Schauder estimates for parabolic and elliptic nondivergence operators of Hörmander type

joint work with Luca Brandolini (Università di Bergamo)

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Introduction

Let $X_1, X_2, \ldots, X_q$ be a system of smooth real vector fields satisfying Hörmander's rank condition in a bounded domain $\Omega \subset \mathbb{R}^n$.

In this setting, "sum of squares" operators (of stationary or evolutionary type)

$$\sum_{i=1}^{q} X_i^2 \quad \text{or} \quad \partial_t - \sum_{i=1}^{q} X_i^2$$

have been widely studied since Hörmander's famous paper (1967, Acta Math.): these operators are hypoelliptic, and share with elliptic and parabolic operators several deep analogies.

In recent years, **nondivergence operators** modeled on the above classes, namely

$$L = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j \quad \text{or} \quad H = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x) X_i X_j$$

have also been studied, assuming that $A = \{a_{ij}\}_{i,j=1}^{q}$ is a symmetric, uniformly positive definite matrix of real functions defined in Ω (stationary case) or in a bounded domain $U \subset \mathbb{R} \times \Omega$ (evolutionary case), and $\lambda > 0$ is a constant such that:

$$\lambda^{-1} |\xi|^2 \leqslant \sum_{i,j=1}^{q} a_{ij}\xi_i\xi_j \leqslant \lambda |\xi|^2 \;\; \forall \xi \in \mathbb{R}^q,$$

uniformly in Ω or U.

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- Motivations to study these operators:
  - geometry in several complex variables (see Montanari-Lanconelli 2004, J. Diff. Eq. and references therein)
  - models of human vision (see Citti-Sarti to appear on J. Math. imaging and vision)

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- If the coefficients a_{ij} are not C^∞, the operator L or H is **no longer hypoelliptic**, and no result can be drawn on it from the classical theory of Hörmander's sums of squares. Nevertheless:
- classical results about elliptic and parabolic operators, which do not require, in principle, high regularity of the coefficients, when properly reformulated in the language of vector fields, look like reasonable -although nontrivial- conjectures.

Two typical instances of this situation are (local) L^p estimates and C^α estimates on the "second order" derivatives $X_i X_j u$.

- In B.-Brandolini (2000, Rend. Torino; 2000, Trans. A.M.S.) we have proved L^p estimates of this kind for stationary operators of type (L) or some more general classes, assuming the coefficients a_{ij} in the space VMO, extending the classical results of Rothschild-Stein (1976, Acta Math.) for Hörmander' sum of squares.

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- In this paper, we prove local C^α estimates of Schauder type for nonvariational parabolic operators of Hörmander's type. To state our main result, we need some more formal definition...

Parabolic Carnot-Carathéodory distance and Hölder spaces

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- For $x, y \in \Omega_0$, let:

$d(x, y) =$

$= \inf\{T(\gamma) \,|\, \gamma : [0, T(\gamma)] \to \mathbb{R}^n \ X\text{-subunit}, \ \gamma(0) = x, \gamma(T(\gamma)) = y\}$

where we call X-subunit any absolutely continuous path γ such that

$$\gamma'(t) = \sum_{j=1}^{m} \lambda_j(t) X_j(\gamma(t))$$

a.e. with $\sum_{j=1}^{m} \lambda_j(t)^2 \leqslant 1$ a.e. For $x \in \Omega$, we set

$$B_r(x) = \{y \in \Omega_o : d(x, y) < r\}.$$

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- A well known result by Nagel-Stein-Weinger (1985, Acta Math.) states that d **is a distance** and **the Lebesgue measure is locally doubling** w.r.t. d:

  $$|B_{2r}(x)| \leqslant c\,|B_r(x)| \, \forall x \in \Omega, r \leqslant r_0.$$

Let us now consider the **parabolic Carnot-Carathéodory distance** d_P corresponding to d, namely

$$d_P\left(\left(t,x\right),\left(s,y\right)\right)=\sqrt{d\left(x,y\right)^2+\left|t-s\right|},$$

defined in the cylinder $\mathbb{R}\times\Omega$. An analogous locally doubling condition holds.

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- We can now define *parabolic Hölder spaces* adapted to this context.
- For any bounded domain $U\subset\mathbb{R}\times\Omega\subset\mathbb{R}^{n+1}$ and any $\alpha>0$, let:

$$\left|u\right|_{C^\alpha(U)}=\sup\left\{\frac{\left|u\left(t,x\right)-u\left(s,y\right)\right|}{d_P\left(\left(t,x\right),\left(s,y\right)\right)^\alpha}:\left(t,x\right),\left(s,y\right)\in U,\left(t,x\right)\neq\left(s,y\right)\right\}$$

$$\left\|u\right\|_{C^\alpha(U)}=\left|u\right|_{C^\alpha(U)}+\left\|u\right\|_{L^\infty(U)}$$

$$C^\alpha\left(U\right)=\left\{u:U\to\mathbb{R}:\left\|u\right\|_{C^\alpha(U)}<\infty\right\}.$$

- For any positive integer k, let

$$C^{k,\alpha}(U) = \left\{ u : U \to \mathbb{R} : \|u\|_{C^{k,\alpha}(U)} < \infty \right\}$$

with

$$\|u\|_{C^{k,\alpha}(U)} = \sum_{|I|+2h\leqslant k} \left\| \partial_t^h X^I u \right\|_{C^{\alpha}(U)}$$

where, for any multiindex $I = (i_1, i_2, ..., i_s)$, with $1 \leqslant i_j \leqslant q$, we say that $|I| = s$ and

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- A known property of CC-distance states that

$$c_1|x-y|\leqslant d(x,y)\leqslant c_2|x-y|^{1/s}\ \ \forall x,y\in\Omega,$$

(where s is the step appearing in Hörmander's condition). Hence:

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  - a function $u \in C^\alpha(U)$ is also continuous on U in Euclidean sense.
  - all the derivatives $\partial_t^h X^I u$ involved in the definition of $C^{k,\alpha}$ are continuous in Euclidean sense

Theorem (Main result)

Under the above assumptions on $X_1, X_2, \ldots, X_q$ *and* $A = \{a_{ij}(t,x)\}_{i,j=1}^{q}$ *,let*

$$H = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x) X_i X_j - \sum_{i=1}^{q} b_i(t,x) X_i - c(t,x)$$

with $a_{ij}, b_i, c \in C^{k,\alpha}(U)$ *for some integer* $k \geqslant 0$ *and* $\alpha \in (0,1)$ *. Then,* $\forall U' \Subset U\ \exists c > 0$ *depending on* $U, U', \{X_i\}, \alpha, k, \lambda$ *and the* $C^{k,\alpha}$ *norms of the coefficients such that* $\forall u \in C^{k+2,\alpha}_{loc}(U)$ *with* $Hu \in C^{k,\alpha}(U)$ *one has*

$$\|u\|_{C^{k+2,\alpha}(U')} \leqslant c \left\{ \|Hu\|_{C^{k,\alpha}(U)} + \|u\|_{L^\infty(U)} \right\}.$$

Analogous Schauder estimates for stationary operators (L) obviously follow from the above theorem, as a particular case.

Comparison with the existing literature.

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  The main feature of the present paper, besides the "evolutionary" case it covers, is that our theory applies to *any* system of Hörmander vector fields.

Plan of the talk

In our proof, *a basic role is played by C^{α} continuity of singular and fractional integrals on spaces of homogeneous type*, that we prove here, coupled with the machinery introduced by Rothschild-Stein (1976, Acta Math.), that we have adapted to nondivergence operators in B.-Brandolini (2000, T.A.M.S.; 2005, Rev. Mat. Iberoam.).

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- A further regularization result;
- Application to regularity of the fundamental solution.

Background: the homogeneous case

- A **homogeneous group** $\mathbb{G}$ is $\mathbb{R}^N$ with a Lie group structure (*translations*), and a one parameter family of automorphisms (*dilations*).
  We write: $u \circ v$ for translation, u^{-1} for the inverse; 0 is the identity.
  Dilations: for suitable exponents $0 < a_1 \leq a_2 \leq \ldots \leq a_N$ we have

  $$D(\lambda) : (u_1, \ldots, u_N) \mapsto (\lambda^{a_1} u_1, \ldots, \lambda^{a_N} u_N) \ \ \forall \lambda > 0$$

  The number

  $$Q = \sum_{i=1}^{n} \alpha_i$$

  is called the *homogeneous dimension* of $\mathbb{G}$.

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  The number
  $$Q = \sum_{i=1}^{n} \alpha_i$$
  is called the *homogeneous dimension* of $\mathbb{G}$.
- **Homogeneous functions and differential operators.** We say that: a *function* f is homogeneous of degree α if
  $$f\ ((D(\lambda)u) = \lambda^{\alpha} f\ (u)\ \forall \lambda > 0, u \in \mathbb{R}^N;$$
  a *differential operator* Y is homogeneous of degree α if
  $$Y[f((D(\lambda)u)] = \lambda^{\alpha} (Yf)(D(\lambda)u)\ \forall \lambda > 0, u \in \mathbb{R}^N, \text{ test function } f.$$

- **Sublaplacian.** Assume $Y_1, Y_2, ..., Y_q$ is a system of Hörmander's vector fields in $\mathbb{R}^N$ such that the Y_i are left invariant and homogeneous of degree 1. Then the sublaplacian:

$$L = \sum_{i=1}^{q} Y_i^2$$

is a left invariant, homogeneous of degree 2, hypoelliptic operator on $\mathbb{G}$.

A fundamental result by Folland (1975, Arkiv för Mat.) states that:

Theorem (Existence of a homogeneous fundamental solution)

Let $\mathcal{L}$ be a left invariant differential operator homogeneous of degree two on a homogeneous group $\mathbb{G}$, such that $\mathcal{L}$ and $\mathcal{L}^T$ are both hypoelliptic. Moreover, assume $Q \geq 3$. Then there is a unique fundamental solution Γ such that:

(a) $\Gamma \in C^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$;
(b) Γ *is homogeneous of degree* $(2-Q)$;
(c) *for every distribution* τ,

$$\mathcal{L}(\tau * \Gamma) = (\mathcal{L}\tau) * \Gamma = \tau.$$

The previous representation formula also implies that, for any test function u:

$$Y_i Y_j u = PV\left((\mathcal{L}u) * Y_i Y_j \Gamma\right) + c_{ij}\mathcal{L}u$$

where c_{ij} are constants.

Theorem (continued)

Moreover, the kernel

$$k = Y_i Y_j \Gamma$$

satisfies the standard assumption of a Calderón-Zygmund-type singular integral kernel:

(a) $Y_i Y_j \Gamma \in C^\infty \left(\mathbb{R}^N \setminus \{0\} \right)$;

(b) $Y_i Y_j \Gamma$ *is homogeneous of degree* $-Q$;

(c)

$$\int_{r_1 < \|u\| < r_2} Y_i Y_j \Gamma (u) \, du = 0 \ \forall r_1 < r_2.$$

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  - results by Koranyi-Vagi (1972) also imply continuity on Hölder spaces C^α.

- “**Parabolic” version** of Folland’s results. Let us consider

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where the Y_i's are, as above, homogeneous left invariant Hörmander's vector fields on $\mathbb{G}$.

- In $\mathbb{R}^{N+1}$, let us define the following structure of homogeneous group $\mathbb{G}'$: for $(t, \xi), (s, \eta) \in \mathbb{R} \times \mathbb{G}$, we set:

$$(t, \xi) \circ (s, \eta) = (t + s, \xi \circ \eta),$$

$$D(\lambda)(t, \xi) = \left(\lambda^2 t, D(\lambda)\xi\right).$$

Then $\mathbb{G}'$ is a homogeneous group with homogeneous dimension $Q' = Q + 2$ (where Q is the homogeneous dimension of $\mathbb{G}$).

- "**Parabolic" version** of Folland's results. Let us consider

$$H = \partial_t - \sum_{i=1}^{q} Y_i^2$$

where the Y_i's are, as above, homogeneous left invariant Hörmander's vector fields on $\mathbb{G}$.

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Then $\mathbb{G}'$ is a homogeneous group with homogeneous dimension $Q' = Q + 2$ (where Q is the homogeneous dimension of $\mathbb{G}$).

- Therefore H has a fundamental solution h, homogeneous of degree $2 - Q' = -Q$.

Background: the non-homogeneous case and the "lifting and approximation" technique

Let us consider now a **generic system** of Hörmander's vector fields $X_1, X_2, ..., X_q$. Rothschild-Stein (1976, Acta Math.) have found a way to exploit the Folland's theory (in homogeneous groups) to study the more general operators

$$L = \sum_{i=1}^{q} X_i^2 \text{ or } H = \partial_t - L.$$

Theorem ("Lifting and approximation")

Assume $X_1, \ldots, X_q$ satisfy Hörmander's condition of step s at some point $x_0 \in \Omega \subset \mathbb{R}^n$,

$$X_i = \sum_{j=1}^{n} b_{ij}(x) \partial_{x_j}.$$

Let $\mathcal{G}(s, q)$ be the free Lie algebra of step s on q generators, $\mathbb{G}$ the corresponding homogeneous group on $\mathbb{R}^N$ ($N = \dim \mathcal{G}(s, q) > n$), Y_i ($i = 1, 2, ..., q$) the canonical generators of the Lie algebra.

Theorem (continued)

Then, the X_i (defined in $\mathbb{R}^n$) can be "lifted" to new vector fields $\widetilde{X}_i$ defined (locally) in $\mathbb{R}^N$:

$$\widetilde{X}_i = X_i + \sum_{j=1}^{N-n} c_{ij}(x,h)\,\partial_{h_j}, \qquad \xi = (x,h) \in \mathbb{R}^N, h \in \mathbb{R}^{N-n}$$

such that the $\widetilde{X}_i$'s satisfy Hörmander's condition of step s, and can be locally approximated by the (left inv. homog. vector fields) Y_i's, that is: there exists a smoothly varying family of diffeomorphisms Θ_ξ, from a neighborhood of ξ to a neighborhood of $0 \in \mathbb{G}$, vector fields R_i^ξ ("remainders") defined in a neighborhood of ξ, s.t.

$$\widetilde{X}_i\,\left(f\left(\Theta_\xi(\cdot)\right)\right)(\eta) = \left(Y_i f + R_i^\xi f\right)\left(\Theta_\xi(\eta)\right) \quad \forall f \in C_0^\infty(\mathbb{G})$$

where, if $f(u)$ is homogeneous of negative degree $-\lambda$, then $R_i^\xi f$ can be written as a sum of homogeneous functions of degrees $\geq -\lambda$, plus a smooth function (the singularity "does not become worse").

Local Schauder estimates: strategy of the proof

We will focus our attention on our main result in its *basic case*: no lower order terms, no higher order derivatives. The general case then follows from this result by a tedious repetition of the same general ideas.

Theorem (Main result in the basic case)

Under the above assumptions on $X_1, X_2, \ldots, X_q$ and the matrix $\{a_{ij}(t,x)\}_{i,j=1}^{q}$ let

$$H = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x) X_i X_j .$$

Then, for every domain $U' \Subset U$ there exists $c > 0$ depending on $U, U', \{X_i\}, \alpha, \lambda$ and $\|a_{ij}\|_{C^\alpha}$ such that for every $u \in C^{2,\alpha}_{loc}(U)$ with $Hu \in C^\alpha(U)$ one has

$$\|u\|_{C^{2,\alpha}(U')} \leqslant c \left\{ \|Hu\|_{C^\alpha(U)} + \|u\|_{L^\infty(U)} \right\}.$$

We now show how the proof is organized.

- First of all, by Rothschild-Stein "lifting Theorem", we lift the vector fields

$$X_i(x) \text{ defined in } \mathbb{R}^n$$

to new vector fields

$$\widetilde{X}_i(\xi) \text{ defined in } \mathbb{R}^N, \text{with } \xi = (x, h), h \in \mathbb{R}^{N-n}.$$

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- We also set $\widetilde{a}_{ij}\left(t, \xi\right) = \widetilde{a}_{ij}\left(t, x, h\right) = a_{ij}\left(t, x\right)$,and

$$\widetilde{H} = \partial_t - \sum_{i,j=1}^{q} \widetilde{a}_{ij}\left(t, \xi\right) \widetilde{X}_i \widetilde{X}_j.$$

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$$\widetilde{H} = \partial_t - \sum_{i,j=1}^{q} \widetilde{a}_{ij}(t, \xi)\widetilde{X}_i\widetilde{X}_j.$$

- The proof of our main Theorem then proceeds in three steps:

Step 1:

Theorem (Schauder estimates in the lifted space for functions with small support)

There exist $r, c > 0$ *such that* $\forall u \in C_0^{2,\alpha}\left(\widetilde{B}_r\left(t_0, \xi_0\right)\right)$,

$$\|u\|_{C^{2,\alpha}\left(\widetilde{B}_r\right)} \leqslant c\left\{\left\|\widetilde{H}u\right\|_{C^{\alpha}\left(\widetilde{B}_r\right)} + \|u\|_{L^{\infty}\left(\widetilde{B}_r\right)}\right\}$$

where c, r *depend on* $\{X_i\}, \alpha, \lambda$ *and* $\|a_{ij}\|_{C^{\alpha}(U)}$.

Step 2:

Theorem (Schauder estimates in the lifted space for any function)

There exist $r, c, \beta > 0$ *such that* $\forall u \in C^{2,\alpha}\left(\widetilde{B}_r(t_0, \xi_0)\right)$, $0 < t < s < r$,

$$\|u\|_{C^{2,\alpha}(\widetilde{B}_t)} \leqslant \frac{c}{(s-t)^\beta}\left\{\left\|\widetilde{H}u\right\|_{C^\alpha(\widetilde{B}_s)} + \|u\|_{L^\infty(\widetilde{B}_s)}\right\}$$

where c, r *depend on* $\{X_i\}, \alpha, \lambda$ *and* $\|a_{ij}\|_{C^\alpha(U)}$, β *depends on* $\{X_i\}, \alpha$.

Step 3:

Theorem (Schauder estimates in the original space)

There exist $r, c, \beta > 0$ *such that* $\forall u \in C^{2,\alpha}(B_r(t_0, x_0))$, $0 < t < s < r$,

$$\|u\|_{C^{2,\alpha}(B_t)} \leqslant \frac{c}{(s-t)^\beta} \left\{ \|Hu\|_{C^\alpha(B_s)} + \|u\|_{L^\infty(B_s)} \right\}$$

where c, r *depend on* $\{X_i\}, \alpha, \lambda$ *and* $\|a_{ij}\|_{C^\alpha(U)}$, β *depends on* $\{X_i\}, \alpha$.

By an easy covering argument, this will imply (the basic case of) our main result.

Step 1. Local Schauder estimates for test functions with small support

- Let us start again with the *lifted operator*:

$$\widetilde{H} = \partial_t - \sum_{i,j=1}^{q} \widetilde{a}_{ij}\left(t, \xi\right) \widetilde{X}_i \widetilde{X}_j .$$

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- We now freeze the coefficients $\widetilde{a}_{ij}$ (but not the vector fields $\widetilde{X}_i$!) at some point $\left(t_0,\xi_0\right) \in \widetilde{U}$, and consider the *frozen lifted operator*:

$$\widetilde{H}_0 = \partial_t - \sum_{i,j=1}^{q} \widetilde{a}_{ij}(t_0,\xi_0)\widetilde{X}_i\widetilde{X}_j.$$

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- To study $\widetilde{H}_0$, we will consider its *approximating operator*, defined on $\mathbb{G}' = \mathbb{R} \times \mathbb{G}$:

$$\mathcal{H}_0 = \partial_t - \sum_{i,j=1}^{q} \widetilde{a}_{ij}(t_0,\xi_0)Y_iY_j.$$

- Since $\{\widetilde{a}_{ij}(t_0, \xi_0)\}$ is a constant positive definite matrix, the operator $\sum_{i,j=1}^{q} \widetilde{a}_{ij}(t_0, \xi_0) Y_i Y_j$ can be rewritten as a "sum of squares" operator, so $\mathcal{H}_0$ is hypoelliptic by Hörmander's theorem. Moreover, $\mathcal{H}_0$ is left invariant and homogenous of degree 2 in $\mathbb{G}' = \mathbb{R} \times \mathbb{G}$, hence by Folland's results it has a fundamental solution, denoted by

$$h(t_0, \xi_0, t, u) \equiv h(t, u)$$

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- Next, we use this fundamental solution $h(t, u)$ to build a parametrix for the lifted operator $\widetilde{H}$:

$$h\left(t, \Theta_{\xi}(\eta)\right).$$

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- Applying a technique introduced by Rothschild-Stein (1976, Acta Math.) and adapted to nondivergence operators by B.-Brandolini (2000, T.A.M.S.) we prove a representation formula for second order derivatives $\widetilde{X}_r\widetilde{X}_s$ of a test function f, in terms of singular integrals applied to $f,\widetilde{X}_kf$ and $\widetilde{H}f$.

Theorem

The following **representation formula for second derivatives of any test function** f*, in terms of* $\widetilde{H}f$*, holds:*

$$\widetilde{X}_r\widetilde{X}_s\,(af)\,(t,\xi) = T\widetilde{H}f(t,\xi) +$$

$$+\,T\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(t_0,\xi_0)-\widetilde{a}_{ij}(t,\xi)\right]\widetilde{X}_i\widetilde{X}_j f\,(t,\xi)\,+$$

$$+\sum_{i,j=1}^{q}\widetilde{a}_{ij}(t_0,\xi_0)\left\{\sum_{k=1}^{q}T_{ij}^{k}\widetilde{X}_k f\,(t,\xi)+T_{ij}f\,(t,\xi)\right\}.$$

where T, T_{ij}, T_{ij}^{k} *are* **frozen singular integrals**, *and* a *is a cutoff function.*

Roughly speaking, a frozen singular integral is a singular integral with a kernel of the kind:

$$\widetilde{X}_i\widetilde{X}_j\left[h\left(t,\Theta_{\xi}\left(\cdot\right)\right)\right](\eta)$$

We can prove the following:

Theorem (C^α continuity of frozen singular integrals)

If T is a frozen singular integral and $\widetilde{B}_r$ a $\widetilde{d}_P$-ball in $\mathbb{R}^{N+1}$, then T is continuous on $C^\alpha\left(\widetilde{B}_r\right)$:

$$\|Tf\|_{C^\alpha\left(\widetilde{B}_r\right)} \leqslant c\,\|f\|_{C^\alpha\left(\widetilde{B}_r\right)}.$$

- Now, taking C^α norms of both sides of our representation formula, applying the above continuity theorem, plus standard properties of Hölder norms (classical "Korn's trick" which is used in Schauder theory) we get, for r small enough :

$$\left\|\widetilde{X}_k\widetilde{X}_h f\right\|_{C^\alpha\left(\widetilde{B}_r\right)} \leqslant c\left\{\left\|\widetilde{H}f\right\|_{C^\alpha\left(\widetilde{B}_r\right)} + \sum_{l=1}^{q}\left\|\widetilde{X}_l f\right\|_{C^\alpha\left(\widetilde{B}_r\right)} + \|f\|_{C^\alpha\left(\widetilde{B}_r\right)}\right\}$$

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- and finally, with some more work,

$$\|f\|_{C^{2,\alpha}\left(\widetilde{B}_r\right)} \leqslant c\left\{\left\|\widetilde{H}f\right\|_{C^{\alpha}\left(\widetilde{B}_r\right)} + \|f\|_{L^{\infty}\left(\widetilde{B}_r\right)}\right\}$$

Frozen singular integrals

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- To prove that our "frozen singular integrals" map $C^{\alpha}\left(\widetilde{B}_r\right)$ in itself:
- we prove abstract theorems of C^{α}-continuity of singular and fractional integrals on "spaces of homogeneous type", in the sense of Coifman-Weiss (1971, L.N.M. n.242); our assumptions are standard growth conditions and suitable cancellation properties of the kernel:

$$\left|\int_{d(x,y)>r} k(x,y)\,dy\right| \leqslant c$$

$\forall r>0$ (with c independent of r) and

$$\lim_{\varepsilon\to 0}\left|\int_{d(x,y)>\varepsilon} k(x,y)\,dy-\int_{d(x_0,y)>\varepsilon} k(x_0,y)\,dy\right| \leqslant c\,d(x,x_0)^{\gamma}$$

for some $\gamma\in(0,1]$.

To check that our "frozen singular kernels" $\widetilde{X}_i \widetilde{X}_j \left[h \left(t, \Theta_{\xi} \left(\cdot \right) \right) \right] \left(\eta \right)$ satisfy the assumptions of our abstract Theorem we use:

- the uniform Gaussian estimates on $h\left(s, u\right)$ and its derivatives, proved by Bonfiglioli-Lanconelli-Uguzzoni (2002, Adv. Diff. Eqts.): the point is that these bounds depend on $a_{ij}\left(t_0, \xi_0\right)$ only through the ellipticity constant λ:

$$\left| \partial_t^k Y_{i_1} \ldots Y_{i_r} h\left(t, u\right) \right| \leq \frac{c}{t^{Q/2+k+r/2}} e^{-c\|u\|^2/t}$$

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- a "mean value inequality"

$$|f(t, \xi) - f(s, \eta)| \leqslant \left(\sup \left| \widetilde{X} f \right| + R \sup |\partial_t f| \right) \widetilde{d}_P\left((t, \xi), (s, \eta)\right)$$

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  $\forall f \in C_0^1 \left(\widetilde{B}_R \left(t_0, \xi_0 \right) \right)$;

- a suitable splitting of the singular integral kernel in a part which has *vanishing integral on spherical shells* (like in the homogeneous case), plus other parts which are less singular, and therefore satisfy the required cancellation properties.

Step 2. Interpolation inequalities for Hölder norms and local Schauder estimates in the lifted variables

For functions *not necessarily vanishing at the boundary* we have to prove:

Theorem

There exist positive constants r, c, β *such that* $\forall u \in C^{2,\alpha}\left(\widetilde{B}_r\left(t_0, \xi_0\right)\right)$, $0 < t < s < r$,

$$\|u\|_{C^{2,\alpha}\left(\widetilde{B}_t\right)} \leqslant \frac{c}{(s-t)^{\beta}}\left\{\left\|\widetilde{H}u\right\|_{C^{\alpha}\left(\widetilde{B}_s\right)} + \|u\|_{L^{\infty}\left(\widetilde{B}_s\right)}\right\}$$

where c, r *depend on* $\{X_i\}, \alpha, \lambda$ *and* $\|a_{ij}\|_{C^{\alpha}(U)}$, β *depends on* $\{X_i\}, \alpha$.

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- Step 2 follows from Step 1 by standard properties of **cutoff functions** and suitable **interpolation inequalities** for Hölder norms.
- The construction of well-shaped cutoff functions is fairly standard, also in this context, and we will not give further details.
- The second tool is much more delicate, due to the *lack of dilations*:

Theorem (Interpolation inequality)

There exist positive constants c, R, γ *depending on* α, $\{X_i\}$ *such that* $\forall f \in C^{2,\alpha}(\widetilde{B}_R)$, $0 < \rho < R, 0 < \delta < 1/3$,

$$\|Df\|_{C^\alpha(\widetilde{B}_\rho)} \leqslant \delta \left[\|D^2 f\|_{C^\alpha(\widetilde{B}_R)} + \|\partial_t f\|_{C^\alpha(\widetilde{B}_R)} \right] + \frac{c}{\delta^\gamma (R-\rho)^{2\gamma}} \|f\|_{L^\infty(\widetilde{B}_R)}.$$

Step 3. Hölder spaces and lifting

We have now to show how we can transfer to the original space $\mathbb{R}^{n+1}$ the Schauder estimates we have proved in the lifted space $\mathbb{R}^{N+1}$.

- In other words, starting from the estimate

$$\|u\|_{C^{2,\alpha}\left(\widetilde{B}_t\right)} \leqslant \frac{c}{(s-t)^{\beta}} \left\{ \left\| \widetilde{H} u \right\|_{C^{\alpha}\left(\widetilde{B}_s\right)} + \|u\|_{L^{\infty}\left(\widetilde{B}_s\right)} \right\}$$

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$$\forall u \in C^{2,\alpha}\left(\widetilde{B}_r\left(t_0, \xi_0\right)\right), \, 0 < t < s < r,$$

- we have to prove the analogue:

$$\|u\|_{C^{2,\alpha}(B_t)} \leqslant \frac{c}{(s-t)^{\beta}} \left\{ \|Hu\|_{C^{\alpha}(B_s)} + \|u\|_{L^{\infty}(B_s)} \right\}$$

$$\forall u \in C^{2,\alpha}\left(B_r\left(t_0, x_0\right)\right), \, 0 < t < s < r.$$

This fact involves the delicate relation between the CC-distance $d(x, y)$ induced by a system $X_1, X_2, ..., X_q$ of Hörmander's vector fields in $\mathbb{R}^n$, and the CC-distance $\widetilde{d}(\xi, \eta)$ induced by the lifted vector fields $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_q$ in $\mathbb{R}^N$.

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- Let $d_P, \widetilde{d}_P$ be the corresponding parabolic distances.
- Denote by $C_X^\alpha(U), C_{\widetilde{X}}^\alpha\left(\widetilde{U}\right)$ the Hölder spaces induced by d_P and $\widetilde{d}_P$, respectively.

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- Denote by $C_X^{\alpha}(U), C_{\widetilde{X}}^{\alpha}\left(\widetilde{U}\right)$ the Hölder spaces induced by d_P and $\widetilde{d}_P$, respectively.
- We are interested in the following question: if, $\forall f : U \rightarrow \mathbb{R}$, we set

$$\widetilde{f} : \widetilde{U} \rightarrow \mathbb{R} \text{ with } \widetilde{f}(t, x, h) = f(t, x),$$

then, can we say that

$$f \in C_X^{\alpha}(U) \Longleftrightarrow \widetilde{f} \in C_{\widetilde{X}}^{\alpha}\left(\widetilde{U}\right) \ ?$$

- It is well-known that

$$\widetilde{d}\left(\left(x,h\right),\left(y,k\right)\right)\geqslant d\left(x,y\right);$$

this obviously implies

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- However, no simple inequality of the reverse kind

$$\widetilde{d}\left(\left(x,h\right),\left(y,h\right)\right)\leqslant cd\left(x,y\right)$$

seems to be known, so an inequality of the kind

$$\left|f\right|_{C^{\alpha}_{X}\left(U\right)}\leqslant c\left|\widetilde{f}\right|_{C^{\alpha}_{\widetilde{X}}\left(\widetilde{U}\right)}$$

is not trivial.

What is known is a *relation between the volume of d-balls and $\widetilde{d}$-balls*, proved by Sanchez-Calle (1984, Inv. Math.):
(the parabolic version we have written is a straightforward adaptation):

Lemma

Given a point $(t, x, h) \in \mathbb{R}^{N+1}$,

$$\left|\widetilde{B}_r(t, x, h)\right| \simeq |B_r(t, x)| \cdot \left|\left\{h' \in \mathbb{R}^{N-n} : (\tau, z, h') \in \widetilde{B}_r(t, x, h)\right\}\right|$$

provided $(\tau, z) \in B_{\delta r}(t, x)$ *for some fixed* $\delta < 1$. *The equivalence holds with respect to* $r > 0$, *and the symbol* $|\cdot|$ *denotes the volume of a set in the suitable dimension.*

To exploit this "integral relation" between d and $\widetilde{d}$, the point is to make use of an *integral formulation of Hölder continuity*, analogous to the classical integral characterization of C^{α}, by Campanato (1963, Ann. Pisa).

The above two ingredients (parabolic adaptation of Sanchez-Calle result + adaptation of Campanato's charachterization of C^{α}) enable us to prove the following:

Lemma

If $f, \widetilde{f}$ are as above, then

$$\left|\widetilde{f}\right|_{C^{\alpha}_{\widetilde{X}}\left(\widetilde{B}_R\right)} \leqslant |f|_{C^{\alpha}_{X}(B_R)} \leqslant c\left|\widetilde{f}\right|_{C^{\alpha}_{\widetilde{X}}\left(\widetilde{B}_R\right)}.$$

Moreover,

$$\left|\widetilde{X}_{i_1}\widetilde{X}_{i_2}...\widetilde{X}_{i_k}\widetilde{f}\right|_{C^{\alpha}_{\widetilde{X}}\left(\widetilde{B}_R\right)} \leqslant |X_{i_1}X_{i_2}...X_{i_k}f|_{C^{\alpha}_{X}(B_R)} \leqslant c\left|\widetilde{X}_{i_1}\widetilde{X}_{i_2}...\widetilde{X}_{i_k}\widetilde{f}\right|_{C^{\alpha}_{\widetilde{X}}\left(\widetilde{B}_R\right)}$$

for $i_j = 1, 2, ..., q$.

Combining "Step 2" with this Lemma, we immediately get "Step 3"

Regularization of solutions

Assume now we have proved our main result also for *higher order derivatives*. We would like to use our a-priori estimates to show that, whenever a function $u \in C^{2,\alpha}_{loc}(U)$ solves $Hu = f$ in U with $C^{k,\alpha}(U)$ coefficients and data, then actually $u \in C^{k+2,\alpha}_{loc}(U)$.
This result is contained in the following:

Theorem

Under the assumptions of Theorem "Main result", $\forall \alpha \in (0,1)$, if $u \in C^{2,\alpha}_{loc}(U)$ and $Hu \in C^{k,\alpha}(U)$ for some even integer k, then $u \in C^{2+k,\alpha}_{loc}(U)$. Moreover, $\forall U' \Subset U$, $\exists c > 0$ depending on $U, U', \{X_i\}, \alpha, k, \lambda$ and $\|a_{ij}\|_{C^{k,\alpha}(U)}$, $\|b_i\|_{C^{k,\alpha}(U)}$, $\|c\|_{C^{k,\alpha}(U)}$ such that

$$\|u\|_{C^{2+k,\alpha}(U')} \leqslant c \left\{ \|Hu\|_{C^{k,\alpha}(U)} + \|u\|_{L^\infty(U)} \right\}.$$

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3. apply some compactness argument to find a sequence u^{ε_n} converging to some $v \in C_{loc}^{k+2,\alpha}$ (provided the constants in the a-priori estimates and the $C^{k,\alpha}$-norm of $a_{ij}^{\varepsilon}, f^{\varepsilon}$ are bounded uniformly in ε);

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- *The point is to provide a suitable mollification technique, allowing to control $C^{k,\alpha}$-norms.*

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- We have done this exploiting the *heat kernel for the model operator*:

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- By known results of Kusuoka-Stroock (1987, J. Fac. Sci. Univ. Tokyo), there exists a fundamental solution $h(t,x,y)$ of $\mathbf{H}$ such that

$$\frac{1}{c\left|B\left(x,\sqrt{t}\right)\right|} e^{-\frac{cd(x,y)^2}{t}} \leqslant h(t,x,y) \leqslant \frac{c}{\left|B\left(x,\sqrt{t}\right)\right|} e^{-\frac{d(x,y)^2}{ct}}$$

$$\left|X_x^I X_y^J h(t,x,y)\right| \leqslant \frac{c}{t^{(|I|+|J|)/2}\left|B\left(x,\sqrt{t}\right)\right|} e^{-\frac{d(x,y)^2}{ct}}$$

$\forall t \in (0,1), x, y \in \mathbb{R}^n$, for every multi-indexes I and J. Moreover:

$$\int_{\mathbb{R}^n} h(t,x,y)\, dy = 1 \quad \forall t \in (0,+\infty), x \in \mathbb{R}^n.$$

We now use this "Gaussian kernel" to build a family of mollifiers adapted to the vector fields X_i.

Theorem (mollifiers)

Let $\eta \in C_0^\infty(\mathbb{R})$ be a positive test function with $\int \eta(t)\,dt = 1$ and let

$$\phi_\varepsilon(t,x,y) = \varepsilon^{-1} h(\varepsilon,x,y)\,\eta\left(\frac{t}{\varepsilon}\right).$$

$\forall f \in C^\alpha\left(\mathbb{R}^{n+1}\right)$, $\varepsilon \in (0,1)$, set

$$f_\varepsilon(t,x) = \int_{\mathbb{R}^{n+1}} \phi_\varepsilon(t-s,x,y)\, f(s,y)\,dsdy.$$

Then, there exists a constant c depending on α, $\{X_i\}$, such that

$$\|f_\varepsilon\|_{C^\alpha} \leqslant c\,\|f\|_{C^\alpha}.$$

Moreover,

$$\lim_{\varepsilon\to 0} \|f_\varepsilon - f\|_{L^\infty(\mathbb{R}^{n+1})} = 0.$$

To prove the above theorem, the idea is to apply again our abstract result on C^{α}-continuity of singular integrals on spaces of homogeneous type, showing that the kernels ϕ_{ε} satisfy the axioms of our theory with constants uniformly bounded with respect to ε. This is possible exploiting the properties of the heat kernel h that we have quoted.
We are also able to prove an analogous control on the $C^{k,\alpha}$-norm of the mollified function, but unfortunately *only for even* k:

Lemma

$\forall \alpha \in (0,1)$, k even integer, U, U' bounded open sets, with $U' \Subset U$, there exists a constant c such that $\forall f \in C^{k,\alpha}(U)$, $\varepsilon \in (0,1)$,

$$\|f_{\varepsilon}\|_{C^{k,\alpha}(U')} \leqslant c\,\|f\|_{C^{k,\alpha}(U)}.$$

With these tools, our regularization result can be proved following the scheme we have recalled.
The limitation "k even" in the control of $C^{k,\alpha}$-norms of mollified functions is the reason of the analogous limitation in our regularization result.

Application to the regularity of fundamental solutions

- In a recent paper in collaboration with Brandolini, Lanconelli, Uguzzoni (2006, an announcement will appear on C.R.A.S.) we prove that operators of the type we have considered so far,

$$H = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x) X_i X_j - \sum_{i=1}^{q} b_i(t,x) X_i - c(t,x)$$

with C^{α} coefficients, possess a fundamental solution $h(t,x;\tau,y)$, which satisfies sharp Gaussian estimates of Kusuoka-Stroock type.

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  with C^α coefficients, possess a fundamental solution $h(t,x;\tau,y)$, which satisfies sharp Gaussian estimates of Kusuoka-Stroock type.
- By construction, this fundamental solution has only a weak regularity (roughly speaking, $\partial_t h$ and $X_i X_j h$ just *exist*, without good continuity properties); however, in the above paper we prove that actually

$$h(\cdot;\tau,y) \in C^{2,\alpha}_{loc}(\mathbb{R}^{n+1} \setminus \{(\tau,y)\})$$

  with norm depending only on the coefficients only through their C^α norms and the ellipticity constant λ.