\eta = \Theta_{(\cdot)}(\xi)^{-1}(u)$ is $C^{1,\alpha}$ in the joint variables (ξ, u). Moreover we have

$$d\eta = c(\xi)(1 + \chi(\xi, u))\, du,$$

where, analogously to Theorem 2.9, $c(\cdot)$ is a C^α function, bounded and bounded away from zero, $\chi(\xi, u)$ is C^α in the joint variables (ξ, u) and for every $\gamma_1, \gamma_2 \geqslant 0$ such that $\gamma_1 + \gamma_2 \leqslant \alpha$ there exists a constant c such that

$$|\chi(\xi_1, u) - \chi(\xi_2, u)| \leqslant c\, |\xi_1 - \xi_2|^{\gamma_1} \|u\|^{\gamma_2}.$$

In particular

$$|\chi(\xi, u)| \leqslant c\, \|u\|^\alpha.$$

Proof. i) This follows with the same proof of [6, Thm. 3.9], under assumption B.

ii) This follows as Corollary 2.11, by point 2.i of Theorem 2.10. Actually, the same proof of point 2.i of Theorem 2.10 implies this stronger conclusion, under the stronger assumption B.

iii) With the notations of [6, section 3.2] let

$$\xi = E(u, \eta) = \exp\left(\sum_{I \in B} u_I S_{[I],\eta}\right)(\eta)$$

and recall that $\Theta_\eta(\xi)$ is defined by $E(\Theta_\eta(\xi), \eta) = \xi$. Observe that, for every fixed ξ, to express η as a function of u is equivalent to solve with respect to η the equation

$$E(u, \eta) - \xi = 0. \tag{5.1}$$

Revising the proof of [6, Thm. 3.9] under the assumption $b_{ij} \in C^{r,\alpha}(\Omega)$, one can see that the smooth vector fields $S_{[I],\eta}$ depend on η in a $C^{1,\alpha}$ way. This implies that $\xi = E(u, \eta)$ depends in a $C^{1,\alpha}$ way on the joint variables (u, η) (see [6, Prop. 30]). Since $E(0, \eta) = \eta$ we have $\frac{\partial E}{\partial \eta}(0, \eta) = I$. The implicit function theorem applied to equation (5.1) shows that $\eta = \eta(u, \xi)$ is at least C^1 in the joint variables. The standard argument used to prove the further regularity of the implicit function allows to prove that this function is indeed $C^{1,\alpha}$ in the joint variables. Also, since

$$E(u,\eta(u,\xi))-\xi=0$$

differentiating with respect to u yields

$$\frac{\partial E}{\partial u}(u,\eta(u,\xi))+\frac{\partial E}{\partial \eta}(u,\eta(u,\xi))\frac{\partial \eta}{\partial u}(u,\xi)=0.$$

Evaluating this identity for $u=0$ (that is $\eta=\xi$) gives

$$\frac{\partial E}{\partial u}(0,\xi)+I\frac{\partial \eta}{\partial u}(0,\xi)=0$$

so that

$$\frac{\partial \eta}{\partial u}(0,\xi)=-\frac{\partial E}{\partial u}(0,\xi)=-\left((S_{[I],\xi})_{\xi}\right)_{I\in B}=-\left(\left(\widetilde{X}_{[I]}\right)_{\xi}\right)_{I\in B}.$$

Since

$$d\eta=J_{\xi}(u)\,du$$

with $J_{\xi}(u)=\left|\det\frac{\partial\eta}{\partial u}(\xi,u)\right|$, we have

$$J_{\xi}(u)=\left|\det\left(\left(\widetilde{X}_{[I]}\right)_{\xi}\right)_{I\in B}\right|+\chi_0(\xi,u). \tag{5.2}$$

Note that $\chi_0(\xi,u)$ is C^{α} in the joint variables (u,ξ) since $\eta(u,\xi)$ is $C^{1,\alpha}$.

Assume now $|\xi_1-\xi_2|<|u|$, then for any $\gamma_1,\gamma_2\geqslant 0$ with $\gamma_1+\gamma_2\leqslant 0$,

$$|\chi_0(u,\xi_1)-\chi_0(u,\xi_2)|\leqslant c\,|\xi_1-\xi_2|^{\alpha}\leqslant c\,|\xi_1-\xi_2|^{\gamma_1}|u|^{\gamma_2}.$$

If $|\xi_1-\xi_2|\geqslant|u|$, since $\chi_0(0,\xi_1)=\chi_0(0,\xi_2)=0$ we have

$$\begin{aligned}|\chi_0(u,\xi_1)-\chi_0(u,\xi_2)|&\leqslant|\chi_0(u,\xi_1)-\chi_0(0,\xi_1)|+|\chi_0(u,\xi_2)-\chi_0(0,\xi_2)|\\&\leqslant c\,|u|^{\alpha}\leqslant c\,|\xi_1-\xi_2|^{\gamma_1}|u|^{\gamma_2}.\end{aligned}$$

Hence in any case

$$|\chi_0(\xi_1,u)-\chi_0(\xi_2,u)|\leqslant c\,|\xi_1-\xi_2|^{\gamma_1}\,\|u\|^{\gamma_2}. \tag{5.3}$$

Then (5.2) can be rewritten as

$$d\eta=c(\xi)\,(1+\chi(\xi,u))\,du$$

where

$$c(\xi)=\left|\det\left(\left(\widetilde{X}_{[I]}\right)_{\xi}\right)_{I\in B}\right|$$

is C^{α} and locally bounded away from zero, while

$$\chi(\xi,u)=\frac{\chi_0(\xi,u)}{c(\xi)}$$

still satisfies (5.3). Hence point (iii) is proved. ■

The following Hölder continuity estimate on the function Φ' will be crucial.

Proposition 5.5 *For any $\varepsilon \in (0, \alpha)$ there exists $c > 0$ such that*

$$|\Phi'(x_1, y) - \Phi'(x_2, y)| \leqslant c d(x_1, x_2)^{\alpha-\varepsilon} \phi_\varepsilon(x_1, y)$$

for any $x_1, x_2, y \in U$ with $d(x_1, y) \geqslant 3d(x_1, x_2)$.

Note that the same result holds if the number 3 is replaced by another constant $k > 1$, with c depending on k.

The following easy variation of the previous result will be also useful:

Corollary 5.6 *For any $\varepsilon \in (0, \alpha)$ there exists $c > 0$ such that*

$$|\Phi'(x_1, y) - \Phi'(x_2, y)| \leqslant c d(x_1, x_2)^{\alpha-\varepsilon} \left[\frac{d(x_1, y)^\varepsilon}{|B(x_1, d(x_1, y))|} + \frac{d(x_2, y)^\varepsilon}{|B(x_2, d(x_2, y))|} \right]$$

for any $x_1, x_2, y \in U$ with $y \neq x_1, x_2$.

Proof of Corollary 5.6. If $d(x_1, y) \geqslant 3d(x_1, x_2)$ by Proposition 5.5 and Lemma 3.3 we can bound

$$|\Phi'(x_1, y) - \Phi'(x_2, y)| \leqslant c d(x_1, x_2)^{\alpha-\varepsilon} \phi_\varepsilon(x_1, y) \leqslant c d(x_1, x_2)^{\alpha-\varepsilon} \frac{d(x_1, y)^\varepsilon}{|B(x_1, d(x_1, y))|}.$$

Analogously if $d(x_2, y) \geqslant 3d(x_1, x_2)$ we can write

$$|\Phi'(x_1, y) - \Phi'(x_2, y)| \leqslant c d(x_1, x_2)^{\alpha-\varepsilon} \phi_\varepsilon(x_2, y) \leqslant c d(x_1, x_2)^{\alpha-\varepsilon} \frac{d(x_2, y)^\varepsilon}{|B(x_2, d(x_1, y))|}.$$

Hence, let us assume $3d(x_1, x_2) > \max(d(x_1, y), d(x_2, y))$. Then by Proposition 4.3 and Lemma 3.3:

$$\begin{aligned}
|\Phi'(x_1, y) - \Phi'(x_2, y)| &\leqslant c \{\phi_\alpha(x_1, y) + \phi_\alpha(x_2, y)\} \\
&\leqslant c \left\{ \frac{d(x_1, y)^\alpha}{|B(x_1, d(x_1, y))|} + \frac{d(x_2, y)^\alpha}{|B(x_2, d(x_2, y))|} \right\} \\
&\leqslant c d(x_1, x_2)^{\alpha-\varepsilon} \left\{ \frac{d(x_1, y)^\varepsilon}{|B(x_1, d(x_1, y))|} + \frac{d(x_2, y)^\varepsilon}{|B(x_2, d(x_2, y))|} \right\}.
\end{aligned}$$

■

Proposition 5.5 will be proved in several steps, establishing first an analogous result for the functions Z_1' and Z_k'.

Lemma 5.7 *For every $x_1, x_2, y \in U$ with $d(x_1, y) \geqslant 2d(x_1, x_2)$ we have*

$$|Z_1'(x_1, y) - Z_1'(x_2, y)| \leqslant c d(x_1, x_2)^\alpha \phi_0(x_1, y). \tag{5.4}$$

Proof. Since

$$Z_1'(x, y) = \frac{1}{c_0(x)} Z_1(x, y)$$

with c_0 Hölder continuous and bounded away from zero, it suffices to prove (5.4) with Z_1' replaced by Z_1. Under assumptions B, the explicit expression of Z_1 given in the proof of Proposition 4.2 shows, by Proposition 5.4, that

$$Z_1(\cdot, y) \in C^{1,\alpha}_{loc}(U \setminus \{y\}).$$

In particular, for fixed y, x_1, we have that $Z_1(\cdot, y) \in C^{1,\alpha}\left(B\left(x_1, \frac{1}{2}d(x_1, y)\right)\right)$ and we can apply Proposition 2.6 with $R = \frac{1}{2}d(x_1, y)$, writing

$$
\begin{aligned}
|Z_1(x_1, y) - Z_1(x_2, y)| \leqslant c d(x_1, x_2) \Bigg(& \sum_{i=1}^{n} \sup_{x \in B\left(x_1, \frac{1}{2}d(x_1, y)\right)} |X_i Z_1(x, y)| + \\
& + d(x_1, x_2) \sup_{x \in B\left(x_1, \frac{1}{2}d(x_1, y)\right)} |X_0 Z_1(x, y)| \Bigg)
\end{aligned}
\tag{5.5}
$$

for $d(x_1, y) \geqslant 2d(x_1, x_2)$. Let us estimate $\sup_{x \in B\left(x_1, \frac{1}{2}d(x_1, y)\right)} |X_i Z_1(x, y)|$. We know that:

$$
Z_1(x, y) = \sum_{i=1}^{3} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} Q_i(y, k;\ x, h)\, \varphi_i(h)\, \varphi(k)\, dh dk,
$$

where the Q_i's are defined in the proof of Proposition 4.2. Let us bound $X_i Z_1$ for one of the terms Q_i, for instance

$$
R_j^{(y,k)} R_j^{(y,k)} \Gamma\left(\Theta_{(y,k)}(x, h)\right)
$$

(since the other terms do not behave worse than this). We have, for $i = 1, 2, ..., n$,

$$
\begin{aligned}
& X_i \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} R_j^{(y,k)} R_j^{(y,k)} \Gamma\left(\Theta_{(y,k)}(x, h)\right) \varphi(h)\, \varphi(k)\, dh dk \\
& = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \widetilde{X}_i \left[R_j^{(y,k)} R_j^{(y,k)} \Gamma\left(\Theta_{(y,k)}(x, h)\right) \varphi(h)\right] \varphi(k)\, dh dk \\
& = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} R_j^{(y,k)} R_j^{(y,k)} \Gamma\left(\Theta_{(y,k)}(x, h)\right) \left(\widetilde{X} \varphi_i\right)(h)\, \varphi(k)\, dh dk \\
& + \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left(Y_i R_j^{(y,k)} R_j^{(y,k)} \Gamma\right) \left(\Theta_{(y,k)}(x, h)\right) \varphi(h)\, \varphi(k)\, dh dk \\
& + \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left(R_i^{(y,k)} R_j^{(y,k)} R_j^{(y,k)} \Gamma\right) \left(\Theta_{(y,k)}(x, h)\right) \varphi(h)\, \varphi(k)\, dh dk.
\end{aligned}
\tag{5.6}
$$

Now, by Proposition 5.4, (ii),

$$
\begin{aligned}
& \left| \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left(Y_i R_j^{(y,k)} R_j^{(y,k)} \Gamma\right) \left(\Theta_{(y,k)}(x, h)\right) \varphi(h)\, \varphi(k)\, dh dk \right| \\
& \leqslant c \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{\varphi(h)\, \varphi(k)}{\left\|\Theta_{(y,k)}(x, h)\right\|^{Q+1-\alpha}} dh dk \leqslant c \int_{d(x,y)}^{R} \frac{r^{\alpha-2}}{|B(x, r)|} dr,
\end{aligned}
$$

and the other two terms in (5.6) are bounded by the same quantity. Next, we have to take the supremum of the last quantity for $x \in B\left(x_1, \frac{1}{2}d(x_1, y)\right)$. Since $d(x_1, y) < 2d(x, y)$, by (3.2), this sup is bounded by

$$
c \int_{d(x_1,y)}^{R} \frac{r^{\alpha-2}}{|B(x, r)|} dr,
$$

hence

$$d(x_1,x_2)\sum_{i=1}^{n}\sup_{x\in B\left(x_1,\frac{1}{2}d(x_1,y)\right)}|X_iZ_1(x,y)|\leqslant cd(x_1,x_2)\int_{d(x_1,y)}^{R}\frac{r^{\alpha-2}}{|B(x,r)|}dr$$

since $d(x_1,x_2)\leqslant\frac{1}{2}d(x_1,y)<r$

$$\begin{aligned}&\leqslant cd(x_1,x_2)^{\alpha}\int_{d(x_1,y)}^{R}r^{1-\alpha}\frac{r^{\alpha-2}}{|B(x,r)|}dr\\&=cd(x_1,x_2)^{\alpha}\int_{d(x_1,y)}^{R}\frac{r^{-1}}{|B(x,r)|}dr\\&=cd(x_1,x_2)^{\alpha}\phi_0(x_1,y).\end{aligned}$$

An analogous computation gives

$$\begin{aligned}d(x_1,x_2)^2\sup_{x\in B\left(x_1,\frac{1}{2}d(x_1,y)\right)}|X_0Z_1(x,y)|&\leqslant cd(x_1,x_2)^2\int_{d(x_1,y)}^{R}\frac{r^{\alpha-3}}{|B(x,r)|}dr\\&=cd(x_1,x_2)^{\alpha}\int_{d(x_1,y)}^{R}r^{2-\alpha}\frac{r^{\alpha-3}}{|B(x,r)|}dr\\&=cd(x_1,x_2)^{\alpha}\phi_0(x_1,y).\end{aligned}$$

Then (5.5) implies

$$|Z_1(x_1,y)-Z_1(x_2,y)|\leqslant cd(x_1,x_2)^{\alpha}\phi_0(x_1,y)$$

and the lemma is proved. ■

Next we need the following:

Lemma 5.8 *For any $\beta>0$, let*

$$A(x_1,x_2,y)=\int_U|Z_1'(x_1,z)-Z_1'(x_2,z)|\phi_\beta(z,y)\,dz.$$

For any $\varepsilon>0$ there exists $c>0$ such that

$$A(x_1,x_2,y)\leqslant cd(x_1,x_2)^{\alpha-\varepsilon}\phi_{\beta+\varepsilon}(x_1,y)$$

for $d(x_1,y)\geqslant 3d(x_1,x_2)$.

Proof. Let us split:

$$A(x_1,x_2,y)=\int_{d(x_1,z)\geqslant 2d(x_1,x_2)}(\ldots)\,dz+\int_{d(x_1,z)<2d(x_1,x_2)}(\ldots)\,dz\equiv I+II.$$

By Lemma 5.7,

$$I\leqslant c\int_{d(x_1,z)\geqslant 2d(x_1,x_2)}d(x_1,x_2)^{\alpha}\phi_0(x_1,z)\,\phi_\beta(z,y)\,dz.$$

Now, for any $\varepsilon > 0$, and $d(x_1, z) \geqslant 2d(x_1, x_2)$, we have

$$d(x_1, x_2)^{\alpha} \phi_0(x_1, z) \leqslant c d(x_1, x_2)^{\alpha-\varepsilon} d(x_1, z)^{\varepsilon} \int_{d(x_1,z)}^{R} \frac{r^{-1}}{|B(x_1, r)|} dr$$

$$\leqslant c d(x_1, x_2)^{\alpha-\varepsilon} \int_{d(x_1,z)}^{R} \frac{r^{\varepsilon-1}}{|B(x_1, r)|} dr = c d(x_1, x_2)^{\alpha-\varepsilon} \phi_{\varepsilon}(x_1, z)$$

hence, by Theorem 3.5

$$I \leqslant c d(x_1, x_2)^{\alpha-\varepsilon} \int_{d(x_1,z) \geqslant 2d(x_1,x_2)} \phi_{\varepsilon}(x_1, z) \phi_{\beta}(z, y) dz \qquad (5.7)$$

$$\leqslant c d(x_1, x_2)^{\alpha-\varepsilon} \phi_{\beta+\varepsilon}(x_1, y).$$

Next,

$$II \leqslant \int_{d(x_1,z)<2d(x_1,x_2)} [\phi_{\alpha}(x_1, z) + \phi_{\alpha}(x_2, z)] \phi_{\beta}(z, y) dz$$

$$\equiv II_A + II_B.$$

From $d(x_1, y) \geqslant 3d(x_1, x_2)$ and $d(x_1, z) < 2d(x_1, x_2)$, we deduce $d(y, z) \geqslant d(x_1, x_2)$, hence $d(x_1, z) \leqslant 2d(y, z)$ and

$$d(x_1, y) \leqslant d(x_1, z) + d(z, y) \leqslant 3d(z, y)$$

which allows us to write

$$II_A \leqslant c\phi_{\beta}(x_1, y) \int_{d(x_1,z)<2d(x_1,x_2)} \phi_{\alpha}(x_1, z) dz$$

by Corollary 3.4

$$\leqslant c\phi_{\beta}(x_1, y) d(x_1, x_2)^{\alpha}.$$

By the same reason,

$$II_B \leqslant c\phi_{\beta}(x_1, y) \int_{d(x_1,z)<2d(x_1,x_2)} \phi_{\alpha}(x_2, z) dz$$

$$\leqslant c\phi_{\beta}(x_1, y) \int_{d(x_2,z)<3d(x_1,x_2)} \phi_{\alpha}(x_2, z) dz$$

$$\leqslant c\phi_{\beta}(x_1, y) d(x_1, x_2)^{\alpha}$$

as above. We conclude, for $d(x_1, y) \geqslant 3d(x_1, x_2)$,

$$II \leqslant c\phi_{\beta}(x_1, y) d(x_1, x_2)^{\alpha} \leqslant$$

$$\leqslant c d(x_1, x_2)^{\alpha-\varepsilon} d(x_1, y)^{\varepsilon} \int_{d(x_1,y)}^{R} \frac{r^{\beta-1}}{|B(x_1, r)|} dr$$

$$\leqslant c d(x_1, x_2)^{\alpha-\varepsilon} \phi_{\beta+\varepsilon}(x_1, y),$$

which together with (5.7) gives the assertion. ■

Proof of Proposition 5.5. Let $x_1, x_2, y \in U$ with $d(x_1, y) \geqslant 3d(x_1, x_2)$. By the identity (4.30) we can write

$$\Phi'(x_1, y) - \Phi'(x_2, y) = Z_1'(x_1, y) - Z_1'(x_2, y) + \int_U [Z_1'(x_1, z) - Z_1'(x_2, z)] \Phi'(z, y) dz$$

which by (4.20) gives

$$|\Phi'(x_1,y)-\Phi'(x_2,y)|\leqslant|Z_1'(x_1,y)-Z_1'(x_2,y)|+c\int_U|Z_1'(x_1,z)-Z_1'(x_2,z)|\,\phi_\alpha(z,y)\,dz.$$

Exploiting Lemmas 5.7 and 5.8, for any $\varepsilon>0$ we get

$$\begin{aligned}|\Phi'(x_1,y)-\Phi'(x_2,y)|&\leqslant cd(x_1,x_2)^\alpha\,\phi_0(x_1,y)+cd(x_1,x_2)^{\alpha-\varepsilon}\,\phi_{\beta+\varepsilon}(x_1,y)\\&\leqslant cd(x_1,x_2)^{\alpha-\varepsilon}\,\phi_\varepsilon(x_1,y)\end{aligned}$$

as desired. ■

5.2 Estimates on the second derivatives of the fundamental solution

We are now going to prove the existence and a sharp bound of Hölder type of the second derivatives of our local fundamental solution.

Theorem 5.9 (Second derivatives of the fundamental solution) *Under Assumptions B, for $i,j=1,2,...,n$ and for $x,y\in U, x\neq y$, the following assertions hold true.*

(i) There exist the second derivatives $X_jX_iJ'(x,y)$, $X_0J'(x,y)$, $X_iX_j\gamma(x,y)$, $X_0\gamma(x,y)$ continuous in the joint variables for $x\neq y$; in particular,

$$\gamma(\cdot,y)\in C_X^2(U\setminus\{y\})\text{ for any }y\in U.$$

(ii) For every $\varepsilon\in(0,\alpha)$, every $U'\Subset U$ there exists $c>0$ such that for every $x\in U'$ and $y\in U$,

$$|X_jX_iJ'(x,y)|\,,|X_0J'(x,y)|\leqslant cR^\varepsilon\frac{d(x,y)^{\alpha-\varepsilon}}{|B(x,d(x,y))|}\tag{5.8}$$

with R as at the beginning of §3, and

$$|X_jX_i\gamma(x,y)|\,,|X_0\gamma(x,y)|\leqslant c\frac{1}{|B(x,d(x,y))|}.\tag{5.9}$$

Note the presence, at the right-hand side of (5.8), (5.9), of the kernels $d(x,y)^{\alpha-\varepsilon}|B(x,d(x,y))|^{-1}$, $|B(x,d(x,y))|^{-1}$, instead of $\phi_{\alpha-\varepsilon}(x,y),\phi_0(x,y)$, which one could expect.

In order to reduce the length of some computation in the proof of this theorem and some of the following ones, it is convenient to introduce first the following abstract definitions, and make a preliminary study of the involved concept.

Definition 5.10 *We say that $R_\ell(x,y)$ is a remainder of type ℓ ($=0,1,2,3$) if for $x\neq y$*

$$R_\ell(x,y)=\sum_{s=1}^m\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}D_{\ell,s}^{(y,k)}\Gamma\left(\Theta_{(y,k)}(x,h)\right)a_s(h)\,b_s(k)\,dhdk$$

where $D_{\ell,s}^{(y,k)}$ are differential operators given by the composition of at most ℓ vector fields of the kind Y_i or $R_i^{(y,k)}$, of total weight $\geqslant\alpha-\ell$, depending on (y,k) in a C^α way and a_s, b_s are cutoff functions. Here and in the following, the number α is fixed, and is the exponent appearing in Assumptions B.

Definition 5.11 *We say that $k_\ell(x,y)$ is a* kernel of type ℓ *($=0,1,2,3$) if for $x\neq y$*

$$k_\ell(x,y)=\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} D_\ell\Gamma\left(\Theta_{(y,k)}(x,h)\right)a_0(h)\,b_0(k)\,dhdk+R_\ell(x,y)$$

where D_ℓ is a left invariant differential operator homogeneous of degree ℓ, a_0, b_0 are cutoff functions and $R_\ell(x,y)$ is a remainder of type ℓ. If $R_\ell(x,y)\equiv 0$, we say that $k_\ell(x,y)$ is a pure kernel of type ℓ.

Theorem 5.12 *Under Assumptions B, let $k_\ell(x,y)$ be a kernel of type ℓ. Then for $x\neq y$, $k_\ell(x,y)$ is jointly continuous and satisfies the bound:*

$$|k_\ell(x,y)|\leqslant c\phi_{2-\ell}(x,y).$$

Moreover, if $\ell\leqslant 2$, then $X_ik_\ell(x,y)$ is a kernel of type $\ell+1$ for $i=1,2,...,n$; if $\ell\leqslant 1$, then $X_0k_\ell(x,y)$ is a kernel of type $\ell+2$.

Let $R_\ell(x,y)$ be a remainder of type $\ell=0,1,2,3$. Then, for $x\neq y$, $R_\ell(x,y)$ is jointly continuous and satisfies the bound:

$$|R_\ell(x,y)|\leqslant c\phi_{2+\alpha-\ell}(x,y).$$

Also, if $\ell\leqslant 2$, then $X_iR_\ell(x,y)$ is a remainder of type $\ell+1$ for $i=1,2,...,n$; if $\ell\leqslant 1$, then $X_0R_\ell(x,y)$ is a remainder of type $\ell+2$.

Proof. The continuity properties follow as in the proof of Proposition 4.1. Also, we have

$$\left|D_\ell\Gamma\left(\Theta_{(y,k)}(x,h)\right)\right|\leqslant\frac{c}{\left\|\Theta_{(y,k)}(x,h)\right\|^{Q-2+\ell}},$$
$$\left|D_{\ell,s}^{(y,k)}\Gamma\left(\Theta_{(y,k)}(x,h)\right)\right|\leqslant\frac{c}{\left\|\Theta_{(y,k)}(x,h)\right\|^{Q-2+\ell-\alpha}}$$

hence by Lemma 3.1 we have

$$|k_\ell(x,y)|\leqslant c\phi_{2-\ell}(x,y),$$
$$|R_\ell(x,y)|\leqslant c\phi_{2-\ell+\alpha}(x,y).$$

Let us compute, for $x\neq y$,

$$X_ik_\ell(x,y)=\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\widetilde{X}_i\left[D_\ell\Gamma\left(\Theta_{(y,k)}(x,h)\right)a_0(h)\right]b_0(k)\,dhdk$$
$$+\sum_{s=1}^m\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\widetilde{X}_i\left[D_{\ell,s}^{(y,k)}\Gamma\left(\Theta_{(y,k)}(x,h)\right)a_s(h)\right]b_s(k)\,dhdk$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} Y_i D_\ell \Gamma\left(\Theta_{(y,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk \\
&+ \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} R_i^{(y,k)} D_\ell \Gamma\left(\Theta_{(y,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk \\
&+ \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} D_\ell \Gamma\left(\Theta_{(y,k)}(x,h)\right) \widetilde{X}_i a_0(h)\, b_0(k)\, dhdk \\
&+ \sum_{s=1}^{m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} Y_i D_{\ell,s}^{(y,k)} \Gamma\left(\Theta_{(y,k)}(x,h)\right) a_s(h)\, b_s(k)\, dhdk \\
&+ \sum_{s=1}^{m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} R_i^{(y,k)} D_{\ell,s}^{(y,k)} \Gamma\left(\Theta_{(y,k)}(x,h)\right) a_s(h)\, b_s(k)\, dhdk \\
&+ \sum_{s=1}^{m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} D_{\ell,s}^{(y,k)} \Gamma\left(\Theta_{(y,k)}(x,h)\right) \widetilde{X}_i a_s(h)\, b_s(k)\, dhdk
\end{aligned}
$$

by Proposition 5.4 and Definition 5.10

$$
\begin{aligned}
&= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} D_{\ell+1} \Gamma\left(\Theta_{(y,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk \\
&+ \sum_{s=1}^{m'}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} D_{\ell+1,s}^{(y,k)} \Gamma\left(\Theta_{(y,k)}(x,h)\right) a_s'(h)\, b_s'(k)\, dhdk
\end{aligned}
$$

which gives the desired result for $X_i k_\ell$; analogously one can handle $X_0 k_\ell$. ■

Definition 5.13 *Let $\Phi_0 : \{(x,y) \in U \times U : x \neq y\} \to \mathbb{R}$. We say that Φ_0 is a function of (ϕ,α)-type if it is continuous (in the joint variables), satisfies*

$$
|\Phi_0(x,y)| \leqslant c\phi_\alpha(x,y)
$$

and for every $\varepsilon \in (0,\alpha)$ there exists a constant c_ε such that for every $x_1, x_2, y \in U$ with $d(x_1,y) \geqslant 3d(x_1,x_2)$

$$
|\Phi_0(x_1,y) - \Phi_0(x_2,y)| \leqslant c_\varepsilon d(x_1,x_2)^{\alpha-\varepsilon}\, \phi_\varepsilon(x_1,y).
$$

Lemma 5.14 *Let Φ_0 be a (ϕ,α)-type function. For every $\varepsilon \in (0,\alpha)$ there exists c_ε such that for every $x_1, x_2, y \in U$ we have*

$$
|\Phi_0(x_1,y) - \Phi_0(x_2,y)| \leqslant c_\varepsilon d(x_1,x_2)^{\alpha-\varepsilon}\left[\frac{d(x_1,y)^\varepsilon}{|B(x_1,d(x_1,y))|} + \frac{d(x_2,y)^\varepsilon}{|B(x_2,d(x_2,y))|}\right].
$$

The Lemma follows from the above definition as in the proof of Corollary 5.6. We will also need the following easy

Lemma 5.15 *If $\beta \in \mathbb{R}$ and $\varepsilon > 0$, then there exists $c > 0$ such that*

$$
\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{\psi(h)\,\varphi(k)}{\left\|\Theta_{(y,k)}(x,h)\right\|^{Q-\beta}} \chi_{\left\{h:\left\|\Theta_{(y,k)}(x,h)\right\|<\delta\right\}} dhdk \leqslant c\delta^\varepsilon \phi_{\beta-\varepsilon}(x,y). \quad (5.10)
$$

Proof. To prove (5.10) it is enough to observe that

$$
\begin{aligned}
&\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{\psi(h)\,\varphi(k)}{\left\|\Theta_{(y,k)}(x,h)\right\|^{Q-\beta}} \chi_{\left\{h:\left\|\Theta_{(y,k)}(x,h)\right\|<\delta\right\}} dhdk \\
&\leqslant \delta^\varepsilon \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{\psi(h)\,\varphi(k)}{\left\|\Theta_{(y,k)}(x,h)\right\|^{Q-\beta+\varepsilon}} \chi_{\left\{h:\left\|\Theta_{(y,k)}(x,h)\right\|<\delta\right\}} dhdk \leqslant c\delta^\varepsilon \phi_{\beta-\varepsilon}(x,y)
\end{aligned}
$$

by Lemma 3.1. ■

Theorem 5.16 *Let k be a kernel of type $\ell = 0$ and let Φ_0 be a function of (ϕ, α)-type. If*

$$J_0(x,y) = \int_U k(x,z)\,\Phi_0(z,y)\,dz,$$

then for $i = 1, 2, ..., n$,

$$X_i J_0(x,y) = \int_U X_i k(x,z)\,\Phi_0(z,y)\,dz$$

and

$$|X_i J_0(x,y)| \leqslant c\phi_{1+\alpha}(x,y).$$

Let $\omega_\delta \in C^\infty(\mathbb{G})$ such that $\omega_\delta(u) = 1$ for $\|u\| > \delta$ and $\omega_\delta(u) = 0$ for $\|u\| < \delta/2$, then, for $i, j = 1, 2, ..., n$, $X_j X_i J_0(x,y)$ exists and is continuous in the joint variables for $x \neq y$ and can be computed as follows

$$\begin{aligned}
&X_j X_i J_0(x,y) \\
&= \lim_{\delta\to 0} \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} Y_j(\omega_\delta D_1\Gamma)\big(\Theta_{(z,k)}(x,h)\big)\,a_0(h)\,b_0(k)\,dhdk\,\Phi_0(z,y)\,dz \\
&+ \int_U R_2'(x,z)\,\Phi_0(z,y)\,dz \\
&= \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (Y_j D_1\Gamma)\big(\Theta_{(z,k)}(x,h)\big)\,a_0(h)\,b_0(k)\,dhdk\,\left[\Phi_0(z,y) - \Phi_0(x,y)\right]dz \\
&+ C(x)\,\Phi_0(x,y) + \int_U R_2''(x,z)\,\Phi_0(z,y)\,dz
\end{aligned} \tag{5.11}$$

where D_1 is a left invariant homogeneous vector field of degree 1, $R_2'(x,z)$ and $R_2''(x,z)$ are suitable remainders of type 2 and $C \in C^\alpha_{X,loc}(U)$. Moreover, for any $U' \Subset U$ there exists $c > 0$ such that for every $x \in U'$, $y \in U$, $x \neq y$

$$|X_j X_i J_0(x,y)| \leqslant c\frac{d(x,y)^{\alpha-\varepsilon}}{|B(x, d(x,y))|}. \tag{5.12}$$

Proof. Since $X_i k \equiv k_1$ is a kernel of type 1 we have

$$|X_i k(x,y)| \leqslant c\phi_1(x,y) \leqslant c\frac{d(x,y)}{|B(x,d(x,y))|},$$

so that we can differentiate under the integral sign. Therefore

$$X_i J_0(x,y) = \int_U k_1(x,z)\,\Phi_0(z,y)\,dz$$

with

$$|X_i J_0(x,y)| \leqslant c\phi_{1+\alpha}(x,y).$$

In order to compute $X_j X_i J_0(x,y)$ we rewrite

$$k_1(x,z) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} D_1\Gamma\big(\Theta_{(z,k)}(x,h)\big)\,a_0(h)\,b_0(k)\,dhdk + R_1(x,z)$$

where $R_1(x,z)$ is a remainder of type 1. Then

$$\begin{aligned}X_i J_0(x,y) &= \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} D_1\Gamma\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk\, \Phi_0(z,y)\, dz \\ &+ \int_U R_1(x,z)\,\Phi_0(z,y)\,dz \\ &\equiv B_1(x,y) + B_2(x,y)\,.\end{aligned}$$

As to B_2 we can simply write

$$\begin{aligned}X_j B_2(x,y) &= \int_U X_j R_1(x,z)\,\Phi_0(z,y)\,dz \\ &\equiv \int_U R_2(x,z)\,\Phi_0(z,y)\,dz\end{aligned}$$

where $R_2(x,z)$ is a remainder of type 2.

To handle $B_1(x,y)$ we consider

$$B_1^\delta(x,y) = \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (\omega_\delta D_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk\, \Phi_0(z,y)\, dz.$$

Due to the presence of this cutoff function, we can compute the derivative

$$\begin{aligned}X_j B_1^\delta(x,y) &= \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \widetilde{X}_j \left[(\omega_\delta D_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\right] b_0(k)\, dhdk\, \Phi_0(z,y)\, dz \\ &= \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} Y_j (\omega_\delta D_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk\, \Phi_0(z,y)\, dz \\ &+ \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} R_j^{(y,k)} (\omega_\delta D_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk\, \Phi_0(z,y)\, dz \\ &+ \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (\omega_\delta D_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right) \widetilde{X}_j a_0(h)\; b_0(k)\, dhdk\, \Phi_0(z,y)\, dz \\ &= B_{1,1}^\delta(x,y) + B_{1,2}^\delta(x,y) + B_{1,3}^\delta(x,y)\,.\end{aligned}$$

An argument similar to one already used shows that for any fixed δ the function $X_j B_1^\delta(x,y)$ is continuous in the joint variables for any $x,y \in U$, $x \neq y$.

First of all we observe that

$$\lim_{\delta\to 0} B_{1,2}^\delta(x,y) = \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left(R_j^{(y,k)} D_1\Gamma\right)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk\, \Phi_0(z,y)\, dz$$

since

$$\left(R_j^{(y,k)}\omega_\delta\right)(u) \neq 0 \text{ for } \frac{\delta}{2} < \|u\| < \delta,$$

hence

$$\begin{aligned}&\left|\int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left(\left(R_j^{(y,k)}\omega_\delta\right) D_1\Gamma\right)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk\, \Phi_0(z,y)\, dz\right| \\ &\leqslant \int_{\mathbb{R}^m} \int_{\left\|\Theta_{(z,k)}(x,h)\right\|<\delta} \frac{c}{\left\|\Theta_{(z,k)}(x,h)\right\|^{Q-\alpha}} \left|a_0(h)\, b_0(k)\, \Phi_0(z,y)\right| dk\, dzdh \\ &\leqslant \int_{\mathbb{R}^m} \int_{\left\|\Theta_{(z,k)}(x,h)\right\|<\delta} \frac{c}{\left\|\Theta_{(z,k)}(x,h)\right\|^{Q-\alpha}} dk\, dz \left|a_0(h)\right| dh \leqslant c\delta^\alpha \to 0 \text{ as } \delta \to 0\end{aligned}$$

for some constant c depending on $d(x,y)$.

Also

$$\lim_{\delta\to 0} B_{1,3}^{\delta}(x,y) = \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (D_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right) \widetilde{X}_j a_0(h)\, b_0(k)\, dhdk\, \Phi_0(z,y)\, dz$$

so that

$$\lim_{\delta\to 0} X_j B_1^{\delta}(x,y) = \lim_{\delta\to 0} B_{1,1}^{\delta}(x,y) + \int_U R_3(x,z)\,\Phi_0(z,y)\,dz$$

where $R_3(x,z)$ is still another remainder of type 2.

Let us now consider

$$B_{1,1}^{\delta}(x,y) = \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} Y_j(\omega_\delta D_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk\, \Phi_0(z,y)\, dz.$$

We write

$$\begin{aligned} B_{1,1}^{\delta}(x,y) &= \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} Y_j(\omega_\delta D_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk\, [\Phi_0(z,y) - \Phi_0(x,y)]\, dz \\ &\quad + \Phi_0(x,y) \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} Y_j(\omega_\delta D_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk\, dz \\ &\equiv B_{1,1,1}^{\delta}(x,y) + B_{1,1,2}^{\delta}(x,y). \end{aligned}$$

We have

$$\begin{aligned} B_{1,1,1}^{\delta}(x,y) &= \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (Y_j\omega_\delta \cdot D_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk\, [\Phi_0(z,y) - \Phi_0(x,y)]\, dz \\ &\quad + \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (\omega_\delta\, Y_j D_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk\, [\Phi_0(z,y) - \Phi_0(x,y)]\, dz. \end{aligned}$$

Since $Y_j\omega_\delta\left(\Theta_{(z,k)}(x,h)\right)$ is supported in $\left\{\frac{\delta}{2} < \left\|\Theta_{(z,k)}(x,h)\right\| < \delta\right\}$ and bounded by δ^{-1}, by Lemma 5.15 , Corollary 5.6 and Lemma 3.3 the first term of $B_{1,1,1}^{\delta}(x,y)$ is bounded by

$$\begin{aligned} & c\int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \chi_{\left\{\delta/2 \leqslant \left\|\Theta_{(z,k)}(x,h)\right\| \leqslant \delta\right\}} \left\|\Theta_{(z,k)}(x,h)\right\|^{-Q} a_0(h)\, b_0(k)\, dhdk\, |\Phi_0(z,y) - \Phi_0(x,y)|\, dz \\ & \leqslant c\int_U \delta^{\varepsilon} \phi_{-\varepsilon}(x,z)\, |\Phi_0(z,y) - \Phi_0(x,y)|\, dz \\ & \leqslant c\delta^{\varepsilon} \int_U \frac{d(x,z)^{\alpha-2\varepsilon}}{|B(x,d(x,z))|} \left(\frac{d(z,y)^{\varepsilon}}{|B(z,d(z,y))|} + \frac{d(x,y)^{\varepsilon}}{|B(x,d(x,y))|} \right) dz. \end{aligned}$$

Since this last integral converges the first term in $B_{1,1,1}^{\delta}(x,y)$ vanishes uniformly in x (as long as x stays away from y) as $\delta \to 0$. We will show now that the second term converges uniformly to

$$\int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (Y_j D_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk\, [\Phi_0(z,y) - \Phi_0(x,y)]\, dz$$

as $\delta \to 0$. Again, by Lemma 5.15, Lemma 5.14 and Lemma 3.3 we have

$$\begin{aligned} & \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left|((1-\omega_\delta)\, Y_j D_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right)\right| a_0(h)\, b_0(k)\, dhdk\, |\Phi_0(z,y) - \Phi_0(x,y)|\, dz \\ & \leqslant c\int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \chi_{\left\{\left\|\Theta_{(z,k)}(x,h)\right\| \leqslant \delta\right\}} \left\|\Theta_{(z,k)}(x,h)\right\|^{-Q} |a_0(h)\, b_0(k)|\, dhdk\, |\Phi_0(z,y) - \Phi_0(x,y)|\, dz \\ & \leqslant c\delta^{\varepsilon} \int_U \phi_{-\varepsilon}(x,z)\, |\Phi_0(z,y) - \Phi_0(x,y)|\, dz \end{aligned}$$

and from this bound we conclude as above that this term converges to 0 as $\delta \to 0$, uniformly as soon as $d(x,y) \geqslant c$.

To handle $B^{\delta}_{1,1,2}(x,y)$, let us first fix some notation. Let $U' \Subset U$, $I \subset \mathbb{R}^m$ and $r > 0$ such that $I \supset \operatorname{sprt} a_0 \cup \operatorname{sprt} b_0$ and:

$$(x,h) \in U' \times \operatorname{sprt} a_0 \text{ and } \left\|\Theta_{(z,k)}(x,h)\right\| < r \Rightarrow (z,k) \in U \times I \equiv \Sigma.$$

Then for any $x \in U'$ we have:

$$B^{\delta}_{1,1,2}(x,y) = \Phi_0(x,y)\int_{\mathbb{R}^m} a_0(h)\left(\int_{\Sigma} Y_j(\omega_\delta D_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right) b_0(k)\,dkdz\right)dh$$

$$= \Phi_0(x,y)\int_{\mathbb{R}^m} a_0(h)\left(\int_{\left\|\Theta_{(z,k)}(x,h)\right\|<r}(...)\,dkdz + \int_{\Sigma,\left\|\Theta_{(z,k)}(x,h)\right\|\geqslant r}(...)\,dkdz\right)dh$$

$$= \Phi_0(x,y)\left[I^{\delta}_1(x) + I^{\delta}_2(x)\right].$$

Next, making the change of variables $(z,k) \mapsto u = \Theta_{(z,k)}(x,h)$ and letting $\widetilde{b}_0(\xi,u) = b_0\left(\Theta_{\cdot}(x,h)^{-1}(u)\right)$,

$$I^{\delta}_1(x) = \int_{\mathbb{R}^m} c(\xi)\,a_0(h)\left(\int_{\|u\|<r}\left[Y_j(\omega_\delta D_1\Gamma)\right](u)\,(1+\chi(\xi,u))\,\widetilde{b}_0(\xi,u)\,du\right)dh$$

$$= \int_{\mathbb{R}^m} c(\xi)\,a_0(h)\left(\int_{\|u\|<r}\left[Y_j(\omega_\delta D_1\Gamma)\right](u)\,\chi(\xi,u)\,\widetilde{b}_0(\xi,u)\,du\right)dh$$

$$+ \int_{\mathbb{R}^m} c(\xi)\,a_0(h)\left(\int_{\|u\|<r}\left[Y_j(\omega_\delta D_1\Gamma)\right](u)\,\widetilde{b}_0(\xi,u)\,du\right)dh$$

$$\equiv \beta^{\delta}_1(x) + \beta^{\delta}_2(x).$$

By Proposition 5.4 we know that $|\chi(\xi,u)| \leqslant c\,\|u\|^{\alpha}$, hence for $\delta \to 0$ (by the same argument used to compute the limit of $B^{\delta}_{1,2}$)

$$\beta^{\delta}_1(x) \to \int_{\mathbb{R}^m} c(\xi)\,a_0(h)\left(\int_{\|u\|<r}(Y_jD_1\Gamma)(u)\,\chi(\xi,u)\,\widetilde{b}_0(\xi,u)\,du\right)dh \equiv \beta_1(x).$$

Note $\beta_1 \in C^{\alpha}(U')$. Namely, the functions $c(\cdot), \chi(\cdot,u)$ are Hölder continuous by 5.4 (iii); since $\Theta_{\cdot}(x,h)^{-1}(u)$ is $C^{1,\alpha}$ also $\widetilde{b}_0(\cdot,u)$ is Hölder continuous.

To handle $\beta^{\delta}_2(x)$ we integrate by parts; writing $Y_j = \sum_{k=1}^{N} a_{jk}(u)\,\partial_{u_k}$ and denoting by $\nu = (\nu_1, \nu_2, ..., \nu_N)$ the unit outer normal we get

$$\beta^{\delta}_2(x) = -\int_{\mathbb{R}^m} c(\xi)\,a_0(h)\left(\int_{\|u\|<r}(\omega_\delta D_1\Gamma)(u)\left(Y_j\widetilde{b}_0(\xi,\cdot)\right)(u)\,du\right)dh$$

$$+ \int_{\mathbb{R}^m} c(\xi)\,a_0(h)\left(\int_{\|u\|=r}(\omega_\delta D_1\Gamma)(u)\,\widetilde{b}_0(\xi,u)\sum_k a_{jk}(u)\,\nu_k\,d\sigma(u)\right)dh$$

$$\to -\int_{\mathbb{R}^m} c(\xi)\,a_0(h)\left(\int_{\|u\|<r}(D_1\Gamma)(u)\left(Y_j\widetilde{b}_0(\xi,\cdot)\right)(u)\,du\right)dh$$

$$+ \int_{\mathbb{R}^m} c(\xi)\,a_0(h)\left(\int_{\|u\|=r}(D_1\Gamma)(u)\,\widetilde{b}_0(\xi,u)\sum_{k=1}^{N} a_{jk}(u)\,\nu_k\,d\sigma(u)\right)dh$$

$$\equiv \beta_2(x)$$

which is again a $C^{\alpha}(U')$ function by Proposition 5.4 (iii). Hence for any $x \in U'$,

$$I_1^{\delta}(x) \to \beta_1(x) + \beta_2(x),$$

which is a $C^{\alpha}(U')$ function.

As to $I_2^{\delta}(x)$, for any $\delta < r$ we have (writing $\eta = (z, k)$)

$$I_2^{\delta}(x) = I_2(x) \equiv \int_{\mathbb{R}^m} a_0(h) \left(\int_{\Sigma, \|\Theta_\eta(x,h)\|>r} (Y_j D_1 \Gamma)(\Theta_\eta(x,h))\, b_0(k)\, d\eta \right) dh.$$

Let us show that I_2 is Hölder continuous. Actually, we will show that

$$|I_2(x_1) - I_2(x_2)| \leqslant c\,|x_1 - x_2|$$

for small $|x_1 - x_2|$. Since I_2 is clearly bounded, this is enough to conclude Hölder continuity in U'.

$$\begin{aligned}
&I_2(x_1) - I_2(x_2)\\
&= \int_{\mathbb{R}^m} a_0(h) \left(\int_{\Sigma, \|\Theta_\eta(x_1,h)\|>r} [Y_j D_1 \Gamma(\Theta_\eta(x_1,h)) - Y_j D_1 \Gamma(\Theta_\eta(x_2,h))]\, b_0(k)\, d\eta \right) dh\\
&+ \int_{\mathbb{R}^m} a_0(h) \left(\int_{\Sigma, \|\Theta_\eta(x_1,h)\|>r} Y_j D_1 \Gamma(\Theta_\eta(x_2,h))\, b_0(k)\, d\eta \right) dh\\
&- \int_{\mathbb{R}^m} a_0(h) \left(\int_{\Sigma, \|\Theta_\eta(x_2,h)\|>r} Y_j D_1 \Gamma(\Theta_\eta(x_2,h))\, b_0(k)\, d\eta \right) dh\\
&\equiv A + B - C.
\end{aligned}$$

Note that for some small $c_1(r), c_2(r) > 0$, if $|x_1 - x_2| < c_1$ and $\|\Theta_\eta(x_1,h)\| > r$ then also $\|\Theta_\eta(x_2,h)\| > c_2 r$. Then

$$|A| \leqslant c(r)\,|x_1 - x_2| \int_{\mathbb{R}^m} a_0(h) \int_{\Sigma, \|\Theta_\eta(x_1,h)\|>r} b_0(k)\, d\eta dh \leqslant c(r)\,|x_1 - x_2|.$$

Moreover, letting

$$\Lambda = \{\eta : \|\Theta_\eta(x_2,h)\| > r, \|\Theta_\eta(x_1,h)\| \leqslant r\ \} \cup \{\eta : \|\Theta_\eta(x_1,h)\| > r, \|\Theta_\eta(x_2,h)\| \leqslant r\} = \Lambda_1 \cup \Lambda_2$$

we have:

$$|B - C| \leqslant \int_{\mathbb{R}^m} a_0(h) \left(\int_{\Sigma \cap \Lambda} |Y_j D_1 \Gamma(\Theta_\eta(x_2,h))|\, b_0(\eta)\, d\eta \right) dh$$

In Λ_1 we have

$$\begin{aligned}
r < \|\Theta_\eta(x_2,h)\| &\leqslant \|\Theta_\eta(x_1,h)\| + \|\Theta_\eta(x_2,h) - \Theta_\eta(x_1,h)\|\\
&\leqslant r + \|\Theta_\eta(x_2,h) - \Theta_\eta(x_1,h)\| \leqslant r + c\,|x_1 - x_2|
\end{aligned}$$

hence

$$\int_{\mathbb{R}^m} a_0(h)\left(\int_{\Sigma\cap\Lambda_1} |Y_j D_1\Gamma(\Theta_\eta(x_2,h))|\, b_0(\eta)\, d\eta\right) dh$$
$$\leqslant c(r)\int_{\mathbb{R}^m} a_0(h)\left(\int_{r<\|\Theta_\eta(x_2,h)\|<r+c|x_1-x_2|} b_0(\eta)\, d\eta\right) dh$$
$$\leqslant c\int_{\mathbb{R}^m} a_0(h)\left(\int_{r<\|u\|<r+c|x_1-x_2|} du\right) dh$$
$$\leqslant c\left[(r+c|x_1-x_2|)^Q - r^Q\right] \leqslant c(r)\,|x_1-x_2|\,.$$

Since for $\eta\in\Lambda_2$ we have $\|\Theta_\eta(x_2,h)\|>c_2 r$ (by the above remark), we can still write

$$\int_{\mathbb{R}^m} a_0(h)\left(\int_{\Sigma\cap\Lambda_2} |Y_j D_1\Gamma(\Theta_\eta(x_2,h))|\, b_0(\eta)\, d\eta\right) dh$$
$$\leqslant c\int_{\mathbb{R}^m} a_0(h)\left(\int_{r<\|\Theta_\eta(x_1,h)\|<r+c|x_1-x_2|} b_0(\eta)\, d\eta\right) dh$$
$$\leqslant c\,|x_1-x_2|\,.$$

We can conclude that

$$B^\delta_{1,1,2}(x,y)\to C(x)\,\Phi_0(x,y) \text{ as } \delta\to 0,$$

where $C(x)$ is a suitable $C^\alpha_{X,loc}(U)$ function. This completes the proof of (5.11). In particular, for any $x\in U', y\in U$,

$$|X_k X_i J_0(x,y)| \leqslant c_1\int_U \phi_0(x,z)\,|\Phi_0(z,y)-\Phi_0(x,y)|\, dz$$
$$+ c_2\phi_\alpha(x,y) + c_3\int_U \phi_\alpha(x,z)\,\phi_\alpha(z,y)\, dz.$$

Let now

$$\int_U \phi_0(x,z)\,|\Phi_0(z,y)-\Phi_0(x,y)|\, dz$$
$$\leqslant c\int_{\{d(x,y)\geqslant 3d(x,z)\}} \phi_0(x,z)\, d^{\alpha-\varepsilon}(x,z)\,\phi_\varepsilon(x,y)\, dz$$
$$+ c\int_{\{d(x,y)<3d(x,z)\}} \phi_0(x,z)\,(\phi_\alpha(z,y)+\phi_\alpha(x,y))\, dz$$
$$= D + E.$$

Then

$$D \leqslant c\phi_\varepsilon(x,y)\int_{\{d(x,y)\geqslant 3d(x,z)\}} \phi_0(x,z)\, d^{\alpha-\varepsilon}(x,z)\, dz$$
$$\leqslant c\phi_\varepsilon(x,y)\int_{\{d(x,y)\geqslant 3d(x,z)\}} \phi_{\alpha-\varepsilon}(x,z)\, dz$$
$$\leqslant c\phi_\varepsilon(x,y)\, d(x,y)^{\alpha-\varepsilon} \leqslant c\frac{d(x,y)^\alpha}{|B(x,d(x,y))|}.$$

and

$$\begin{aligned}E &\leqslant \frac{c}{d(x,y)^{\varepsilon}} \int_U \left(\phi_\alpha(z,y) + \phi_\alpha(x,y)\right) \phi_\varepsilon(x,z)\, dz \\ &\leqslant \frac{c}{d(x,y)^{\varepsilon}} \left[\phi_{\alpha+\varepsilon}(x,y) + \phi_\alpha(x,y) \int_U \phi_\varepsilon(x,z)\, dz\right] \\ &\leqslant \frac{c}{d(x,y)^{\varepsilon}} \left[\phi_{\alpha+\varepsilon}(x,y) + \phi_\alpha(x,y) R^{\varepsilon}\right] \\ &\leqslant \frac{c}{d(x,y)^{\varepsilon}} \left[\frac{d(x,y)^{\alpha+\varepsilon}}{|B(x,d(x,y))|} + \frac{d(x,y)^{\alpha} R^{\varepsilon}}{|B(x,d(x,y))|}\right] \\ &\leqslant cR^{\varepsilon} \frac{d(x,y)^{\alpha-\varepsilon}}{|B(x,d(x,y))|}.\end{aligned}$$

It follows that

$$\begin{aligned}|X_k X_i J_0(x,y)| &\leqslant cR^{\varepsilon} \frac{d(x,y)^{\alpha-\varepsilon}}{|B(x,d(x,y))|} + c_2 \phi_\alpha(x,y) + c_3 \phi_{2\alpha}(x,y) \\ &\leqslant c \frac{d(x,y)^{\alpha-\varepsilon}}{|B(x,d(x,y))|},\end{aligned}$$

which proves (5.12). ■

Proof of Theorem 5.9. It is enough to prove (5.8) and the continuity of $X_i X_j J'(x,y)$, $X_0 J'(x,y)$ in the joint variables, for $x \neq y$, because these facts together with Proposition 4.1 imply (5.9) and the continuity properties of $X_i X_j \gamma(x,y)$, $X_0 \gamma(x,y)$. The results about $X_i X_j J'(x,y)$ immediately follow by Theorem 5.16 choosing $\Phi_0 = \Phi'$. The proof of the analog result for $X_0 J'(x,y)$ is very similar: we can start from

$$J'_\delta(x,y) = \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (\omega_\delta \Gamma)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\, b_0(k)\, dh dk\, \Phi'(z,y)\, dz$$

and compute

$$X_j J'_\delta(x,y) = \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \widetilde{X}_j \left[(\omega_\delta \Gamma)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\right] b_0(k)\, dh dk\, \Phi'(z,y)\, dz$$

From this point the computation of $X_0 J'(x,y)$ proceeds as above. ■

We can now refine the previous analysis of the second derivatives of our local fundamental solution and prove a sharp bound of Hölder type on $X_i X_j \gamma$. This is both interesting in its own, and will be a basic ingredient to deduce, via the theory of singular integrals, local Hölder estimates for the second derivatives of the local solution to the equation $Lw = f$ that we will build in the next section.

Theorem 5.17 *For every $\varepsilon \in (0,\alpha)$ and $U' \Subset U$ there exists $c > 0$ such that* *for every* $x_1, x_2 \in U'$, $y \in U$ *such that* $d(x_1, y) \geqslant 2d(x_1, x_2)$, $i, j = 1, 2, ..., n$,

$$|X_i X_j P(x_1, y) - X_i X_j P(x_2, y)| \leqslant c \frac{d(x_1, x_2)}{d(x_1, y)} \frac{1}{|B(x_1, d(x_1, y))|}; \tag{5.13}$$

$$|X_i X_j J'(x_1, y) - X_i X_j J'(x_2, y)| \leqslant c \left(\frac{d(x_1, x_2)}{d(x_1, y)} \right)^{\alpha - 2\varepsilon} \frac{d(x_1, y)^{\alpha - \varepsilon}}{|B(x_1, d(x_1, y))|}; \tag{5.14}$$

$$|X_i X_j \gamma(x_1, y) - X_i X_j \gamma(x_2, y)| \leqslant c \left(\frac{d(x_1, x_2)}{d(x_1, y)} \right)^{\alpha - \varepsilon} \frac{1}{|B(x_1, d(x_1, y))|}; \tag{5.15}$$

$$|X_0 \gamma(x_1, y) - X_0 \gamma(x_2, y)| \leqslant c \left(\frac{d(x_1, x_2)}{d(x_1, y)} \right)^{\alpha - \varepsilon} \frac{1}{|B(x_1, d(x_1, y))|}. \tag{5.16}$$

In particular, for every $\varepsilon \in (0, \alpha)$ *and* $y \in U$,

$$\gamma(\cdot, y) \in C^{2, \alpha - \varepsilon}_{X, loc}(U \setminus \{y\}).$$

Proof. The proof will be achieved in several steps. We know that

$$X_i X_j \gamma(x, y) = \frac{1}{c_0(y)} [X_i X_j P(x, y) + X_i X_j J'(x, y)], \tag{5.17}$$

hence (5.15) will follow from (5.13) and (5.14). Also (5.16) will follow from (5.15) since $X_0 \gamma(x, y) = -\sum_{j=1}^n X_j^2 \gamma(x, y)$ for $x \neq y$.

Let us first prove (5.13). To do this, let us apply "Lagrange theorem" (Proposition 2.6) to the function

$$f(x) = X_i X_j P(x, y) \text{ for } x \in B\left(x_1, \frac{1}{2} d(x_1, y)\right):$$

$$|X_i X_j P(x_1, y) - X_i X_j P(x_2, y)| \leqslant c d(x_1, x_2) \left(\sum_{k=1}^n \sup_{B\left(x_1, \frac{1}{2} d(x_1, y)\right)} |X_k X_i X_j P(\cdot, y)| + \right.$$
$$\left. + d(x_1, x_2) \sup_{B\left(x_1, \frac{1}{2} d(x_1, y)\right)} |X_0 X_i X_j P(\cdot, y)| \right).$$

Note that since, under our assumptions, the coefficients of the X_i's belong to $C^{r,\alpha}$, with $r \geqslant 2$, the compositions $X_k X_i X_j$, $X_0 X_i X_j$ are actually well defined. Reasoning like in the proof of Proposition 4.1 we get, for $x \in B\left(x_1, \frac{1}{2} d(x_1, y)\right)$

$$|X_k X_i X_j P(x, y)| \leqslant c \phi_{-1}(x, y) \leqslant c \phi_{-1}(x_1, y) \leqslant \frac{c}{d(x_1, y) |B(x_1, d(x_1, y))|}$$

by Lemma 3.3. Analogously,

$$|X_0 X_i X_j P(x, y)| \leqslant c \phi_{-2}(x, y) \leqslant \frac{c}{d(x_1, y)^2 |B(x_1, d(x_1, y))|}$$

so that, for $2d(x_1,x_2) \leqslant d(x_1,y)$,

$$\left|X_iX_jP(x_1,y) - X_iX_jP(x_2,y)\right| \leqslant c\frac{d(x_1,x_2)}{d(x_1,y)}\frac{1}{|B(x_1,d(x_1,y))|}$$

and (5.13) is proved. Applying Theorem 5.16 with $\Phi_0 = \Phi'$, we know that for any $x \in U', y \in U$

$$\begin{aligned}
&X_jX_iJ'(x,y) \\
&= \int_U \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (Y_jD_1\Gamma)\left(\Theta_{(z,k)}(x,h)\right) a_0(h)\, b_0(k)\, dhdk\, \left[\Phi'(z,y) - \Phi'(x,y)\right] dz \\
&+ C(x)\,\Phi'(x,y) + \int_U R_2(x,z)\,\Phi'(z,y)\,dz \\
&\equiv A(x,y) + B(x,y) + C(x,y).
\end{aligned}$$

Let us start from the last two terms, which are easier. By Proposition 5.5 and the local Hölder continuity of $C(x)$ we have

$$\begin{aligned}
|B(x_2,y) - B(x_1,y)| &\leqslant |C(x_2) - C(x_1)|\,\Phi'(x_2,y) + |C(x_1)|\,|\Phi'(x_2,y) - \Phi'(x_1,y)| \\
&\leqslant cd(x_1,x_2)^{\alpha}\,\phi_\alpha(x_2,y) + cd(x_1,x_2)^{\alpha-\varepsilon}\,\phi_\varepsilon(x_1,y) \\
&\leqslant cd(x_1,x_2)^{\alpha-\varepsilon}\,\phi_\varepsilon(x_1,y) \leqslant c\left(\frac{d(x_1,x_2)}{d(x_1,y)}\right)^{\alpha-\varepsilon}\frac{d(x_1,y)^{\alpha}}{|B(x_1,d(x_1,y))|}.
\end{aligned}$$

As to C,

$$\begin{aligned}
C(x_2,y) - C(x_1,y) &= \int_U \left[R_2(x_2,z) - R_2(x_1,z)\right]\Phi'(z,y)\,dz \\
&= \int_{U,d(z,x_1)>2d(x_1,x_2)} (...)\,dz + \int_{U,d(z,x_1)\leqslant 2d(x_1,x_2)} (...)\,dz \\
&\equiv C_1 + C_2.
\end{aligned}$$

To bound C_1 we apply Lagrange theorem:

$$\begin{aligned}
|R_2(x_2,z) - R_2(x_1,z)| &\leqslant cd(x_1,x_2)\left(\sum_{k=1}^{n}\sup_{B\left(x_1,\frac{1}{2}d(x_1,z)\right)}|X_kR_2(\cdot,z)|\right. \\
&\quad \left.+d(x_1,x_2)\sup_{B\left(x_1,\frac{1}{2}d(x_1,z)\right)}|X_0R_2(\cdot,z)|\right) \\
&\leqslant cd(x_1,x_2)\,\phi_{-1+\alpha}(x_1,z)
\end{aligned}$$

where the bounds on $|X_kR_2(\cdot,z)|, |X_0R_2(\cdot,z)|$ exploit Proposition 5.4 (ii). Hence

$$\begin{aligned}
|C_1| &\leqslant cd(x_1,x_2)\int_{U,d(z,x_1)>2d(x_1,x_2)} \phi_{-1+\alpha}(x_1,z)\,\phi_\alpha(z,y)\,dz \\
&\leqslant cd(x_1,x_2)^{\alpha-\varepsilon}\int_{U,d(z,x_1)>2d(x_1,x_2)} d(x_1,z)^{1-\alpha+\varepsilon}\,\phi_{-1+\alpha}(x_1,z)\,\phi_\alpha(z,y)\,dz \\
&\leqslant cd(x_1,x_2)^{\alpha-\varepsilon}\int_U \phi_\varepsilon(x_1,z)\,\phi_\alpha(z,y)\,dz \leqslant cd(x_1,x_2)^{\alpha-\varepsilon}\,\phi_{\alpha+\varepsilon}(x_1,y),
\end{aligned}$$

while

$$|C_2| \leqslant \int_{U, d(z,x_1) \leqslant 2d(x_1,x_2)} [\phi_\alpha(x_2,z) + \phi_\alpha(x_1,z)] \phi_\alpha(z,y)\, dz.$$

Since $d(x_1,z) \leqslant 2d(x_1,x_2)$ and $d(x_1,y) \geqslant 3d(x_1,x_2)$ implies $d(x_1,y) \leqslant 3d(z,y)$

$$\begin{aligned}
|C_2| &\leqslant c \int_{U, d(z,x_1) \leqslant 2d(x_1,x_2)} [\phi_\alpha(x_2,z) + \phi_\alpha(x_1,z)] \phi_\alpha(x_1,y)\, dz \\
&\leqslant c\phi_\alpha(x_1,y) \left(\int_{U, d(z,x_2) \leqslant 3d(x_1,x_2)} \phi_\alpha(x_2,z)\, dz + \int_{U, d(z,x_1) \leqslant 2d(x_1,x_2)} \phi_\alpha(x_1,z)\, dz \right) \\
&\leqslant c\phi_\alpha(x_1,y)\, d(x_1,x_2)^\alpha
\end{aligned}$$

by Corollary 3.4. Hence

$$\begin{aligned}
&|C(x_2,y) - C(x_1,y)| \leqslant cd(x_1,x_2)^{\alpha-\varepsilon} \phi_{\alpha+\varepsilon}(x_1,y) + c\phi_\alpha(x_1,y)\, d(x_1,x_2)^\alpha \\
&\leqslant c \left(\frac{d(x_1,x_2)}{d(x_1,y)} \right)^{\alpha-\varepsilon} \frac{d(x_1,y)^{2\alpha}}{|B(x_1,d(x_1,y))|} + c \left(\frac{d(x_1,x_2)}{d(x_1,y)} \right)^{\alpha} \frac{d(x_1,y)^{2\alpha}}{|B(x_1,d(x_1,y))|} \\
&\leqslant c \left(\frac{d(x_1,x_2)}{d(x_1,y)} \right)^{\alpha-\varepsilon} \frac{d(x_1,y)^{2\alpha}}{|B(x_1,d(x_1,y))|}.
\end{aligned}$$

As to A, let

$$k(x,z) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (Y_j D_1 \Gamma) \left(\Theta_{(z,k)}(x,h) \right) a_0(h)\, b_0(k)\, dh dk,$$

then

$$\begin{aligned}
A(x_2,y) - A(x_1,y) &= \int_U \{ k(x_2,z) [\Phi'(z,y) - \Phi'(x_2,y)] - k(x_1,z) [\Phi'(z,y) - \Phi'(x_1,y)] \}\, dz \\
&= \int_{U, d(x_1,z) \geqslant 2d(x_1,x_2)} \{\cdots\}\, dz + \int_{U, d(x_1,z) < 2d(x_1,x_2)} \{\cdots\}\, dz \\
&\equiv A_1(x_1,x_2,y) + A_2(x_1,x_2,y).
\end{aligned}$$

$$\begin{aligned}
A_1(x_1,x_2,y) = &\int_{U, d(x_1,z) \geqslant 2d(x_1,x_2)} [k(x_2,z) - k(x_1,z)] [\Phi'(z,y) - \Phi'(x_2,y)]\, dz \\
&+ [\Phi'(x_1,y) - \Phi'(x_2,y)] \int_{U, d(x_1,z) \geqslant 2d(x_1,x_2)} k(x_1,z)\, dz \\
&\equiv A_{1,1}(x_1,x_2,y) + A_{1,2}(x_1,x_2,y).
\end{aligned}$$

Since, for $d(x_1,z) \geqslant 2d(x_1,x_2)$ we have

$$|k(x_2,z) - k(x_1,z)| \leqslant d(x_1,x_2)\, \phi_{-1}(x_1,z)$$

we obtain

$$|A_{1,1}(x_1,x_2,y)| \leqslant cd(x_1,x_2) \int_{U, d(x_1,z) \geqslant 2d(x_1,x_2)} \phi_{-1}(x_1,z) |\Phi'(z,y) - \Phi'(x_2,y)|\, dz.$$

We now split the domain of integration $\{z \in U : d(x_1, z) \geqslant 2d(x_1, x_2)\}$ into two pieces

$$U_1 = \{z : U : d(x_1, z) \geqslant 2d(x_1, x_2),\ d(z, y) \geqslant 3d(z, x_2)\},$$
$$U_2 = \{z : U : d(x_1, z) \geqslant 2d(x_1, x_2),\ d(z, y) < 3d(z, x_2)\},$$

so that

$$|A_{1,1}(x_1, x_2, y)| \leqslant cd(x_1, x_2) \int_{U_1} (\cdots)\, dz + d(x_1, x_2) \int_{U_2} (\cdots)\, dz$$
$$\equiv A_{1,1,1}(x_1, x_2, y) + A_{1,1,2}(x_1, x_2, y).$$

Note that $d(x_1, z)$ and $d(x_2, z)$ are equivalent on U_1 and U_2. Also on U_1 (since $d(z, y) \geqslant 3d(z, x_2)$) we have $|\Phi'(z, y) - \Phi'(x_2, y)| \leqslant d(z, x_2)^{\alpha-\varepsilon} \phi_\varepsilon(z, y)$ and therefore

$$\begin{aligned}
|A_{1,1,1}(x_1, x_2, y)| &\leqslant cd(x_1, x_2) \int_{U, d(x_1,z) \geqslant 2d(x_1,x_2)} \phi_{-1}(x_1, z)\, d(z, x_2)^{\alpha-\varepsilon} \phi_\varepsilon(z, y)\, dz \\
&\leqslant cd(x_1, x_2) \int_{U, d(x_1,z) \geqslant 2d(x_1,x_2)} \phi_{-1}(x_1, z)\, d(z, x_1)^{\alpha-\varepsilon} \phi_\varepsilon(z, y)\, dz \\
&\leqslant cd(x_1, x_2)^{\alpha-2\varepsilon} \int_{U, d(x_1,z) \geqslant 2d(x_1,x_2)} \phi_{-1}(x_1, z)\, d(z, x_1)^{1+\varepsilon} \phi_\varepsilon(z, y)\, dz \\
&\leqslant cd(x_1, x_2)^{\alpha-2\varepsilon} \int_U \phi_\varepsilon(x_1, z)\, \phi_\varepsilon(z, y)\, dz \\
&\leqslant cd(x_1, x_2)^{\alpha-2\varepsilon} \phi_{2\varepsilon}(x_1, y) \leqslant c \left(\frac{d(x_1, x_2)}{d(x_1, y)} \right)^{\alpha-2\varepsilon} \frac{d(x_1, y)^\alpha}{|B(x_1, d(x_1, y))|}.
\end{aligned}$$

We now consider the second term. We have

$$|A_{1,1,2}(x_1, x_2, y)| \leqslant cd(x_1, x_2) \int_{U_2} \phi_{-1}(x_1, z)\, (\phi_\alpha(z, y) + \phi_\alpha(x_2, y))\, dz$$
$$\equiv A'_{1,1,2} + A''_{1,1,2}.$$

Since $d(y, z) \leqslant \frac{1}{2} d(x_1, y)$ implies $d(x_1, y) \leqslant 2d(x_1, z)$,

$$\begin{aligned}
A'_{1,1,2} &\leqslant cd(x_1, x_2) \frac{d(x_1, y)^{1+\varepsilon}}{d(x_1, y)^{1+\varepsilon}} \int_{U_2 \cap \left\{ d(y,z) \leqslant \frac{1}{2} d(x_1,y) \right\}} \phi_{-1}(x_1, z)\, \phi_\alpha(z, y)\, dz \\
&+ cd(x_1, x_2)^{\alpha-\varepsilon} \int_{U_2 \cap \left\{ d(y,z) > \frac{1}{2} d(x_1,y) \right\}} d(x_1, x_2)^{1-\alpha+\varepsilon} \phi_{-1}(x_1, z) \frac{d(z, y)^\alpha}{|B(z, d(z, y))|} dz \\
&\leqslant c \frac{d(x_1, x_2)}{d(x_1, y)^{1+\varepsilon}} \int_U \phi_\varepsilon(x_1, z)\, \phi_\alpha(z, y)\, dz \\
&+ \frac{cd(x_1, x_2)^{\alpha-\varepsilon}}{|B(y, d(x_1, y))|} \int_{U_2 \cap \left\{ d(y,z) > \frac{1}{2} d(x_1,y) \right\}} d(x_1, z)^{1-\alpha+\varepsilon} \phi_{-1}(x_1, z)\, d(z, x_1)^\alpha\, dz \\
&\leqslant c \frac{d(x_1, x_2)}{d(x_1, y)^{1+\varepsilon}} \phi_{\alpha+\varepsilon}(x_1, y) + \frac{cd(x_1, x_2)^{\alpha-\varepsilon}}{|B(y, d(x_1, y))|} \int_U \phi_\varepsilon(x_1, z)\, dz \\
&\leqslant cd(x_1, x_2) \frac{d(x_1, y)^{\alpha-1}}{|B(y, d(x_1, y))|} + c \frac{d(x_1, x_2)^{\alpha-\varepsilon}}{|B(x_1, d(x_1, y))|} R^\varepsilon
\end{aligned}$$

$$\leqslant c\left(\frac{d(x_1,x_2)}{d(x_1,y)}\right)^{\alpha-\varepsilon}\left(\frac{d(x_1,y)^{\alpha-1}d(x_1,x_2)}{d(x_1,x_2)^{\alpha-\varepsilon}}\frac{d(x_1,y)^{\alpha-\varepsilon}}{|B(y,d(x_1,y))|}+\frac{d(x_1,y)^{\alpha-\varepsilon}}{|B(x_1,d(x_1,y))|}\right)$$
$$\leqslant c\left(\frac{d(x_1,x_2)}{d(x_1,y)}\right)^{\alpha-\varepsilon}\left(R^{\varepsilon}\frac{d(x_1,y)^{\alpha-\varepsilon}}{|B(y,d(x_1,y))|}+\frac{d(x_1,y)^{\alpha-\varepsilon}}{|B(x_1,d(x_1,y))|}\right).$$

Since in U_2

$$d(x_2,y)\leqslant d(x_2,z)+d(z,y)\leqslant cd(z,x_2)\leqslant cd(z,x_1)$$

we have

$$A''_{1,1,2}\leqslant d(x_1,x_2)^{\alpha-\varepsilon}\frac{\phi_\alpha(x_2,y)}{d(x_2,y)^{\alpha}}\int_{U_2}\phi_\varepsilon(x_1,z)\,dz\leqslant c\frac{d(x_1,x_2)^{\alpha-\varepsilon}}{|B(x_2,d(x_2,y))|}R^{\varepsilon}$$
$$\leqslant c\left(\frac{d(x_1,x_2)}{d(x_2,y)}\right)^{\alpha-\varepsilon}\frac{d(x_2,y)^{\alpha-\varepsilon}}{|B(x_2,d(x_2,y))|}.$$

Hence

$$|A_{1,1,2}(x_1,x_2,y)|\leqslant c\left(\frac{d(x_1,x_2)}{d(x_1,y)}\right)^{\alpha-\varepsilon}\frac{d(x_1,y)^{\alpha-\varepsilon}}{|B(x_1,d(x_1,y))|}$$

and

$$|A_{1,1}(x_1,x_2,y)|\leqslant c\left(\frac{d(x_1,x_2)}{d(x_1,y)}\right)^{\alpha-2\varepsilon}\frac{d(x_1,y)^{\alpha-\varepsilon}}{|B(x_1,d(x_1,y))|}.$$

We now consider $A_{1,2}$. Observe that for $d(x_1,y)>3d(x_1,x_2)$ we have

$$|A_{1,2}(x_1,x_2,y)|\leqslant|\Phi'(x_1,y)-\Phi'(x_2,y)|\int_{U,d(x_1,z)\geqslant 2d(x_1,x_2)}\phi_0(x_1,z)\,dz$$
$$\leqslant cd(x_1,x_2)^{\alpha-\varepsilon}\phi_\varepsilon(x_1,y)\int_{U,d(x_1,z)\geqslant 2d(x_1,x_2)}\phi_0(x_1,z)\,dz$$
$$\leqslant cd(x_1,x_2)^{\alpha-2\varepsilon}\phi_\varepsilon(x_1,y)\int_{U,d(x_1,z)\geqslant 2d(x_1,x_2)}\phi_\varepsilon(x_1,z)\,dz$$
$$\leqslant cd(x_1,x_2)^{\alpha-2\varepsilon}\phi_\varepsilon(x_1,y)R^{\varepsilon}\leqslant c\left(\frac{d(x_1,x_2)}{d(x_1,y)}\right)^{\alpha-2\varepsilon}\frac{d(x_1,y)^{\alpha-\varepsilon}}{|B(x_1,d(x_1,y))|}.$$

Finally we have to bound $A_2(x_1,x_2,y)$. We have

$$|A_2(x_1,x_2,y)|\leqslant\int_{U,d(x_2,z)<3d(x_1,x_2)}\phi_0(x_2,z)|\Phi'(z,y)-\Phi'(x_2,y)|\,dz$$
$$+\int_{U,d(x_1,z)<2d(x_1,x_2)}\phi_0(x_1,z)|\Phi'(z,y)-\Phi'(x_1,y)|\,dz.$$

Since the two terms are similar it is enough to bound the second. We have

$$\int_{U,d(x_1,z)<2d(x_1,x_2)}\phi_0(x_1,z)|\Phi'(z,y)-\Phi'(x_1,y)|\,dz$$
$$=\int_{U,d(x_1,z)<2d(x_1,x_2),d(y,z)\leqslant\frac{1}{2}d(x_1,y)}\{\cdots\}\,dz+\int_{U,d(x_1,z)<2d(x_1,x_2),d(y,z)>\frac{1}{2}d(x_1,y)}\{\cdots\}\,dz$$
$$\equiv A_{2,1}(x_1,x_2,y)+A_{2,2}(x_1,x_2,y).$$

As to $A_{2,1}(x_1,x_2,y)$, we note that, under the assumption $d(x_1,y) \geqslant 3d(x_1,x_2)$, in the domain of integration the following equivalences hold:

$$d(x_1,y) \simeq d(z,y) \simeq d(x_1,z).$$

Therefore

$$|\Phi'(z,y) - \Phi'(x_1,y)| \leqslant \phi_\alpha(z,y) + \phi_\alpha(x_1,y) \leqslant c\phi_\alpha(z,y)$$

and

$$\begin{aligned}
&A_{2,1}(x_1,x_2,y) \leqslant c\int_{\left\{d(x_1,z)<2d(x_1,x_2),d(y,z)\leqslant\frac{1}{2}d(x_1,y)\right\}} \phi_0(x_1,z)\,\phi_\alpha(z,y)\,dz \\
&\leqslant \frac{c}{d(x_1,y)^\alpha}\int_{\left\{d(x_1,z)<2d(x_1,x_2),d(y,z)\leqslant\frac{1}{2}d(x_1,y)\right\}} d(x_1,z)^\alpha\,\phi_0(x_1,z)\,\phi_\alpha(z,y)\,dz \\
&\leqslant \frac{c}{d(x_1,y)^\alpha}\phi_\alpha(x_1,y)\int_{\{d(x_1,z)<2d(x_1,x_2)\}} \phi_\alpha(x_1,z)\,dz \\
&\leqslant c\left(\frac{d(x_1,x_2)}{d(x_1,y)}\right)^\alpha \phi_\alpha(x_1,y) \leqslant c\left(\frac{d(x_1,x_2)}{d(x_1,y)}\right)^\alpha \frac{d(x_1,y)^\alpha}{|B(x_1,d(x_1,y))|}.
\end{aligned}$$

On the other hand, since $d(x_1,z) \leqslant 2d(x_1,x_2) \leqslant \frac{2}{3}d(x_1,y) < 3d(x_1,y)$, by Proposition 5.5

$$\begin{aligned}
&A_{2,2}(x_1,x_2,y) \leqslant \\
&\leqslant c\int_{U,d(x_1,z)<2d(x_1,x_2),d(y,z)>\frac{1}{2}d(x_1,y)} \phi_0(x_1,z)\,d(x_1,z)^{\alpha-\varepsilon}\,\phi_\varepsilon(z,y)\,dz \\
&\leqslant cd(x_1,x_2)^{\alpha-2\varepsilon}\int_{U,d(x_1,z)<2d(x_1,x_2),d(y,z)>\frac{1}{2}d(x_1,y)} d(x_1,z)^\varepsilon\,\phi_0(x_1,z)\,\phi_\varepsilon(z,y)\,dz \\
&\leqslant cd(x_1,x_2)^{\alpha-2\varepsilon}\int_U \phi_\varepsilon(x_1,z)\,\phi_\varepsilon(z,y)\,dz \\
&\leqslant cd(x_1,x_2)^{\alpha-2\varepsilon}\,\phi_{2\varepsilon}(x_1,y) \leqslant c\left(\frac{d(x_1,x_2)}{d(x_1,y)}\right)^{\alpha-2\varepsilon} \frac{d(x_1,y)^\alpha}{|B(x_1,d(x_1,y))|}.
\end{aligned}$$

We can conclude that

$$|A(x_2,y) - A(x_1,y)| \leqslant c\left(\frac{d(x_1,x_2)}{d(x_1,y)}\right)^{\alpha-2\varepsilon} \frac{d(x_1,y)^{\alpha-\varepsilon}}{|B(x_1,d(x_1,y))|}.$$

This completes the proof of (5.14). ■

5.3 Local solvability and Hölder estimates on the highest derivatives of the solution

Throughout this section we keep Assumptions B, stated at the beginning of §5. We can now prove one of the main results in this paper:

Theorem 5.18 (Local solvability of L) *Under Assumptions B, the function γ is a solution to the equation*

$$L\gamma(\cdot,y) = 0 \text{ in } U\setminus\{y\}, \text{ for any } y \in U.$$

Moreover, for any $\beta > 0$, $f \in C_X^\beta(U)$, the function

$$w(x) = -\int_U \gamma(x,y) f(y)\, dy \tag{5.18}$$

is a $C_X^2(U)$ solution to the equation $Lw = f$ in U (in the sense of Definition 5.1). Hence the operator L is locally solvable in Ω.

Moreover, if $X_0 \equiv 0$, choosing U small enough, we have the following positivity property: if $f \in C_X^\beta(U), f \leqslant 0$ in U, then the equation $Lw = f$ has at least a $C_X^2(U)$ solution $w \geqslant 0$ in U.

Proof. By Theorem 4.8 and Theorem 5.9 we already know that $\gamma(\cdot, y) \in C_X^2(U \setminus \{y\})$. For fixed $y \in U$ and $r > 0$, let $\omega \in C_0^\infty(U)$, ω vanishing in the ball $B(y,r)$. Then, by Theorem 4.8 we have

$$0 = \int \gamma(x,y) L^*\omega(x)\, dx = \int L\gamma(x,y)\, \omega(x)\, dx$$

with $L\gamma(\cdot, y)$ continuous in the support of ω. Since r and ω are arbitrary, we get $L\gamma(x,y) = 0$ for every $x \in U \setminus \{y\}$, any $y \in U$.

Let now w be as in (5.18) for some $f \in C^\beta(U)$, $\beta > 0$; for any $\psi \in C_0^\infty(U)$ we can write, by Theorem 4.8,

$$\begin{aligned}
\int_U w(x) L^*\psi(x)\, dx &= \int_U \left(-\int_U \gamma(x,y) f(y)\, dy\right) L^*\psi(x)\, dx \\
&= -\int_U \left(\int_U \gamma(x,y) L^*\psi(x)\, dx\right) f(y)\, dy \\
&= \int_U \psi(y) f(y)\, dy. \qquad (5.19)
\end{aligned}$$

Hence if we show that Lw actually exists and is continuous in U, we can write

$$\int_U w(x) L^*\psi(x)\, dx = \int_U Lw(x)\, \psi(x)\, dx \ \forall \psi \in C_0^\infty(U),$$

which coupled with (5.19) gives $Lw = f$. Actually, we will prove that $w \in C_X^2(U)$.

By the results in §4 it is easy to see that $w \in C_X^1(U)$. Namely, by Proposition 3.7 (ii), $w \in C(U)$ by the estimate (4.33) while

$$X_i w(x) = -\int_U X_i \gamma(x,y) f(y)\, dy$$

is continuous in U by the estimate (4.34).

Let us write:

$$\begin{aligned}
X_j X_i w(x) &= -X_j X_i \int_U \gamma(x,y) f(y)\, dy = \\
&= -X_j X_i \int_U \frac{1}{c_0(y)} \left[P(x,y) + J'(x,y)\right] f(y)\, dy \equiv A(x) + B(x).
\end{aligned}$$

By Theorem 5.9 we can write

$$B(x) = -\int_U X_j X_i J'(x,y) \widetilde{f}(y)\, dy, \tag{5.20}$$

having set

$$\widetilde{f}(y) = \frac{f(y)}{c_0(y)} \tag{5.21}$$

and again by Proposition 3.7 (ii), and the bound (5.9), B is continuous in U.

Let us now consider

$$A(x) = -X_j X_i \int_U P(x,y)\, \widetilde{f}(y)\, dy. \tag{5.22}$$

From the computation in the proof of Theorem 5.16 we read that

$$-X_i P(x,y) = k_1(x,y)$$

with $k_1(x,y)$ kernel of type 1 in the sense of Definition 5.11, hence

$$A(x) = X_j \int_U k_1(x,y)\, \widetilde{f}(y)\, dy \tag{5.23}$$

where the function $\widetilde{f}$ is Hölder continuous in U. To show that $A(x)$ exists and is continuous we can now proceed as we did in the proof of Theorem 5.16 for the term $X_j B(x,y)$, getting, analogously to (5.11) and with the same notation,

$$\begin{aligned} A(x) = & \int_{\mathbb{R}^m} a_0(h) \int_\Sigma Y_j D_1 \Gamma\left(\Theta_\eta(\xi)\right) b_0(k) \left[\widetilde{f}(z) - \widetilde{f}(x)\right] d\eta dh \\ & + c_1(x)\, \widetilde{f}(x) + \int_U R_2(x,z)\, \widetilde{f}(z)\, dz \end{aligned}$$

where $\xi = (x,h)$, $\eta = (z,k)$, $\Sigma = U \times I$, $I \subset \mathbb{R}^m$ such that $I \supset \operatorname{sprt} a_0 \cup \operatorname{sprt} b_0$. Note that here $\widetilde{f}$ plays the role of the function $\Phi_0(\cdot, y)$ in the proof of Theorem 5.16; since $\widetilde{f} \in C_X^\beta(U)$ for some $\beta > 0$, it obviously satisfies the properties required in the definition of $\Phi_0(\cdot, y)$. Hence

$$\begin{aligned} X_j X_i w(x) = & \int_{\mathbb{R}^m} a_0(h) \int_\Sigma Y_j D_1 \Gamma\left(\Theta_\eta(\xi)\right) b_0(k) \left[\widetilde{f}(z) - \widetilde{f}(x)\right] d\eta dh \\ & + c_1(x)\, \widetilde{f}(x) + \int_U R_2(x,z)\, \widetilde{f}(z)\, dz - \int_U X_j X_i J'(x,z)\, \widetilde{f}(z)\, dz, \end{aligned}$$

and this function is continuous in U.

To complete the proof we should prove the existence and continuity of

$$X_0 \int_U P(x,z)\, \widetilde{f}(z)\, dz.$$

However, this is very similar to what we have just done.

Finally, the positivity property of L when $X_0 \equiv 0$ and U is small enough immediately follows from (5.18) and (4.36). So we have finished. ■

From the proof of the above theorem we read in particular a representation formula for the second derivatives $X_i X_j w$ of our solution. In view of the proof of local Hölder continuity of $X_i X_j w$, we have to localize our representation formula.

For $\overline{x} \in U$ and $B(\overline{x}, R) \subset U$, pick a cutoff function

$$b \in C_0^\infty (B(\overline{x}, R)) \text{ such that } b = 1 \text{ in } B\left(\overline{x}, \frac{3}{4}R\right). \tag{5.24}$$

For any $\beta > 0$, $f \in C_X^\beta(U)$, let w be the solution to $Lw = f$ in U assigned by (5.18). Then, for any $x \in B(\overline{x}, R)$ we can write:

$$w(x) = -\int_{B(\overline{x},R)} \gamma(x,y)\, b(y)\, f(y)\, dy + \int_U \gamma(x,y)\,[b(y) - 1]\, f(y)\, dy. \tag{5.25}$$

We also have:

Corollary 5.19 *With the notation and assumptions just recalled, for every $x \in B\left(\overline{x}, \frac{R}{2}\right)$ and $i, j = 1, 2, ..., n$, we have:*

$$\begin{aligned} X_j X_i w(x) &= \int_U X_i X_j \gamma(x,y)\,[b(y) - 1]\, f(y)\, dy + c_1(x)\, \widetilde{f}(x) \\ &+ \int_{B(\overline{x},R)} k_2(x,z) \left[\widetilde{f}(z) - \widetilde{f}(x)\right] b(z)\, dz \\ &+ \int_{B(\overline{x},R)} R_2(x,z)\, b(z)\, \widetilde{f}(z)\, dz - \int_{B(\overline{x},R)} X_j X_i J'(x,z)\, b(z)\, \widetilde{f}(z)\, dz \\ &\equiv \sum_{k=1}^{5} T_k f(x), \end{aligned}$$

where $c_1 \in C_X^\alpha\left(B\left(\overline{x}, \frac{R}{2}\right)\right)$, k_2 and R_2 are a pure kernel and a remainder of type 2, respectively, in the sense of Definition 5.11 and $\widetilde{f}$ is defined in (5.21).

Proof. Let us write

$$\begin{aligned} w(x) &= -\int_{B(\overline{x},R)} \gamma(x,y)\, b(y)\, f(y)\, dy + \int_U \gamma(x,y)\,[b(y) - 1]\, f(y)\, dy \\ &\equiv K_1 f(x) + K_2 f(x). \end{aligned}$$

Note that for $x \in B(\overline{x}, R/2)$ the integral defining $K_2 f(x)$ can be freely differentiated since $[b(y) - 1] \neq 0$ only if $d(x,y) \geqslant R/4$, so

$$X_i X_j K_2 f(x) = \int_U X_i X_j \gamma(x,y)\,[b(y) - 1]\, f(y)\, dy.$$

Arguing as in the proof of Theorems 5.18 and 5.16 we have therefore (with $\eta = (z,k)$, $\xi = (x,h)$, $\Sigma = U \times I$ for $I \supset \operatorname{sprt} a_0 \cup \operatorname{sprt} b_0$)

$$\begin{aligned} X_j X_i w(x) &= \int_U X_i X_j \gamma(x,y)\,[b(y) - 1]\, f(y)\, dy + c_1(x)\, \widetilde{f}(x) \\ &+ \int_{\mathbb{R}^m} a_0(h) \int_\Sigma Y_j D_1 \Gamma(\Theta_\eta(\xi)) \left[\widetilde{f}(z)\, b(z) - \widetilde{f}(x)\, b(x)\right] b_0(k)\, d\eta dh \\ &+ \int_{B(\overline{x},R)} R_2(x,z)\, b(z)\, \widetilde{f}(z)\, dz - \int_{B(\overline{x},R)} X_j X_i J'(x,z)\, b(z)\, \widetilde{f}(z)\, dz. \end{aligned}$$

Let us rewrite the third term as

$$
\begin{aligned}
&\int_{\mathbb{R}^m} a_0(h) \int_{\Sigma} Y_j D_1 \Gamma(\Theta_\eta(\xi)) \left[\widetilde{f}(z) - \widetilde{f}(x)\right] b_0(k)\, b(z)\, d\eta dh \\
&+ \widetilde{f}(x) \int_{\mathbb{R}^m} a_0(h) \int_{\Sigma} Y_j D_1 \Gamma(\Theta_\eta(\xi)) [b(z) - b(x)]\, b_0(k)\, d\eta dh \\
&= \int_{B(\overline{x},R)} k_2(x,z) \left[\widetilde{f}(z) - \widetilde{f}(x)\right] b(z)\, dz \\
&+ \widetilde{f}(x)\, c_2(x)
\end{aligned}
$$

where k_2 is a kernel of type 2, while

$$
\begin{aligned}
c_2(x) &= \int_U k_2(x,z) [b(z) - b(x)]\, dz \\
&= \int_U k_2(x,z) [b(z) - 1]\, dz
\end{aligned}
$$

is another $C_X^\alpha\left(B\left(\overline{x}, \frac{R}{2}\right)\right)$ function. Namely, recalling that $b = 1$ in $B\left(\overline{x}, \frac{3}{4}R\right)$, for any $x_1, x_2 \in B(\overline{x}, R/2)$, we have

$$
|c_2(x_2) - c_2(x_1)| \leqslant \int_U |k_2(x_2,z) - k_2(x_1,z)|\, [1 - b(z)]\, dz. \tag{5.26}
$$

Note that, from

$$
k_2(x,y) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} D_2 \Gamma\left(\Theta_{(y,k)}(x,h)\right) a_0(h)\, b_0(k)\, dh dk,
$$

by Proposition 5.4 (ii) we read that

$$
\begin{aligned}
|k_2(x,y)| &\leqslant c\phi_0(x,y); \\
|X_i k_2(x,y)| &\leqslant c\phi_{-1}(x,y) \text{ for } i = 1,2,...,n; \\
|X_0 k_2(x,y)| &\leqslant c\phi_{-2}(x,y),
\end{aligned} \tag{5.27}
$$

hence by Lagrange theorem (Proposition 2.6),

$$
|k_2(x_2,z) - k_2(x_1,z)| \leqslant c \frac{d(x_1,x_2)}{d(x_1,z)} \frac{1}{|B(x_1, d(x_1,z))|} \text{ for } d(x_1,z) \geqslant 2d(x_1,x_2). \tag{5.28}
$$

Now, note that the integrand function in (5.26) does not vanish only for $d(x_1,z) \geqslant R/4$, $d(x_2,z) \geqslant R/4$. Hence if $d(x_1,x_2) \leqslant R/8$ by (5.28) we get

$$
|c_2(x_2) - c_2(x_1)| \leqslant c(R)\, d(x_1,x_2).
$$

On the other hand, if $d(x_1,x_2) > R/8$,

$$
|c_2(x_2) - c_2(x_1)| \leqslant |c_2(x_2)| + |c_2(x_1)| \leqslant c(R) \leqslant c(R)\, d(x_1,x_2),
$$

and $c_2 \in C_X^\alpha\left(B\left(\overline{x}, \frac{R}{2}\right)\right)$. This completes the proof. ■

The rest of this section will be devoted to the proof of the following:

Theorem 5.20 *For any $\beta \in (0, \alpha)$ and $f \in C_X^\beta(U)$, let $w \in C_X^2(U)$ be the solution to $Lw = f$ in U assigned by (5.18). Then $w \in C_{X,loc}^{2,\beta}(U)$. More precisely, for any $U' \Subset U$ there exists $c > 0$ (depending on U, U', β and on the vector fields as specified at the beginning of section 5) such that*

$$\|w\|_{C_X^{2,\beta}(U')} \leqslant c \|f\|_{C_X^\beta(U)}. \tag{5.29}$$

Corollary 5.21 ($C_X^{2,\beta}$ local solvability) *Under assumptions B, for every $\beta \in (0, \alpha)$ the operator L is locally $C_X^{2,\beta}$ solvable in Ω in the following senses:*

(i) for every $\overline{x} \in \Omega$ there exists a neighborhood U of $\overline{x}$ such that for every $f \in C_X^\beta(U)$ there exists a solution $u \in C_{X,loc}^{2,\beta}(U)$ to $Lu = f$ in U.

(ii) for every $\overline{x} \in \Omega$ there exists a neighborhood U of $\overline{x}$ such that for every $f \in C_{X,0}^\beta(U)$ there exists a solution $u \in C_X^{2,\beta}(U)$ to $Lu = f$ in U.

Proof. Point (i) immediately follows by the above theorem and Theorem 5.18. As to point (ii), let U be the neighborhood of $\overline{x}$ given by point (i), and let U' be another neighborhood of $\overline{x}$ such that $U' \Subset U$. For any $f \in C_{X,0}^\beta(U')$ we can regard f also as a function in $C_{X,0}^\beta(U)$, and solve $Lu = f$ in U getting a $u \in C_{X,loc}^{2,\beta}(U)$ by point (i); hence in particular $u \in C_X^{2,\beta}(U')$. Then U' is the required neighborhood. ■

Since, in order to prove the above theorem, we will apply several abstract results about singular and fractional integrals, it is time to explain what is the suitable abstract context for the present situation. Recall that in our neighborhood U we have the distance d, such that the Lebesgue measure is *locally* doubling (see Theorem 2.3). However, we cannot assure the validity of a *global* doubling condition in U, which should mean:

$$|B(x, 2r) \cap U| \leqslant c |B(x, r) \cap U| \text{ for any } x \in U, r > 0. \tag{5.30}$$

Actually, even for the Carnot-Carathéodory distance induced by smooth Hörmander's vector fields, condition (5.30) is known when U is for instance a metric ball and the drift term X_0 is lacking; in presence of a drift, however, the distance d does not satisfy the segment property, and the validity of a condition (5.30) on some reasonable U seems to be an open problem (fur further details on this issue we refer to the introduction of [7]). This means that in our situation (U, d, dx) is not a space of homogeneous type in the sense of Coifman-Weiss. However, (U, d, dx) fits the assumptions of *locally homogeneous spaces* as defined in [7]. We will apply some results proved in [7] which assure the local C^α continuity of singular and fractional integrals defined by a kernel of the kind

$$a(x) k(x, y) b(y)$$

(with a, b smooth cutoff functions) provided that the kernel k satisfies natural assumptions which never involve integration over domains of the kind $B(x, r) \cap U$, but only over balls $B(x, r) \Subset U$, which makes our local doubling condition usable. Before starting the proof of the above theorem we need the following

Definition 5.22 *We say that the a kernel $k(x, y)$ satisfies the standard estimates of fractional integrals with (positive) exponents ν, β in $B(\overline{x}, R)$ if*

$$|k(x, y)| \leqslant c \frac{d(x, y)^\nu}{|B(x, d(x, y))|}$$

for every $x, y \in B(\overline{x}, R)$, and

$$|k(x,y) - k(x_0,y)| \leqslant c \frac{d(x_0,y)^{\nu}}{|B(x_0, d(x_0,y))|} \left(\frac{d(x_0,x)}{d(x_0,y)} \right)^{\beta}$$

for every $x_0, x, y \in B(\overline{x}, R)$ such that $d(x_0,y) \geqslant M d(x_0,x)$ for suitable $M > 1$.

We say that $k(x,y)$ satisfies the standard estimates of singular integrals if the previous estimates hold with $\nu = 0$ and some positive β.

Proof of Theorem 5.20, first part. Fix $U' \Subset U$ and choose $R_0 > 0$ such that for any $\overline{x} \in U'$ one has $B(\overline{x}, KR_0) \subset U$, for some large number $K > 1$ which is not important to specify (it comes out from some proofs in [7]). For any $R \leqslant R_0$, pick a cutoff function $b \in C_0^{\infty}(B(\overline{x}, R))$ such that $b \equiv 1$ in $B\left(\overline{x}, \frac{3}{4}R\right)$. Then for any $x \in B(\overline{x}, R/2)$ the representation formula proved in Corollary 5.19 holds:

$$X_i X_j w(x) = \sum_{k=1}^{5} T_k f(x) \text{ for } i,j = 1,2,....,n.$$

Our proof will mainly consist in showing that for any $\beta \in (0,\alpha)$ and $f \in C_X^{\beta}(U)$,

$$|X_i X_j w(x_1) - X_i X_j w(x_2)| \leqslant c d(x_1,x_2)^{\beta} \|f\|_{C^{\beta}(U)}$$

for any $x_1, x_2 \in B\left(\overline{x}, \frac{R}{2}\right)$. We are going to show how to bound the $C^{\beta}\left(B\left(\overline{x}, \frac{R}{2}\right)\right)$ seminorm of each term in this formula, starting with the easier ones.

Consider the operator

$$T_1 f(x) = \int_U X_i X_j \gamma(x,y) [b(y) - 1] f(y) \, dy.$$

Then by our choice of the cutoff function b, we have, for $x_1, x_2 \in B(\overline{x}, R/2)$,

$$\begin{aligned}
&|T_1 f(x_1) - T_1 f(x_2)| \\
&\leqslant \|f\|_{C^0(U)} \int_{U, d(\overline{x},y) > \frac{3}{4}R, d(x_1,y) > \frac{R}{4}, d(x_2,y) > \frac{R}{4}} |X_i X_j \gamma(x_1,y) - X_i X_j \gamma(x_2,y)| \, dy \\
&= \|f\|_{C^0(U)} \left(\int_{2d(x_1,x_2) < d(x_1,y), d(x_1,y) > \frac{R}{4}} (...) \, dy + \int_{2d(x_1,x_2) \geqslant d(x_1,y), d(x_1,y) > \frac{R}{4}, d(x_2,y) > \frac{R}{4}} (...) \, dy \right)
\end{aligned}$$

by (5.15) and (5.9)

$$\begin{aligned}
&\leqslant c \|f\|_{C^0(U)} \left\{ d(x_1,x_2)^{\alpha-\varepsilon} \int_{d(x_1,y) > \frac{R}{4}} \frac{dy}{d(x_1,y)^{\alpha-\varepsilon} |B(x_1, d(x_1,y))|} \right. \\
&\left. + \frac{1}{R} \int_{2d(x_1,x_2) \geqslant d(x_1,y)} \frac{d(x_1,y)}{|B(x_1, d(x_1,y))|} dy + \frac{1}{R} \int_{3d(x_1,x_2) \geqslant d(x_2,y)} \frac{d(x_2,y)}{|B(x_2, d(x_2,y))|} dy \right\} \\
&\leqslant \|f\|_{C^0(U)} \left\{ c(R) \, d(x_1,x_2)^{\alpha-\varepsilon} + \frac{d(x_1,x_2)}{R} \right\} = c d(x_1,x_2)^{\alpha-\varepsilon} \|f\|_{C^0(U)},
\end{aligned}$$

so that

$$\|T_1 f\|_{C_X^{\beta}(B(\overline{x}, R/2))} \leqslant c(\beta, R) \|f\|_{C^0(U)} \ \forall \beta < \alpha.$$

Next we introduce a second cutoff function $a \in C_0^\infty \left(B\left(\overline{x}, \frac{3}{4}R\right)\right)$ such that $a \equiv 1$ in $B\left(\overline{x}, \frac{R}{2}\right)$. For $x \in B\left(\overline{x}, \frac{R}{2}\right)$ we have $T_k f(x) = \widetilde{T}_k f(x)$, $k = 4, 5$ with

$$\widetilde{T}_4 f(x) = a(x) \int_{B(\overline{x},R)} R_2(x,z)\, b(z)\, \widetilde{f}(z)\, dz$$
$$\widetilde{T}_5 f(x) = -a(x) \int_{B(\overline{x},R)} X_j X_i J'(x,z)\, b(z)\, \widetilde{f}(z)\, dz.$$

These new operators have the form

$$\widetilde{T}_j f(x) = \int_{B(\overline{x},R)} a(x)\, k_j(x,y)\, \frac{f(y)}{c_0(y)} b(y)\, dy \text{ for } j = 4, 5,$$

where the kernels $k_j(x,y)$ satisfy the standard estimates of fractional integrals. Indeed, by Definition 5.10 and Proposition 5.4 (ii), the kernel k_4 satisfies

$$|k_4(x,z)| \leqslant c\phi_\alpha(x,z) \leqslant c \frac{d(x,z)^\alpha}{|B(x, d(x,z))|};$$
$$|X_k k_4(x,z)| \leqslant c\phi_{\alpha-1}(x,z);$$
$$|X_0 k_4(x,z)| \leqslant c\phi_{\alpha-2}(x,z).$$

If $d(x_1,z) \geqslant 2d(x_1,x_2)$, then by Lagrange theorem we can bound

$$\begin{aligned}|k_4(x_1,z) - k_4(x_2,z)| &\leqslant c\left\{d(x_1,x_2)\,\phi_{\alpha-1}(x_1,z) + d(x_1,x_2)^2\,\phi_{\alpha-2}(x_1,z)\right\} \\ &\leqslant c d(x_1,x_2)^{\alpha-\varepsilon}\,\phi_\varepsilon(x_1,z). \qquad (5.31)\end{aligned}$$

Then k_4 satisfies the standard estimates of fractional integrals with exponents ν, α, for any $\nu < \alpha$;

The kernel k_5 satisfies, by (5.8) and (5.14) (note that the cutoff function $a(x)$ compensates the local charachter of those bounds), the standard estimates of fractional integrals with exponents ν, β, for any ν and β both $< \alpha$, hence by [7, Thm. 5.8], for any $\beta < \alpha$

$$\|T_j f\|_{C_X^\beta(B(\overline{x},R/2))} = \left\|\widetilde{T}_j f\right\|_{C_X^\beta(B(\overline{x},R/2))} \leqslant c\,\|f\|_{C_X^\beta(B(\overline{x},R))} \text{ for } j = 4, 5,$$

with c depending on R and β.

Next, $T_2 f(x) = \frac{c_1(x)}{c_0(x)} f(x)$, with c_1, c_0 Hölder continuous functions of exponent α and c_0 bounded away from zero.

We are left to handle the term

$$T_3 f(x) = \int_{B(\overline{x},R)} k_2(x,z)\left[\widetilde{f}(z) - \widetilde{f}(x)\right] b(z)\, dz$$

with k_2 pure kernel of order 2, satisfying the standard estimates of singular integrals (see (5.27), (5.28)). Moreover, the same is true for the kernel

$$\widetilde{k}_2(x,y) = a(x)\, k_2(x,y)\, b(y).$$

In order to deduce an Hölder estimate for $T_3 f$ we still need to establish a suitable cancellation property for $\widetilde{k}_2$. So, let us pause for a moment this proof and pass to this auxiliary result. ■

Proposition 5.23 (Cancellation property) *There exists $C > 0$ such that for a.e. $x \in B(\overline{x}, R)$ and $0 < \varepsilon_1 < \varepsilon_2 < \infty$*

$$\left| \int_{\varepsilon_1 < d(x,y) < \varepsilon_2} a(x) k_2(x,y) \, b(y) \, dy \right| \leqslant C. \tag{5.32}$$

Proof. By Proposition 5.1 in [7], it is enough to prove the following cancellation property for k_2: there exists $C > 0$ such that for a.e. $x \in B(\overline{x}, R_0)$ and every $\varepsilon_1, \varepsilon_2$ such that $0 < \varepsilon_1 < \varepsilon_2$ and $B(x, \varepsilon_2) \subset U$,

$$\left| \int_{\varepsilon_1 < d(x,y) < \varepsilon_2} k_2(x,y) \; dy \right| \leqslant C. \tag{5.33}$$

According to Definition 5.11 of kernel of type 2 we write

$$\int_{\varepsilon_1 < d(x,y) < \varepsilon_2} k_2(x,y) \; dy =$$
$$= \int_{\varepsilon_1 < d(x,y) < \varepsilon_2} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} D_2 \Gamma \left(\Theta_{(y,k)}(x,h) \right) a_0(h) \, b_0(k) \, dh dk \, dy + \int_{\varepsilon_1 < d(x,y) < \varepsilon_2} R_2(x,y) \; dy,$$

where the last integral is uniformly bounded in $\varepsilon_1, \varepsilon_2$ since the remainder R_2 is locally integrable.

We can assume $\varepsilon_2 < 1$. Let us recall that

$$c \left\| \Theta_{(y,k)}(x,h) \right\| \geqslant d_{\widetilde{X}}((x,h),(y,k)) \geqslant d(x,y),$$

then

$$\int_{\varepsilon_1 < d(x,y) < \varepsilon_2} \left(\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} D_2 \Gamma \left(\Theta_{(y,k)}(x,h) \right) a_0(h) \, b_0(k) \, dh dk \right) dy$$
$$= \int_{\mathbb{R}^m} a_0(h) \left(\int_{\varepsilon_1 < c \left\| \Theta_{(y,k)}(x,h) \right\| < \varepsilon_2} D_2 \Gamma \left(\Theta_{(y,k)}(x,h) \right) b_0(k) \, dk dy \right) dh$$
$$+ \int_{\mathbb{R}^m} a_0(h) \left(\int_{c \left\| \Theta_{(y,k)}(x,h) \right\| > \varepsilon_2, d(x,y) < \varepsilon_2} D_2 \Gamma \left(\Theta_{(y,k)}(x,h) \right) b_0(k) \, dk dy \right) dh$$
$$- \int_{\mathbb{R}^m} a_0(h) \left(\int_{c \left\| \Theta_{(y,k)}(x,h) \right\| > \varepsilon_1, d(x,y) < \varepsilon_1} D_2 \Gamma \left(\Theta_{(y,k)}(x,h) \right) b_0(k) \, dk dy \right) dh$$
$$\equiv C^{\varepsilon_1, \varepsilon_2}(x) + D^{\varepsilon_2}(x) - E^{\varepsilon_1}(x).$$

To handle $C^{\varepsilon_1, \varepsilon_2}(x)$ we start rewriting

$$C^{\varepsilon_1, \varepsilon_2}(x) = \int_{\mathbb{R}^m} a_0(h) \left(\int_{\varepsilon_1 < c \left\| \Theta_{(y,k)}(x,h) \right\| < \varepsilon_2} D_2 \Gamma \left(\Theta_{(y,k)}(x,h) \right) [b_0(k) - b_0(h)] \, dk dy \right) dh$$
$$+ \int_{\mathbb{R}^m} a_0(h) \, b_0(h) \left(\int_{\varepsilon_1 < c \left\| \Theta_{(y,k)}(x,h) \right\| < \varepsilon_2} D_2 \Gamma \left(\Theta_{(y,k)}(x,h) \right) dk dy \right) dh$$
$$\equiv C_1^{\varepsilon_1, \varepsilon_2}(x) + C_2^{\varepsilon_1, \varepsilon_2}(x).$$

As to $C_1^{\varepsilon_1, \varepsilon_2}(x)$, since

$$|b_0(k) - b_0(h)| \leqslant c \, |k - h| \leqslant c \left\| \Theta_{(y,k)}(x,h) \right\|,$$

we have

$$\begin{aligned}\left|C_1^{\varepsilon_1,\varepsilon_2}(x)\right| &\leqslant \int_{\mathbb{R}^m} |a_0(h)| \left(\int_{\left\|\Theta_{(y,k)}(x,h)\right\|<\varepsilon_2} \frac{c}{\left\|\Theta_{(y,k)}(x,h)\right\|^{Q-1}} dkdy\right) dh \\ &\leqslant c\varepsilon_2 \int_{\mathbb{R}^m} |a_0(h)|\, dh \leqslant c.\end{aligned}$$

As to $C_2^{\varepsilon_1,\varepsilon_2}(x)$, by the change of variables $(y,k) \mapsto u = \Theta_{(y,k)}(x,h)$ and Proposition 5.4 we have, letting $\xi = (x,h)$,

$$C_2^{\varepsilon_1,\varepsilon_2}(x) = \int_{\mathbb{R}^m} a_0(h)\, b_0(h)\, c(\xi) \left(\int_{\varepsilon_1 < c\|u\| < \varepsilon_2} D_2\Gamma(u)\,(1+\chi(\xi,u))\, du\right) dh.$$

Keeping in mind the vanishing property of $D_2\Gamma$, that is

$$\int_{\varepsilon_1 < c\|u\| < \varepsilon_2} D_2\Gamma(u)\, du = 0,$$

we have

$$C_2^{\varepsilon_1,\varepsilon_2}(x) = \int_{\mathbb{R}^m} a_0(h)\, b_0(h)\, c(\xi) \left(\int_{\varepsilon_1 < c\|u\| < \varepsilon_2} D_2\Gamma(u)\,\chi(\xi,u)\, du\right) dh$$

which is uniformly bounded in $\varepsilon_1, \varepsilon_2$ since

$$\int_{\varepsilon_1 < c\|u\| < \varepsilon_2} |D_2\Gamma(u)\,\chi(\xi,u)|\, du \leqslant \int_{c\|u\| < \varepsilon_2} \frac{c}{\|u\|^{Q-\alpha}} du \leqslant c\varepsilon_2^{\alpha} \leqslant c.$$

Let us come to the terms $D^{\varepsilon_2}(x)$ and $E^{\varepsilon_1}(x)$. Choosing some small $\delta > 0$ we can write, by Corollary 3.4,

$$\begin{aligned}\left|D^{\varepsilon_2}(x)\right| &\leqslant \int_{\mathbb{R}^m} a_0(h) \left(\int_{c\left\|\Theta_{(y,k)}(x,h)\right\|>\varepsilon_2, d(x,y)<\varepsilon_2} \left\|\Theta_{(y,k)}(x,h)\right\|^{-Q} b_0(k)\, dkdy\right) dh \\ &\leqslant \frac{1}{\varepsilon_2^{\delta}} \int_{d(x,y)\leqslant\varepsilon_2} \left(\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \left\|\Theta_{(y,k)}(x,h)\right\|^{-Q+\delta} a_0(h)\, b_0(k)\, dkdh\right) dy \\ &\leqslant \frac{c}{\varepsilon_2^{\delta}} \int_{d(x,y)\leqslant\varepsilon_2} \phi_\delta(x,y)\, dy \leqslant \frac{c}{\varepsilon_2^{\delta}} \cdot \varepsilon_2^{\delta} = c\end{aligned}$$

and the term $E^{\varepsilon_1}(x)$ can be bounded at the same way. ∎

Conclusion of the proof of Theorem 5.20. We are left to prove the C_X^{β} continuity of the operator T_3. Let us consider first

$$\widetilde{T}_3 f(x) = \int_{B(\overline{x},R)} \widetilde{k}_2(x,y) \left[\widetilde{f}(y) - \widetilde{f}(x)\right] dy.$$

We know that the kernel $\widetilde{k}_2(x,y)$ satisfies the standard estimates of singular integrals with exponent $\beta = 1$ (see the end of the first part of this proof) and the cancellation property (5.33). This is enough to repeat *verbatim* the proof of Theorem 2.7 in [3]: the quantity

$$\widetilde{T}_3 f(x) - \widetilde{T}_3 f(x_0)$$

is exactly the quantity which is called A in that proof, see [3, p.183], and the proof of the bound

$$\left|T_3 f(x) - T_3 f(x_0)\right| = \left|\widetilde{T}_3 f(x) - \widetilde{T}_3 f(x_0)\right| \leqslant c d(x, x_0)^{\beta} \|f\|_{C_X^{\beta}(B(\overline{x}, R))} \quad \forall \beta < 1 \tag{5.34}$$

for any $x, x_0 \in B(\overline{x}, R/2)$ only relies on the properties of the kernel that we have already pointed out. In particular, since the integral defining $\widetilde{T}_3 f$ is over $B(\overline{x}, R)$ and $B(\overline{x}, 3R) \subset U$, we can safely apply the local doubling condition on the small balls which are involved in that proof. Combining (5.34) with the first part of the proof of this theorem, we can write

$$\left|X_i X_j w(x_1) - X_i X_j w(x_2)\right| \leqslant c d(x_1, x_2)^{\beta} \|f\|_{C_X^{\beta}(U)} \quad \forall \beta < \alpha$$

for any $x_1, x_2 \in B\left(\overline{x}, \frac{R}{2}\right)$, with some constant c also depending on R.

An analogous, but easier, inspection of each term $T_j f$ also shows that

$$\sup_{x \in B\left(\overline{x}, \frac{R}{2}\right)} \left|X_i X_j w(x)\right| \leqslant c \|f\|_{C_X^{\beta}(U)}. \tag{5.35}$$

By a covering argument this implies

$$\sup_{x \in U'} \left|X_i X_j w(x)\right| \leqslant c \|f\|_{C_X^{\beta}(U)} \tag{5.36}$$

so that for each couple of points $x_1, x_2 \in U'$ we can write

$$\left|X_i X_j w(x_1) - X_i X_j w(x_2)\right| \leqslant c d(x_1, x_2)^{\beta} \|f\|_{C_X^{\beta}(U)}$$

if $d(x_1, x_2) < R_0/2$, and, by (5.36),

$$\left|X_i X_j w(x_1) - X_i X_j w(x_2)\right| \leqslant 2 \sup_{x \in U'} \left|X_i X_j w(x)\right| \leqslant c \left(\frac{d(x_1, x_2)}{R_0}\right)^{\beta} \|f\|_{C_X^{\beta}(U)}$$

if $d(x_1, x_2) \geqslant R_0/2$. Hence

$$\|X_i X_j w\|_{C_X^{\beta}(U')} \leqslant c \|f\|_{C_X^{\beta}(U)}.$$

The norms $\|X_i w\|_{C_X^{\beta}(U')}$, $i = 1, \ldots n$, and $\|w\|_{C_X^{\beta}(U')}$ can be more easily handled and (5.29) follows. ■

6 Appendix. Examples of nonsmooth Hörmander's operators satisfying assumptions A or B

Example 6.1 (Nonsmooth sublaplacian of Heisenberg type) *In* $\mathbb{R}^3 \ni (x, y, t)$, *let*

$$X_1 = \frac{\partial}{\partial x} + y(1 + |y|) \frac{\partial}{\partial t}; \; X_2 = \frac{\partial}{\partial y} - x(1 + |x|) \frac{\partial}{\partial t};$$

$$[X_1, X_2] = -2(1 + |x| + |y|) \frac{\partial}{\partial t};$$

$$L = X_1^2 + X_2^2.$$

The vector fields X_1, X_2 *are* $C^{1,1}$ *and satisfy Hörmander's condition with* $r = 2$, *hence Assumptions A hold. Replacing* $|x|, |y|$ *with* $x|x|, y|y|$ *we find* $C^{2,1}$ *vector fields, satisfying Assumptions B.*

Example 6.2 (Nonsmooth operator of Kolmogorov type) *In* $\mathbb{R}^3 \ni (x, y, t)$, *with* $\alpha \in (0, 1]$, *let:*

$$X_1 = \frac{\partial}{\partial x};\ X_0 = x\left(1 + |x|^\alpha\right)\frac{\partial}{\partial y} + \frac{\partial}{\partial t};\ [X_1, X_0] = \left(1 + (\alpha + 1)|x|^\alpha\right)\frac{\partial}{\partial y};$$
$$L = X_1^2 + X_0.$$

X_1, X_0 *satisfy Hörmander's condition at weighted step* $r = 3$*;* $X_1 \in C^{2,\alpha}$, $X_0 \in C^{1,\alpha}$, *hence Assumptions A hold. Replacing* $|x|^\alpha$ *with* $x|x|^\alpha$, *Assumptions B hold.*

Example 6.3 (Nonsmooth operators of Grushin type with high step r**)** *In* $\mathbb{R}^2 \ni (x, y)$, *with* $\alpha \in (0, 1]$, $r \geqslant 2$ *positive integer, let*

$$X_1 = \frac{\partial}{\partial x};\ X_2 = x^{r-1}\left(1 + |x|^\alpha\right)\frac{\partial}{\partial y};$$
$$L = X_1^2 + X_2^2.$$

X_1, X_2 *satisfy Hörmander's condition at step* r*;* $X_2 \in C^{r-1,\alpha}$, *hence Assumptions A hold (if* $r = 2$ *we need to take* $\alpha = 1$*). Replacing* $|x|^\alpha$ *with* $x|x|^\alpha$, *Assumptions B hold.*

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