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Green's Function and Potential Theory for the Schrödinger Operator: a Nonprobabilistic Approach (**)

Funzione di Green e teoria del potenziale per l'operatore di Schrödinger

SOMMARIO. — Si considera l'operatore di Schrödinger stazionario $Lu \equiv Au + Vu$, con parte principale A in forma di divergenza e potenziale V in una classe di Kato, su domini lipschitziani. Con tecniche analitiche (non probabilistiche) viene ottenuta, sotto un'opportuna condizione integrale su V, la stima per la funzione di Green per L:

$$c_1 \cdot G_A(x,y) \leqslant G_L(x,y) \leqslant c_2 \cdot G_A(x,y)\,.$$

Si stabilisce poi un'analoga stima per la misura L-armonica, e se ne deducono risultati di teoria del potenziale per L: regolarità dei punti alla frontiera, comportamento alla frontiera per soluzioni positive, in particolare esistenza di limiti non tangenziali (quasi ovunque rispetto alla misura L-armonica).

INTRODUCTION

In this paper we consider the Schrödinger operator (in steady state)

$$Lu \equiv Au + Vu \equiv -(a_{ij}u_{x_i})_{x_j} + V \cdot u$$

where A is a uniformly elliptic operator in devergence form, with bounded measurable coefficients, V is assumed in the Kato class $K(\Omega)$, and L is defined on a bounded Lipschitz domain Ω. (Precise definitions will be given later). This operator, or its particular case $-\Delta + V$, has been studied in recent years, especially with probabilistic methods: Aizenman-Simon [1] proved a Harnack's inequality and some subsolution estimates for $-\Delta + V$, using the Feynman-Kac formalism; Simon [9] developed a theory for the same operator from the point of view of semigroups of operators; Zhao [12] proved

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a fundamental estimate for the Green's function of $-\Delta + V$ on $C^{1,1}$ domains. More recently, Cranston-Fabes-Zhao [4], using the results of Caffarelli-Fabes-Mortola-Salsa [2], extended many of these results to the operator $A + V$ on Lipschitz domains, and developed a potential theory for it. On the other hand, Chiarenza-Fabes-Garofalo [3], with a non probabilistic approach, proved for L a Harnack's inequality and the continuity of solutions of $Lu = 0$. Following this line, our aim is to develop, by typical methods of P.D.E., a theory for the operator L.

Since, in the usual variational theory, V is assumed in $\mathfrak{L}^p(\Omega)$ with $p > n/2$, a smaller class than $K(\Omega)$, we shall at first prove—section 1—that Dirichlet's problem for L can still be formulated, and is well posed, if V is, in a proper sense, rather small.

Then—section 2—we shall investigate properties of the Green's function for L; particularly, we will show (Th. 2.6) that it is controlled from above and from below by the Green's function for A. A « weak theory » for the operator L will be also developed, extending to L some results of [6] about A.

In sections 3-4 we shall obtain some potential theoretical results, i.e. properties of solutions of $Lu = 0$. First we shall state—section 3— solvability of Dirichlet's problem for L when the datum is a continuous function defined only on the boundary; from this fact will follow existence of the L-harmonic measure. Then we shall obtain an estimate about harmonic measures (Th. 3.8), analogue to the one involving the Green's functions for L and A. From this fact a comparison principle for positive solutions of $Lu = 0$, vanishing on a part of the boundary, will follow (section 4). Then we shall state regularity of boundary points for L. Finally a « Fatou's theorem » will be obtained, i.e. existence of nontangential boundary limits (a.e. with respect to the L-harmonic measure) for positive solutions of $Lu = 0$.

Note that the two facts we have borrowed from [4] (Theorems 1.4 and 4.5) have analytical proofs. So our work relies on no probabilistic argument.

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1. - Some definitions and known results. Dirichelet's problem for L.

Let Ω be a bounded Lipschitz domain of $\mathbf{R}^n$ ($n \geqslant 3$). This means that there exists a pair of positive numbers r_0 and M such that for every $z \in \partial\Omega$, local coordinates can be selected so that $B(z, r_0) \cap \partial\Omega$ is the graph of a Lipschitz function φ with $|D\varphi| \leqslant M$. The constants r_0 and M determine what will be called the Lipschitz character of Ω.

The operator A is supposed to have bounded, measurable, real valued coefficients a_{ij}. We also suppose $a_{ij} = a_{ji}$ and A uniformly elliptic. So there is a positive constant λ such that

$$\lambda^{-1} |\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant \lambda|\xi|^2 \qquad \text{for every } \xi \in \mathbf{R}^n\,, \text{ for a.e. } x \in \Omega\,.$$

Let us now define the Kato class $K(\Omega)$.

$$K(\Omega)=\left\{f\in \mathfrak{L}^1_{\mathrm{loc}}(\Omega):\lim_{r\downarrow 0}\sup_{x\in R^n}\int\limits_{\Omega\cap B(x,r)}\frac{|f(y)|}{|x-y|^{n-2}}\,dy=0\right\}.$$

If $V\in K(\Omega)$, put

$$\eta(r)=\sup_{x\in R^n}\int\limits_{\Omega\cap B(x,r)}\frac{|V(y)|}{|x-y|^{n-2}}\,dy\,.$$

Sometimes we shall call η « the Kato norm of V ». Now, all the informations we need about L are contained in λ and η.

Note that if $V\in\mathfrak{L}^p(\Omega)$ with $p>n/2$, then by Hölder's inequality, $V\in K(\Omega)$ and $\eta(r)\leqslant c\cdot\|V\|_p\cdot r^\alpha$ with c, α only depending on n and p. So our assumptions generalize the case which is studied in standard variational approach.

If $V\in K(\Omega)$, it is easy to prove the following properties:

(i) $V\in\mathfrak{L}^1(\Omega)$;

(ii) $\eta(r)$ is finite for every r, monotone non decreasing;

(iii) $\|V\|_1\leqslant d^{n-2}\cdot\eta(d)$ where $d=\operatorname{diam}\Omega$;

(iv) $\sup\limits_{\Omega}\int\limits_{\Omega}\frac{|V(y)|}{|x-y|^{n-2}}\,dy\leqslant\eta(d)$;

(v) η is bounded and definitively constant, $\eta(r)\leqslant\eta(2d)$ for every r;

(vi) if $f(x)=\int\limits_{\Omega}\frac{|V(y)|}{|x-y|^{n-2}}\,dy$, f is continuous in Ω.

Note that, if $f\in\mathfrak{L}^1_{\mathrm{loc}}(\Omega)$ and $\eta(r)<\infty$ for some r, then properties (i)-(v) hold, but f must not necessarily belong to $K(\Omega)$. (A counterexample is given in [1]). So the crucial property in defining $K(\Omega)$ is that $\eta(r)\to 0$.

A fundamental result, due to Schechter (See [8], p. 138) is the following:

THEOREM 1.1: If $V\in K(\Omega)$, there exists a constant $k=k(n)$ and, for every $\delta>0$, a constant $c_\delta=c(\delta,n)$, such that for all $\varphi\in W_0^{1,2}(\Omega)$:

$$\int\limits_{\Omega}|V|\varphi^2\leqslant k\cdot\eta(\delta)\|\varphi\|^2_{W^{1,2}}+c_\delta\cdot\eta(1)\cdot\|\varphi\|^2_{\mathfrak{L}^2}\,.$$

REMARK 1.2: From Theorem 1.1 it follows that, since Ω is Lipschitz, the bilinear form associated to L is well defined and continuous on $W^{1,2}(\Omega)$, and it is coercive on $W_0^{1,2}(\Omega)$ provided that a condition

$$c_1(n,\lambda)\cdot\eta(2d)\leqslant 1 \tag{1.1}$$

holds, with $d=\operatorname{diam}\Omega$.

Let us recall also two basic estimates regarding the Green's function G for A.

THEOREM 1.3 (see [6]):

$$G(x,y)\leqslant\frac{c_2}{|x-y|^{n-2}}\qquad\text{for all } x,y\in\Omega\,,\ \text{for some constant } c_2(n,\lambda)\,. \tag{1.2}$$

THEOREM 1.4 (see [4]): There exists a constant c_3 depending on λ, n and the Lipschitz character of Ω such that:

$$\frac{G(x,y)\,G(y,z)}{G(x,z)}\leqslant c_3\cdot\left\{\frac{1}{|x-y|^{n-2}}+\frac{1}{|y-z|^{n-2}}\right\}\qquad\text{for all } x,y,z\in\Omega\,. \tag{1.3}$$

Now, put $c=\max(c_1, c_2, 2c_3)$ where c_1, c_2, c_3 are as in (1.1), (1.2), (1.3). Note that $c=c(\lambda, n, r_0, M)$. Put:

$$\delta=c\cdot\eta(2d)\,. \tag{1.4}$$

Henceforth we shall suppose the Kato norm of V so small to have:

$$\delta<\tfrac{1}{2}\,. \tag{1.5}$$

Then (1.1) holds, and Lax-Milgram's lemma implies:

THEOREM 1.5: The problem

$$\begin{cases} Lu=T & \text{in } \Omega\,,\\ u\ =g & \text{on } \partial\Omega\,,\end{cases}$$

for $T\in W^{-1,2}$ and $g\in W^{1,2}$ assigned, is well posed. The constants in the continuous dependence estimate depend on n, λ, η, r_0, M.

(If $g\equiv 0$, the constant only depends on n, λ, η.)

Let us state also a maximum principle for L. From coerciveness of the bilinear form associated to L it follows:

THEOREM 1.6: If $u\in W^{1,2}(\Omega)$ is a supersolution for L and $u\geqslant 0$ on $\partial\Omega$ (in sense $W^{1,2}$) then $u\geqslant 0$ a.e. in Ω.

2. - GREEN'S FUNCTION FOR L

In the following, we shall indicate with G and G_L the Green's functions for A and L, respectively. Existence of G_L will follow from theorem 2.3. First, we state a lemma regarding G.

LEMMA 2.1. Let $f \in \mathfrak{L}^1(\Omega)$ such that $\sup_x \int_\Omega G(x,y)|f(y)|\,dy \leqslant c < \infty$ for some constant c. Then there exists a unique $u \in W_0^{1,2}(\Omega)$ satisfying $Au = f$ in Ω, in the sense that

$$\int_\Omega a_{ij} u_{x_i} \varphi_{x_j}\,dx = \int_\Omega f\varphi\,dx \qquad \text{for all } \varphi \in C_0^\infty(\Omega)\,.$$

Moreover $u(x) = \int_\Omega G(x,y) f(y)\,dy$.

PROOF: Put

$$f_n(x) = \begin{cases} n & \text{if } f(x) \geqslant n\,, \\ f(x) & \text{if } |f(x)| < n\,, \\ -n & \text{if } f(x) \leqslant -n\,, \end{cases}$$

and let u_n be the solution of

$$\begin{cases} Au_n = f_n & \text{in } \Omega\,, \\ u_n \in W_0^{1,2}(\Omega)\,. \end{cases}$$

Then

$$u_n(x) = \int_\Omega G(x,y) f_n(y)\,dy \quad \text{and} \quad \|u_n\|_\infty \leqslant c\,.$$

From the equation $Au_n = f_n$ it follows:

$$\int_\Omega |Du_n|^2 \leqslant \lambda \int_\Omega a_{ij}(u_n)_{x_i}(u_n)_{x_j}\,dx = \lambda \int_\Omega u_n f_n \leqslant \lambda \|f_n\|_1 \|u_n\|_\infty \leqslant \lambda c \|f\|_1\,.$$

Therefore $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ and there exists a subsequence u_n converging to some u weakly in $W_0^{1,2}$ and strongly in $\mathfrak{L}^2$. So, taking limits in:

$$\int a_{ij}(u_n)_{x_i}\varphi_{x_j}\,dx = \int f_n \varphi$$

for a fixed $\varphi \in C_0^\infty(\Omega)$, we have:

$$\int a_{ij} u_{x_i} \varphi_{x_j}\,dx = \int f\varphi\,.$$

On the other hand, choosing a subsequence $f_n \to f$ a.e., since

$$G(x,\cdot)|f_n| \leqslant G(x,\cdot)|f| \in \mathfrak{L}^1\,,$$

by Lebesgue's theorem one has:

$$u_n(x) \to \int_\Omega G(x, y) f(y)\, dy$$

that is

$$u(x) = \int_\Omega G(x, y) f(y)\, dy \,. \qquad //$$

REMARK 2.2: When $f = V \in K(\Omega)$, by (1.3) and definition of $K(\Omega)$ we have:

$$\int_\Omega G(x, y)\,|V(y)|\, dy \leqslant c_1 \int_\Omega \frac{|V(y)|}{|x-y|^{n-2}}\, dy \leqslant c_1 \eta(d) \leqslant \delta \,, \tag{2.1}$$

and the previous lemma holds. Moreover, by Theorem 1.1, the functional $\varphi \to \int V\varphi$ is continuous on $W_0^{1,2}$. So we have:

$$\int_\Omega a_{ij} u_{x_i} \varphi_{x_j} = \int_\Omega V\varphi \qquad \text{for all } \varphi \in W_0^{1,2} \,.$$

Next theorem is taken from [3].

THEOREM 2.3: If $u \in W_0^{1,2}$ is the solution of $Lu = f$ with $f \in \mathfrak{L}^p(\Omega)$, $p > n/2$, then:

$$\|u\|_\infty \leqslant c \|f\|_p \qquad \text{with } c = \frac{c(n, \lambda, p, |\Omega|)}{1-\delta} \,.$$

REMARK 2.4: As a consequence of Theorem 2.3, there exists the Green's function G_L for L, i.e. for every $f \in \mathfrak{L}^p$ $(p > n/2)$, the solution of:

$$\begin{cases} Lu = f \,, \\ u \in W_0^{1,2} \,, \end{cases}$$

is given by:

$$u(x) = \int_\Omega G_L(x, y) f(y)\, dy \tag{2.2}$$

with:

$$\sup_x \|G_L(x, \cdot)\|_q \leqslant c(n, \lambda, q, |\Omega|, \delta) \qquad \text{for all } q < \frac{n}{n-2} \,. \tag{2.3}$$

Moreover G_L is symmetric and, by the maximum principle (Theorem 1.6), nonnegative.

The maximum principle also implies the following:

COROLLARY 2.5: Let $V_1, V_2 \in K(\Omega)$ (both V_i satisfying our assumption $\delta < \frac{1}{2}$) and let G_{L_1}, G_{L_2} be the Green's functions for $A + V_1$, $A + V_2$, respectively. If $V_1 \leqslant V_2$, $G_{L_1} \geqslant G_{L_2}$.

Now we can state our basic result:

THEOREM 2.6 (Comparison between G and G_L):

$$\left(\frac{1-2\delta}{1-\delta}\right) \cdot G(x,y) \leqslant G_L(x,y) \leqslant \frac{1}{1-\delta} \cdot G(x,y) \qquad \text{for a.e. } x, y \in \Omega . \tag{2.4}$$

PROOF: Using the representation formula (2.2) one can find the following identity:

$$G_L(x,y) = G(x,y) - \int_\Omega G_L(x,w) V(w) G(w,y)\, dw \qquad \text{for a.e. } x, y \in \Omega .$$

Now, let us consider the space B defined by:

$$B = \left\{ f\colon\ \Omega \times \Omega \to \boldsymbol{R},\ f \text{ measurable such that } \|f\|_B \equiv \sup_x \int_\Omega |f(x,y)|\, dy < +\infty \right\}.$$

B is a Banach space. If we define the operator T as:

$$Tf(x,y) \equiv \int_\Omega f(w,y) V(w) G(x,w)\, dw$$

it is easy to verify that T is a well defined, linear continuous operator from B to B, with $\|T\|_{\mathfrak{L}(B)} \leqslant \delta$. Let us consider the integral equation:

$$f + Tf = G \tag{2.6}$$

where the unknown function f is sought in B. Then $(I + T)$ can be inverted by Neumann series:

$$(I + T)^{-1} = \sum_0^\infty (-)^n T^n \tag{2.7}$$

(where the series converges in $\mathfrak{L}(B)$). Since, by (2.5), the solution of (2.6) is G_L, (2.7) gives:

$$G_L = \sum_0^\infty (-)^n T^n G \tag{2.8}$$

(where the series converges in B). Using Theorem 1.4 we have:

$$|TG(x,y)| = \left| \int_\Omega G(w,y)\, V(w)\, G(x,w)\, dw \right| \leqslant$$

$$\leqslant G(x,y) \cdot \int_\Omega \frac{G(x,w)\, G(w,y)}{G(x,y)} \, |V(w)|\, dw \leqslant G(x,y) \cdot c_3 \cdot$$

$$\cdot \left\{ \int_\Omega \frac{|V(w)|}{|y-w|^{n-2}}\, dw + \int_\Omega \frac{|V(w)|}{|x-w|^{n-2}}\, dw \right\} \leqslant G(x,y) \cdot c_3 \cdot 2\eta(d) \leqslant \delta \cdot G(x,y)\,.$$

By iteration:

$$|T^n G(x,y)| \leqslant \delta^n \cdot G(x,y)\,. \tag{2.9}$$

Since convergence in B implies convergence a.e. of a subsequence, it follows from (2.8) and (2.9) that:

$$G_L(x,y) \leqslant \frac{1}{1-\delta} \cdot G(x,y) \qquad \text{for a.e. } x, y \in \Omega\,, \tag{2.10}$$

and so we have the right hand inequality in (2.4).

On the other hand, again from (2.5) we have:

$$G_L(x,y) = G(x,y) - \int_\Omega G_L(w,y)\, V(w)\, G(x,w)\, dw =$$

$$= G(x,y) \cdot \left\{ 1 - \int_\Omega \frac{G_L(w,y)\, G(x,w)}{G(x,y)}\, V(w)\, dw \right\} \geqslant \qquad \text{(by (2.10))}$$

$$G(x,y) \cdot \left\{ 1 - \frac{1}{1-\delta} \int_\Omega \frac{G(w,y)\, G(x,w)}{G(x,y)}\, |V(w)|\, dw \geqslant \qquad \text{(by Theorem 1.4)}\right.$$

$$\geqslant G(x,y) \cdot \left\{ 1 - \frac{1}{1-\delta} \cdot c_3 \cdot 2\eta(d) \right\} \geqslant \frac{1-2\delta}{1-\delta} \cdot G(x,y)$$

and the proof is complete. //

Let us see some consequences of Theorem 2.6. Combining this fact with results in [6], we have:

THEOREM 2.7: Let Σ be a simply connected, bounded Lipschitz domain which can be mapped smoothly onto a sphere, and let G_L, g be the Green's functions for L and $-\Delta$, respectively, in Σ. Then, for any compact subset C of Σ, there exists a constant k only depending on C, Σ and δ, such that:

$$k^{-1} \cdot g(x,y) \leqslant G_L(x,y) \leqslant k \cdot g(x,y) \qquad \text{for a.e. } x, y \in C\,.$$

COROLLARY 2.8:

$$G_L(x, y) \leqslant \frac{c(\delta)}{|x-y|^{n-2}} \qquad \text{for a.e. } x, y \in \Omega . \tag{2.11}$$

REMARK 2.9: We can obtain from (2.11) an integrability property of G_L (and G). Let us recall that $f \in \mathfrak{L}^{p,\infty}(\Omega)$ if, by definition, $f \in \mathfrak{L}^1_{loc}(\Omega)$ and:

$$\|f\|_{p,\infty} \equiv \sup_K \frac{\int_K |f(x)|\,dx}{|K|^{1/q}} < +\infty$$

where the sup is taken among all compact subsets of Ω and $p^{-1}+q^{-1}=1$. It is clear that $\mathfrak{L}^p(\Omega)$ is continuously embedded in $\mathfrak{L}^{p,\infty}(\Omega)$. We already know that $G_L(x,\cdot) \in \mathfrak{L}^q$ for $q < n/(n-2)$. Since it is known that the function $f_x(y) = |x-y|^{2-n}$ belongs to $\mathfrak{L}^{n/(n-2),\infty}$, uniformly in x, by (2.11) the same is true for G_L:

$$\sup_x \|G_L(x,\cdot)\|_{n/(n-2),\infty} \leqslant c(\delta) .$$

REMARK 2.10: Let $V = V^+ - V^-$, let η be the Kato norm of V^-, η satisfying our assumptions (1.4)-(1.5), and $V^+ \in K(\Omega)$. Then the bilinear form associated to L is still coercive, and there exists the Green's function G_L. By Corollary 2.5, $G_L \leqslant G_{A-V^-}$, while G_{A-V^-} clearly satisfies Theorem 2.7. So it is still true that

$$G_L(x, y) \leqslant \frac{1}{1-\delta} \cdot G(x, y) .$$

Green's function can be seen also as the solution of $LG_L(x,\cdot) = \delta_x$ (where δ_x is the Dirac mass concentrated in x) in a weak sense, by developing a « weak theory » for the equation $Lu = \mu$ (where μ is a measure). This can be done, following [6], by transferring to L some properties which are known to hold for A.

THEOREM 2.11: If u is the solution of

$$\begin{cases} Lu = (f_i)_{x_i} & \text{in } \Omega , \\ u \;= h & \text{on } \partial\Omega , \end{cases} \tag{2.12}$$

with $f_i \in \mathfrak{L}^p(\Omega)$, $h \in W^{1,p}(\Omega)$ and $p > n$, then u is continuous on $\bar{\Omega}$.

THEOREM 2.12. Under the same assumptions of the previous theorem, if $h \equiv 0$ then

$$\max_{\Omega} |u(x)| \leqslant c(n, \lambda, p, |\Omega|, \delta) \cdot \|f_i\|_p .$$

Proof: Theorems 2.11-2.12 hold for A; these results are due to Stampacchia (see [6] for references). Since u is bounded, $Vu \in K(\Omega)$. So, by Lemma 2.1 and Remark 2.2, u can be seen as the sum of a function satisfying (2.12) with L substituted by A, and a function expressed by:

$$\tilde{u}(x) = -\int_{\Omega} G(x, y)\, V(y)\, u(y)\, dy\,.$$

Now, $\tilde{u}$ is continuous, by definition of Kato class and (1.2), so Theorem 2.11 is proved. Moreover:

$$\|u\|_\infty \leqslant c \cdot \|f_i\|_p + \delta \cdot \|u\|_\infty$$

and Theorem 2.12 follows. //

Definition 2.13: For a measure μ of bounded variation on Ω, we say that $u \in \mathfrak{L}^1(\Omega)$ is a *weak solution of the equation* $Lu = \mu$ *vanishing at the boundary* $\partial\Omega$ if it satisfies

$$\int_{\Omega} u \cdot L\varphi\, dx = \int_{\Omega} \varphi\, d\mu$$

for every $\varphi \in W_0^{1,2}(\Omega) \cap C(\bar{\Omega})$ such that $L\varphi \in C(\bar{\Omega})$.

Once one knows the results of Theorems 2.11-2.12, the arguments contained in sections 5-6 of [6] can be repeated. We summarize the main results in the following:

Theorem 2.14: For any measure μ of bounded variation, a unique solution u of $Lu = \mu$ vanishing at $\partial\Omega$ exists, and lies in $W_0^{1,p'}(\Omega)$ for every $p' < n/(n-1)$; moreover u satisfies

$$\|u\|_{W_0^{1,p'}} \leqslant c(n, \lambda, p', |\Omega|, \delta) \int_{\Omega} |d\mu|$$

and u is assigned by the integral (a.e. converging)

$$u(x) = \int_{\Omega} G_L(x, y)\, d\mu(y)\,.$$

Finally, $G_L(x, \cdot)$ is the weak solution vanishing at $\partial\Omega$ of $Lu = \delta_x$.

Remark 2.15: If u is the weak solution of $Lu = \mu$ ($Au = \mu$) vanishing at $\partial\Omega$, then $Vu \in \mathfrak{L}^1(\Omega)$ and

$$\|Vu\|_1 \leqslant c \int_{\Omega} |d\mu|$$

with $c = \delta/(1-\delta)$ $(c = \delta)$. In fact we have:

$$\int |Vu| \leq \int |V(x)|\, dx \int G_L(x,y)|d\mu(y)| = \int |d\mu(y)| \int G_L(x,y)|V(x)|\, dx \leq \frac{\delta}{1-\delta}\int |d\mu|$$

where we have used (2.1) and (2.4).

Let us point out also the following regularity properties of G_L:

THEOREM 2.16:

(i) $G_L(x,\cdot) \in W_0^{1,p'}(\Omega)$ for every $p' < n/(n-1)$;

(ii) $G_L(x,\cdot) \in W^{1,2}_{\mathrm{loc}}(\Omega - \{x\})$ and is a local solution of $Lu = 0$ in $\Omega - \{x\}$;

(iii) $G_L(x,\cdot) \in C(\Omega - \{x\})$.

PROOF: (i) follows from 2.14, while (ii)-(iii) can be stated as in [6], using two results about L (a «Caccioppoli's inequality» and the continuity of solutions of $Lu = 0$) contained in [3]. //

3. - Dirichelet's problem with continuous boundary data. L-harmonic measure

In order to define the concept of L-harmonic measure and develop a potential theory for L, we need a sharper version of maximum principle.

THEOREM 3.1: If u is a supersolution (subsolution) for L in Ω, then, respectively:

$$\text{(3.1)}\qquad a)\ \min_{\Omega} u \geq \frac{1}{1-\delta}\cdot \min_{\partial\Omega} u_-\,,\qquad b)\ \max_{\Omega} u \leq \frac{1}{1-\delta}\cdot \max_{\partial\Omega} u^+\,.$$

If u is a solution of $Lu = 0$ in Ω, then

$$\text{(3.2)}\qquad \max_{\Omega} |u| \leq \frac{1}{1-\delta}\cdot \max_{\partial\Omega} |u|\,.$$

PROOF: Let $u \geq k$ on $\partial\Omega$ for some $k \leq 0$. Then

$$\begin{aligned} L(u-k) &= Lu - Vk \geq -Vk && \text{in } \Omega\,,\\ (u-k) &\geq 0 && \text{on } \partial\Omega\,. \end{aligned}$$

Let w be the solution of

$$\begin{cases} Lw = -Vk & \text{in } \Omega\,, \\ w \;= 0 & \text{on } \partial\Omega\,. \end{cases}$$

Then $w(x) = -k\int_\Omega G_L(x,y)V(y)\,dy$ and, by (2.1) and (2.4).

$$|w(x)| \leqslant -\frac{k}{1-\delta}\int_\Omega G(x,y)|V(y)|\,dy \leqslant -k\cdot\frac{\delta}{1-\delta}\,.$$

Put $v = (u-k)-w$. By the maximum principle (Th. 1.6), $v \geqslant 0$ in Ω and so

$$u(x) \geqslant k - |w(x)| \geqslant \frac{k}{1-\delta}\,. \tag{3.3}$$

Now, $\min_{\partial\Omega} u = \sup\{k\colon u \geqslant k \text{ on } \partial\Omega\}$. If $\min_{\partial\Omega} u > 0$, (3.1.$a$) holds by Theorem 1.6 while if $\min u \leqslant 0$, (3.1.a) follows from (3.3).

Changing u in $-u$ it follows (3.1.b), and combining (3.1.a) with (3.1.b) one has (3.2). //

Remark 3.2: The constant δ in (3.1)-(3.2) actually depends only on V^-, so that when $V \geqslant 0$ one has:

$$\max_\Omega |u| \leqslant \max_{\partial\Omega} |u|\,. \tag{3.4}$$

In the general case one cannot expect (3.4) to be true. To see this, it is sufficient to consider Ω the unit ball, V a small negative constant, $A = -\Delta$: then the solution u with boundary value 1 is a positive function assuming a strong maximum at the origin.

Now, using Theorem 2.1 and Caccioppoli's inequality of [3], one can repeat an argument used in [6] and give sense to Dirichelet's problem for L when the datum is a continuous function defined on $\partial\Omega$. Namely, the following holds:

Theorem 3.3: There exists a mapping B which to any continuous function f defined on $\partial\Omega$ associates a local solution of $Lu = 0$ (which is continuous by [3]) such that whenever f is the trace of a $C^1(\overline{\Omega})$ function, Bf coincides with the variational solution of

$$\begin{cases} Lu = 0 & \text{in } \Omega\,, \\ u \;= f & \text{on } \partial\Omega\,. \end{cases}$$

Moreover, if $u = Bf$, one has:

$$\sup_{K \subset\subset \Omega} [\operatorname{dist}(K, \partial\Omega) \cdot \|Du\|_{\mathfrak{L}^2(K)}] + \max_{\Omega} |u| < +\infty ;$$

$$\max_{\Omega} |u| \leqslant \frac{1}{1-\delta} \cdot \max_{\partial\Omega} |f| .$$

This theorem makes it possible to give the following:

DEFINITION 3.4: A point $y \in \partial\Omega$ is said to be *regular* for L iff for every $h \in C(\partial\Omega)$ one has:

$$\lim_{\substack{x \to y \\ x \in \Omega}} Bh(x) = h(y) .$$

DEFINITION 3.5: For a fixed $x \in \Omega$, let us consider the functional $f \to Bf(x)$ defined on $C(\partial\Omega)$. By Riesz' theorem there exists a positive regular Borel measure w_L^x representing it:

$$Bf(x) = \int_{\partial\Omega} f(y)\, dw_L^x(y) .$$

We call w_L^x the L-harmonic measure evaluated at x. By the Harnack principle of [3] it follows that, given $x_1, x_2 \in \Omega$, there exists a constant c such that:

$$w_L^{x_1}(E) \leqslant c \cdot w_L^{x_2}(E) \qquad \text{for any Borel set } E \subset \partial\Omega .$$

We shall indicate with w_A^x the Borel measure obtained with the same construction for the operator A (A-harmonic measure evaluated at x).

The main result we shall prove in this section is the following: there exist constants c_1, c_2 such that for any Borel set E and $x \in \Omega$:

$$c_1 w_A^x(E) \leqslant w_L^x(E) \leqslant c_2 w_A^x(E) . \tag{3.5}$$

From this fact and the results contained in [2] it will follow potential theory for L. To obtain (3.5) two facts are used: the estimate (2.4) involving the Green's functions for A and L; the notion of kernel function for A and some relative results contained in [2].

DEFINITION 3.6: Fix $x \in \Omega$, $z \in \partial\Omega$. A function $K_A^x(w, z)$ defined in Ω is called a kernel function at z for the operator A, normalized at x, if:

(i) $K_A^x(\cdot, z)$ is a solution of $Au = 0$ in Ω;

(ii) $K_A^x(\cdot, z) \in C(\bar{\Omega} - \{z\})$ and $\lim_{\substack{w \to z' \neq z \\ z' \in \partial\Omega}} K_A^x(w, z) = 0$;

(iii) $K_A^x(w, z) > 0$ for each $w \in \Omega$ and $K_A^x(x, z) = 1$.

For x and z fixed, there exists one and only one kernel function $K_A^x(\cdot, z)$ and it is:

$$K_A^x(w, z) = \frac{dw_A^w}{dw_A^x}(z) \tag{3.6}$$

(Radon-Nikodym derivative of the A-harmonic measures).

It can be proved also that, for any $w \in \Omega$, $z \in \partial\Omega$, there exists

$$\lim_{\substack{y \to z \in \partial\Omega \\ y \in \Omega}} \frac{G(w, y)}{G(x, y)} = K_A^x(w, z) . \tag{3.7}$$

THEOREM 3.7: For $x \in \Omega$, $z \in \partial\Omega$, there exists

$$F(x, z) = \lim_{\substack{y \to z \\ y \in \Omega}} \frac{G_L(x, y)}{G(x, y)} . \tag{3.8}$$

Moreover F is continuous on $\Omega \times \partial\Omega$, and:

$$F(x, z) = 1 - \int_\Omega K_A^x(w, z) V(w) G_L(x, w)\, dw . \tag{3.9}$$

PROOF: Let us consider (2.5), written as:

$$\frac{G_L(x, y)}{G(x, y)} = 1 - \int_\Omega \frac{G(w, y)}{G(x, y)} \cdot G_L(x, w) V(w)\, dw . \tag{3.10}$$

By (3.7), what we have to prove is that in (3.10) the limit can be taken under the integral sign. Put:

$$f_r(y) = \int_{|y-w|>r} G_L(x, w) \cdot \frac{G(w, y)}{G(x, y)} \cdot V(w)\, dw .$$

By Lebesgue's theorem, Theorems 2.4 and 2.6, one has:

$$\lim_{y \to z} f_r(y) = \int_{|w-z|>r} K_A^x(w, z) G_L(x, w) V(w)\, dw$$

and, by definition of Kato class,

$$\lim_{r \to 0} \lim_{y \to z} f_r(y) = \int_\Omega K_A^x(w, z) G_L(x, w) V(w)\, dw \qquad \text{uniformly in } z .$$

On the other hand, for $r \to 0$, $f_r(y)$ converges uniformly to:

$$\int_\Omega \frac{G(w, y)}{G(x, y)} \cdot G_L(x, w) V(w)\, dw\,.$$

So, exchanging the limits it follows (3.9) and the continuity of F. //

THEOREM 3.8 (Comparison between harmonic measures): For each $x \in \Omega$, $z \in \partial\Omega$, one has:

$$dw_L^x(z) = F(x, z)\, dw_A^x(z)\,. \tag{3.11}$$

Moreover, there exist constants c_1, c_2 depending on δ such that:

$$c_1 \cdot w_A^x(E) \leqslant w_L^x(E) \leqslant c_2 \cdot w_A^x(E) \qquad \text{for every Borel set } E\,,\ x \in \Omega\,. \tag{3.12}$$

PROOF: Let $f \in C(\partial\Omega)$. By (3.9):

$$\int_{\partial\Omega} f(z) F(x, z)\, dw_A^x(z) = \int_{\partial\Omega} f(z)\, dw_A^x(z) - \int_{\partial\Omega} f(z)\, dw_A^x(z) \int_\Omega K_A^x(w, z) V(w) G_L(x, w)\, dw =$$

$$\text{(by Tonelli and (3.6))} = \int_{\partial\Omega} f(z)\, dw_A^x(z) - \int_\Omega G_L(x, w) V(w)\, dw \int_{\partial\Omega} f(z)\, dw_A^w(z)\,.$$

Now, let v, u be the solutions of

$$\begin{cases} Lv = 0 & \text{in } \Omega\,, \\ v \;= f & \text{on } \partial\Omega\,, \end{cases} \qquad \begin{cases} Au = 0 & \text{in } \Omega\,, \\ u \;= f & \text{on } \partial\Omega\,. \end{cases}$$

Then the previous identity becomes:

$$\int_{\partial\Omega} f(z) F(x, z)\, dw_A^x(z) = u(x) - \int_\Omega G_L(x, w) V(w) u(w)\, dw = v(x) = \int_{\partial\Omega} f(z)\, dw_L^x(z)\,.$$

Since this is true for every $f \in C(\partial\Omega)$, it follows (3.11). Now, by Theorem 3.7, (2.4) implies:

$$\frac{1 - 2\delta}{1 - \delta} \leqslant F(x, z) \leqslant \frac{1}{1 - \delta} \qquad \text{for all } x \in \Omega\,,\ z \in \partial\Omega\,.$$

So (3.12) follows from (3.11). //

4. - POTENTIAL THEORY FOR L

THEOREM 4.1 (Boundary Harnack principle): Let $z_0 \in \partial\Omega$, $r > 0$, $x_r \in \Omega$ such that $|x_r - z_0| = r$ and $\operatorname{dist}(x_r, \partial\Omega) \simeq r$. If v is a positive solution of

$Lv = 0$ in Ω, vanishing continuously on $\partial\Omega \cap B(z_0, 2r)$, then

$$\sup_{B(z_0,r)} v \leqslant c \cdot v(x_r)$$

for some constant c depending on n, λ, δ and the Lipschitz character of Ω.

PROOF: Let $\Omega' = \Omega \cap B(z_0, 2r)$. Then $v \in C(\bar{\Omega}')$. Let us call w_L^x, w_A^x the harmonic measures for Ω'. Then, by the boundary Harnack principle for A (see [2]) and (3.12), one has, for $x \in B(z_0, r)$:

$$v(x) = \int_{\partial\Omega'} v(z)\, dw_L^x(z) \leqslant c_2 \int_{\partial\Omega'} v(z)\, dw_A^x(z) \leqslant \text{const} \int_{\partial\Omega'} v(z)\, dw_A^{x_r}(z) \leqslant$$

$$\leqslant \text{const} \int_{\partial\Omega'} v(z)\, dw_L^{x_r}(z) = c \cdot v(x_r)\,. \qquad //$$

In the same way the following can be obtained:

THEOREM 4.2 (Comparison principle): Let u, v be positive solutions of $Lw = 0$ in Ω, vanishing continuously on $\partial\Omega \cap B(z, 2r)$. Then

$$\sup_{B(z,r)} \frac{u}{v} \leqslant c \cdot \frac{u}{v}(x_r)\,.$$

THEOREM 4.3 (Comparison between solutions of L and A): Let u, v be positive solutions of $Lv = 0$, $Au = 0$ in Ω, vanishing continuously on $\partial\Omega \cap$ $\cap B(z, 2r)$. Then

$$\sup_{B(z,r)} \frac{u}{v} \leqslant c \cdot \frac{u}{v}(x_r)\,.$$

Once one knows these results, one can repeat the arguments contained in section 3 of [2]: the kernel function for the operator L can be defined, and its existence and uniqueness can be proved. Moreover, from the formula

$$K_L^x(w, z) = \frac{dw_L^w}{dw_L^x}(z)$$

it follows, by (3.11), that

$$K_L^x(w, z) = \frac{F(w, z)\, dw_A^w(z)}{F(x, z)\, dw_A^x(z)} = \frac{F(w, z)\, K_A^x(w, z)}{F(x, z)}\,.$$

Hence we have that $K_L^x(w, \cdot) \in C(\partial\Omega)$ and:

$$c_1 K_A^x(w, z) \leqslant K_L^x(w, z) \leqslant c_2 K_A^x(w, z)$$

for all $x, w \in \Omega$, $z \in \partial\Omega$, for some constants c_1, c_2 depending on δ.

Now we are interested in stating regularity of boundary points for L. From this fact and the previous theorems, by the same arguments of [2], some results about boundary behavior of nonnegative solutions will follow. First, we want to point out a consequence of comparison theorem.

THEOREM 4.4: Let u, v be positive solutions of $Lu = 0$ in Ω, vanishing continuously on $\partial\Omega \cap B(z_0, 2r_0)$ (for a fixed $z_0 \in \partial\Omega$, $r_0 > 0$). Then the quotient u/v can be extended as a Holder continuous function on $\bar{\Omega} \cap B(z_0, 2r_0)$. (Note that, by [3], the solutions u, v are in general continuous but not Holder).

PROOF: For every $r > 0$, put $M_r = \sup\limits_{B_r} u/v$, $m_r = \inf\limits_{B_r} u/v$. Since v and $M_{2r} \cdot v - u$ are positive solutions in B_{2r}, by comparison theorem we have:

$$\sup_{B_r}\left(M_{2r} - \frac{u}{v}\right) \leqslant c \cdot \inf_{B_r}\left(M_{2r} - \frac{u}{v}\right)$$

that is

$$M_{2r} - m_r \leqslant c \cdot (M_{2r} - M_r) . \tag{4.1}$$

Considering now $u - m_{2r} v$, with the same reasoning one obtains

$$M_r - m_{2r} \leqslant c \cdot (m_r - m_{2r}) \tag{4.2}$$

and from (4.1), (4.2), with a standard technique (see [7]) Holder continutiy of u/v follows. //

The following lemma is taken from [4].

LEMMA 4.5: For any $z \in \bar{\Omega}$, $w \in \Omega$, $w \neq z$

$$\lim_{\substack{x,y \to z \\ x,y \in \Omega}} \frac{G(x, w)\, G(w, y)}{G(x, y)} = 0 . \tag{4.3}$$

Also, if $z, z' \in \partial\Omega$, then

$$\lim_{x \to z} \frac{G(x, w)\, G(w, y)}{G(x, y)} = K(z, w, z') \tag{4.4}$$

exists and is a continuous function of (z, z') on $\partial\Omega \times \partial\Omega$.

Let us now consider $F(x, y) = G_L(x, y)/G(x, y)$. We have already proved that F can be extended continuously on $\Omega \times \bar{\Omega}$. We now state the following:

LEMMA 4.6: F can be extended continuously on $\bar{\Omega} \times \bar{\Omega}$. For $z, z' \in \partial\Omega$, it is

$$F(z, z') = 1 - \int_{\Omega} F(w, z')\, K(z, w, z')\, V(w)\, dw . \tag{4.5}$$

(Note that, since $K(z, w, z) = 0$, $F(z, z) = 1$.)

PROOF: By (3.9):

$$\lim_{x\to z'} F(x,z) = 1 - \lim_{x\to z'} \int_\Omega \lim_{y\to z} \frac{G(w,y)}{G(x,y)} \cdot G_L(x,w) V(w)\, dw\,.$$

Note that

$$\lim_{\substack{y\to z\\ x\to z'}} \frac{G(w,y)}{G(x,y)} \cdot G_L(x,w) = \lim_{\substack{y\to z\\ x\to z'}} \frac{G_L(x,w)}{G(x,w)} \cdot \frac{G(x,w)\,G(w,y)}{G(x,y)} =$$
$$= F(w,z')\,K(z,w,z') \qquad \text{by (3.8) and (4.4)}\,.$$

Also, by (1.3), we have that $\int_\Omega F(w,z')\,K(z,w,z')\,V(w)\,dw$ converges. Put:

$$f_r(x) = \int_{|w-x|>r} K_A^x(w,z)\, G_L(x,w)\, V(w)\, dw\,.$$

By Lebesgue's theorem, (1.2), (1.3) we see that

$$\lim_{x\to z'} f_r(x) = \int_{|w-z'|>r} F(w,z')\,K(z,w,z')\,V(w)\,dw$$

and the right hand side is a continuous function of z', uniformly converging, when $r\to 0$, to

$$\int_\Omega F(w,z')\,K(z,w,z')\,V(w)\,dw\,.$$

So this is a continuous function. Furthermore, when $r\to 0$:

$$f_r(x) \to \int_\Omega K_A^x(w,z)\, G_L(x,w)\, V(w)\, dw$$

uniformly in x (again by (1.2), (1.3) and definition of Kato class). So (4.5) holds. //

THEOREM 4.7 (Regularity of boundary points): For every $f\in C(\partial\Omega)$, $z_0\in\partial\Omega$, when $x\to z_0$ $(x\in\Omega)$

$$\int_{\partial\Omega} f(z)\, dw_L^x(z) \to f(z_0)\,.$$

PROOF: Let $\{x_n\}$ be a sequence in Ω converging to z_0, $g_n(z) = f(z)\cdot$ $\cdot F(x_n, z)$. Then $g_n\in C(\partial\Omega)$ and $\|g_n\|_\infty \leqslant c\cdot\|f\|_\infty$. By (3.11):

$$\int_{\partial\Omega} f(z)\, dw_L^{x_n}(z) = \int_{\partial\Omega} g_n(z)\, dw_A^{x_n}(z) =$$
$$= \int_{\partial\Omega} [g_n - f\cdot F(z_0,\cdot)]\,(z)\, dw_A^{x_n}(z) + \int_{\partial\Omega} f(z)\,F(z_0,z)\, dw_A^{x_n}(z)\,.$$

The first term tends to zero by uniform continuity of $F(\cdot,\cdot)$ on $\partial\Omega\times\partial\Omega$, while the second term, by regularity of boundary points for $\mathcal{A}$ (see [6]) converges to $f(z_0)F(z_0,z_0)=f(z_0)$ (by (4.5)). So we are done. //

From the facts we have stated up to this point, the arguments contained in section 4 of [2] can be repeated for the operator L. Let Σ be the unit ball in $\boldsymbol{R}^n$, $K_L(\cdot,\cdot)$ the kernel function for L in Σ evaluated at the origin. Then the following hold:

THEOREM 4.8: Let u be a nonnegative solution of $Lu=0$ in Σ. Then there exists a finite Borel measure ν on $\partial\Sigma$ such that:

$$u(x)=\int_{\partial\Sigma} K_L(x,z)\,d\nu(z)\,. \tag{4.6}$$

THEOREM 4.9 (Existence of nontangential limits): Let u be a nonnegative solution of $Lu=0$ in Σ. Then almost everywhere on $\partial\Sigma$ with respect to the L-harmonic measure $w_L\equiv w_L^0$, the nontangential limit of u exists.

If ν is as in (4.6), let us consider the Lebesgue decomposition of ν with respect to w_L:

$$d\nu=d\nu_s+f\,dw_L\,.$$

Then the limit is given by f. Moreover, if f is bounded, the following representation holds:

$$u(x)=\int_{\partial\Sigma} K_L(x,z)f(z)\,dw_L(z)=\int_{\partial\Sigma} f(z)\,dw_L^x(z)\,.$$

As in [2], Theorems 4.8-4.9 still hold when Σ is a bounded Lipschitz starshaped domain.

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