On the gradient of Schwarz symmetrization of functions in Sobolev spaces

Marco Bramanti

Dipartimento di Matematica, Politecnico di Milano. Via Bonardi 9. 20133 Milano. Italy.

Sunto. Sia S uno spazio di Sobolev o Orlicz-Sobolev di funzioni non necessariamente nulle al bordo del dominio. Si danno condizioni sufficienti su una funzione non negativa in S affinché la sua simmetrizzata di Schwarz appartenga ancora ad S. Questi risultati sono ottenuti per mezzo di disuguaglianze isoperimetriche relative e generalizzano in un certo senso un noto teorema di Polya-Szegö. Si dimostra anche che il riarrangiamento di una qualsiasi funzione in S è localmente in S.

Abstract. Let S be a Sobolev or Orlicz-Sobolev space of functions not necessarily vanishing at the boundary of the domain. We give sufficient conditions on a nonnegative function in S in order that its spherical rearrangement ("Schwartz symmetrization") still belongs to S. These results are obtained via relative isoperimetric inequalities and somewhat generalize a wellknown Polya-Szegö's theorem. We also prove that the rearrangement of any function in S is locally in S.

If u is a nonnegative function in $H^{1,2}(\Re^n)$, u has compact support, and \tilde{u} denotes the Schwarz symmetrization of u, then a well known theorem by Polya-Szegö states that \tilde{u} belongs to $H^{1,2}(\Re^n)$ and:

$$\int |D\widetilde{u}|^2 dx \leq \int |Du|^2 dx. \qquad (*)$$

(Henceforth, we will indicate with D the gradient of a function of n variables or the derivative of a function of one real variable).

In particular, this formula holds for $u \in H_0^{1,2}(\Omega)$, where Ω is a bounded domain of \Re^n , the first integral is taken on the ball $\tilde{\Omega}$ having the same measure of Ω and the second is taken on Ω .

If u is a function in $H^{1,2}(\Omega)$, not necessarily vanishing at the boundary, or if u belongs to $H_0^{1,2}(\Omega)$ but assumes also negative values (and so does \tilde{u}), then inequality (*) can actually fail, and \tilde{u} does not necessarily belong to $H^{1,2}(\Omega)$ (see examples below). So, a natural question is under which additional assumptions a nonnegative function in $H^{1,2}(\Omega) \setminus H_0^{1,2}(\Omega)$ has Schwarz symmetrization in $H^{1,2}(\tilde{\Omega})$. In section 1 we will prove some different sufficient conditions (in terms of the size of the set on which u vanishes) in order to a Polya-Szegö-type estimate holds, that is:

$$\int_{\widetilde{\Omega}} |D\widetilde{u}|^2 dx \leq (\text{const.}) \int_{\Omega} |Du|^2 dx.$$

Moreover, we will prove that whenever u is an $H^{1,2}(\Omega)$ function (even of changing sign), \tilde{u} belongs to $H^{1,2}_{loc}(\tilde{\Omega})$ and for any ball $\tilde{\Omega}_{\epsilon}$ concentric to $\tilde{\Omega}$ and with measure $|\Omega| - \epsilon$, one has:

$$\int_{\widetilde{\Omega}} |D\widetilde{u}|^2 dx \leq c(\epsilon) \cdot \int_{\Omega} |Du|^2 dx.$$

where c does not depend on u. (See section 2). All these results can naturally be generalized to Orlicz-Sobolev spaces. This will be done in section 3.

The interest in studying properties of the rearrangement of functions in $H^{1,2}(\Omega)$, or vanishing on part of the boundary, comes from the application of symmetrization techniques to elliptic or parabolic P.D.E. with boundary conditions of Neumann or mixed type: so thm. 2.1 and corollary 2.2 have been used in investigating parabolic Neumann problems, see [2]. We also mention [8], in which a similar result to thm. 1.3 is stated, in a different context: this result is related to the study of elliptic mixed problems, which is carried out in [13].

Some notations and examples

If u is a real measurable function defined on Ω , we define: the distribution function of u:

$$\mu(t) = \left| \left\{ \mathbf{x} \in \Omega : \mathbf{u}(\mathbf{x}) > \mathbf{t} \right\} \right| \quad \text{for } \mathbf{t} \in \Re$$
 (0.1)

(| denotes Lebesgue measure);

the decreasing rearrangement of u:

$$u^*(s) = \inf \{ t \in \Re: \mu(t) \le s \} \text{ for } s \in [0, |\Omega|];$$
 (0.2)

the Schwarz symmetrization of u:

 $\widetilde{u}(x) = u^* (c_n | x |^n) \text{ for } x \in \widetilde{\Omega},$ (0.3)

where $\tilde{\Omega}$ is the sphere centred at the origin with the same measure of Ω ; c_n is the measure of the unit ball in \Re^n .

For general properties of these functions, see [12]; note that, in our definition, u* and ũ assume also

negative values, if u is a function of changing sign, whereas rearrangements are sometimes defined for |u|.

From (0.3) it follows:

$$| D\tilde{u}(\mathbf{x}) | = \mathbf{n} \mathbf{c}_{n} | D\mathbf{u}^{*}(\mathbf{c}_{n} | \mathbf{x} |^{n}) | \cdot | \mathbf{x} |^{n-1}$$

$$\int_{\tilde{\Omega}} | D\tilde{u}(\mathbf{x}) |^{2} d\mathbf{x} = (\mathbf{n} \mathbf{c}_{n}^{1/n})^{2} \int_{0}^{|\Omega|} | D\mathbf{u}^{*}(\mathbf{s}) |^{2} \mathbf{s}^{2-2/n} d\mathbf{s}.$$
(0.4)

Hence, if $\tilde{u} \in H^{1,2}(\tilde{\Omega})$, $u^* \in H^{1,2}(\epsilon, |\Omega|)$ for any $\epsilon > 0$, so that $u^* \in AC(\epsilon, |\Omega|)$ for any $\epsilon > 0$.

For better understanding the problem of assuring integrability of $|D\tilde{u}|^2$, let us consider the case of a radially symmetric and *increasing* function u defined on a ball Ω , i.e.:

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^* (\mid \Omega \mid -\mathbf{c}_n \mid \mathbf{x} \mid ^n). \tag{0.5}$$

In this case one has:

 $\int_{\Omega} |\operatorname{Du}(x)|^2 dx = (n c_n^{1/n})^2 \int_0^{|\Omega|} |\operatorname{Du}^*(s)|^2 (|\Omega| - s)^{2-2/n} ds.$ (0.6) Comparing (0.4) and (0.6) one sees how it may happen that $u \in H^{1,2}(\Omega)$ but $\tilde{u} \notin H^{1,2}(\tilde{\Omega})$. Take, for instance, $u^*(s) = \sqrt{|\Omega| - s}$ and u as in (0.5). Then:

$$\int_{\Omega} | \mathrm{Du}(x) |^{2} dx = \frac{(n c_{n}^{1/n})^{2}}{4} \int_{0}^{|\Omega|} s^{1-2/n} ds < \infty \text{ for every } n \geq 2, \text{ while:}$$

$$\int_{\widetilde{\Omega}} | D\widetilde{u}(x) |^2 dx = \frac{(n c_n^{1/n})^2}{4} \int_0^{|\Omega|} \frac{s^{2-2/n}}{|\Omega|-s} ds = \infty \text{ for every } n$$

Similarly, if one defines: $u^*(s) = \sqrt{|\Omega| - s} - \sqrt{|\Omega|}$ and u as in (0.5), one has an example of a (negative) function $u \in H_0^{1,2}(\Omega)$ such that $\tilde{u} \notin H^{1,2}(\tilde{\Omega})$.

Remark 0.1. The above example works for n > 2. If n = 1 inequality (*) can actually be proved for any nonnegative function in $H^{1,2}(\Omega)$. (See [6], p.35). So in this paper we will always consider n > 2.

1. Isoperimetric inequalities and \mathcal{L}^2 norm of the gradient of \tilde{u}

Here we want to obtain a proof of integrability of $|D\tilde{u}|^2$ without assuming that u vanishes at the boundary of Ω . In what follows u will be a *nonnegative* function defined on Ω . A first basic tool we need is Federer's "coarea formula", as appears in [11]:

if $f \in \mathcal{L}^1(\mathfrak{R}^n)$ and v is a nonnegative Lipschitz function with compact support, then:

$$\int_{\Re^n} \quad f(x) \mid Dv(x) \mid dx = \int_0^{+\infty} dt \int_{\{x: v(x)=t\}} \quad f(x) dH_{n-1}(x).$$
(1.1)

(Here and below, H_{n-1} stands for (n - 1)-dimensional Hausdorff measure).

Let us consider a nonnegative Lipschitz function u defined on Ω . If Ω is Lipschitz, we can extend u to a compact supported Lipschitz function on \Re^n . Then, if $f \in \mathcal{L}^1(\Omega)$ and we put $f \equiv 0$ outside Ω , (1.1) becomes:

$$\int_{\Omega} f(x) | Du(x) | dx = \int_{0}^{+\infty} dt \int_{\{x \in \Omega: u(x) = t\}} f(x) dH_{n-1}(x).$$
(1.2)

From (1.2) it follows in particular:

$$\int_{\{x \in \Omega: \, u(x) > t\}} \, | \, \mathrm{D}u(x) \, | \, dx \quad = \quad \int_{t}^{+\infty} H_{n-1}\{x \in \Omega: \, u(x) = \xi\} \, d\xi. \tag{1.3}$$

Note that:

$$\begin{split} \left\{ x \in \Omega : u(x) = \xi \right\} &\supseteq \ \partial \Big\{ x \in \Omega : u(x) > \xi \Big\} \cap \Omega, \text{ and:} \\ H_{n-1} \Big\{ x \in \Omega : u(x) = \xi \Big\} &\ge \ P_{\Omega} \Big\{ x \in \Omega : u(x) > \xi \Big\}. \end{split}$$

$$(1.4)$$

Here P_{Ω} stands for the perimeter, in the sense of De Giorgi, relative to Ω . For a definition of this concept in the general case, see [9]. However, we will only use the fact that $P_{\Omega}(E) \leq H_{n-1}(\partial E) \cap \Omega$ for every measurable subset E of 0Ω , and, if ∂E is sufficiently smooth, this is an equality. (See [4]). The perimeter of E, P(E), is equal to $P_{\Omega}(E)$ when $\Omega = \Re^n$. We recall De Giorgi's isoperimetric inequality in \Re^n :

$$P(E) \geq n c_n^{1/n} | E | ^{1-1/n}$$

The next theorem points out the role of isoperimetric inequalities in Polya-Szegö-type estimates.

Theorem 1.1. Let Ω be a bounded Lipschitz domain in \Re^n , $n \ge 2$, $u \in Lip(\Omega)$, $u \ge 0$ in Ω , and assume that u satisfies:

$$P_{\Omega}\left\{x\in\Omega:u(x)>t\right\} \geq \gamma \cdot \mu(t)^{1-1/n}$$
(1.5)

for some positive constant γ , any t ≥ 0 . (Here and below, μ is the distribution function of u, defined in (0.1)).

Then $\tilde{u} \in \operatorname{Lip}(\tilde{\Omega})$, and:

$$\int_{\widetilde{\Omega}} | D\widetilde{u} |^2 dx \leq \left(\frac{n c_n^{1/n}}{\gamma} \right)^2 \int_{\Omega} | Du |^2 dx.$$
(1.6)

Proof. (Here we revise an argument of [11]). Let us prove that \tilde{u} is Lipschitz. If L is a constant such that $|Du(x)| \leq L$ in Ω , and t, h such that 0 < h < t, then:

$$L[\mu(t-h) - \mu(t)] \geq \int_{\{x \in \Omega: t-h < u(x) \le t\}} |Du(x)| dx = (by (1.3), (1.4))$$

$$= \int_{t-h}^{t} P_{\Omega}\left\{ x \in \Omega : u(x) > \xi \right\} d\xi \ge (by (1.5)) \quad \gamma \int_{t-h}^{t} \mu(\xi)^{1-1/n} d\xi \ge$$

 \geq (by monotonicity of μ) $\gamma \cdot \mathbf{h} \cdot \mu(\mathbf{t})^{1-1/n}$.

Hence μ is strictly decreasing in $(0, \|\mathbf{u}\|_{\infty})$, so that \mathbf{u}^* is continuous and satisfies:

 $u^*(s) - u^*(s+k) \; \le \; \tfrac{L}{\gamma} \; s^{-1+1/n} \cdot k$

for any k > 0, s, $s + k \in (0, |\Omega|)$. Therefore $u^* \in AC(\epsilon, |\Omega|)$ for any $\epsilon > 0$ and:

$$0 \leq -\frac{du^{*}}{ds}(s) \leq \frac{L}{\gamma} s^{-1+1/n}.$$
 (1.7)

By the definition of \tilde{u} and (1.7) one can compute:

$$|\widetilde{u}(\mathbf{x}) - \widetilde{u}(\mathbf{y})| = |\int_{c_n|\mathbf{y}|^n}^{c_n|\mathbf{x}|^n} \frac{\mathrm{d}\mathbf{u}^*}{\mathrm{d}\mathbf{s}}(\mathbf{s}) \,\mathrm{d}\mathbf{s}| \leq L \frac{\mathrm{n}c_n^{1/n}}{\gamma} |\mathbf{y} - \mathbf{x}|$$

that is \tilde{u} is Lipschitz in Ω .

Let us prove now that (1.6) holds. From (1.3)-(1.4) it follows:

$$-\frac{d}{dt}\int_{\{x\in\Omega:\,u(x)>t\}} |Du(x)| dx = P_{\Omega}\left\{x\in\Omega:u(x)>t\right\} \ge (by\,(1.5))\,\,\gamma\cdot\mu(t)^{1-1/n}.$$
 (1.8)

From (1.2) it follows that:

$$\varphi(t) \equiv \int_{\{x \in \Omega: \ u(x) > t\}} | Du(x) |^2 dx = \int_t^{+\infty} d\xi \int_{\{x \in \Omega: \ u(x) = \xi\}} | Du | dH_{n-1}(x)$$

from which one reads that φ is absolutely continuous, so that:

 $\int_{\Omega} |\operatorname{Du}|^2 dx = \varphi(0) = \int_0^{+\infty} -\varphi'(t) dt.$ (1.9)

Writing differential quotients and applying Holder's inequality one has:

$$-\varphi'(t) \geq \frac{1}{-\mu'(t)} \left[-\frac{d}{dt} \int_{\{x \in \Omega: \ u(x) > t\}} |Du(x)| \ dx \right]^2.$$
(1.10)

From (1.8), (1.9), (1.10) it follows:

$$\int_{\Omega} | \mathbf{D}\mathbf{u} |^{2} d\mathbf{x} \geq \gamma^{2} \int_{0}^{+\infty} \frac{\mu(t)^{2-2/n}}{-\mu(t)} dt.$$
(1.11)

Now consider \tilde{u} . Since its level sets are balls, in (1.10) the equal sign holds, and (1.8) becomes:

$$- \frac{\mathrm{d}}{\mathrm{dt}} \int_{\{\mathbf{x} \in \widetilde{\Omega}: \, \mathfrak{U}(\mathbf{x}) > t\}} | \mathbf{D}\widetilde{\mathbf{u}} | d\mathbf{x} = \mathbf{n} \, \mathbf{c}_n^{1/n} \, \mu(\mathbf{t})^{1-1/n}$$

Hence:

$$\int_{\tilde{\Omega}} | D\tilde{u} |^{2} dx = \left(n c_{n}^{1/n} \right)^{2} \int_{0}^{+\infty} \frac{\mu(t)^{2-2/n}}{-\mu'(t)} dt.$$
(1.12)

From (1.11)-(1.12) it follows estimate (1.6).

Now we are interested in discussing sufficient conditions in order that (1.5) holds. In the following the function u is still supposed nonnegative and Lipschitz in $\overline{\Omega}$.

(*i*) If u = 0 on $\partial \Omega$, we obtain Polya-Szegö's theorem, since:

$$\mathsf{P}_{\Omega}\Big\{\mathbf{x}\in\Omega:\mathbf{u}(\mathbf{x})>\mathbf{t}\Big\} = \mathsf{P}\Big\{\mathbf{x}:\mathbf{u}(\mathbf{x})>\mathbf{t}\Big\} \geq \mathsf{n}\,\mathsf{c}_n^{1/n}\cdot\boldsymbol{\mu}(\mathbf{t})^{1-1/n}$$

by the isoperimetric inequality in \Re^n . So $\gamma = n c_n^{1/n}$, and (1.6) holds with constant equal to 1.

(*ii*) Suppose that: | support of $u | \leq \frac{|\Omega|}{2}$. The *relative* isoperimetric inequality of Ω says that:

$$\mathbf{Q} \cdot \mathbf{P}_{\Omega}(\mathbf{E}) \geq \min\left(\mid \mathbf{E} \mid , \mid \Omega \setminus \mathbf{E} \mid \right)^{1-/n}$$
(1.13)

for some constant Q > 0, any measurable set $E \subseteq \Omega$. (Such an inequality certainly holds if Ω is Lipschitz). Then:

$$P_{\Omega} \Big\{ x \in \Omega : u(x) > t \Big\} \ge Q^{-1} \mu(t)^{1-1/n},$$
(1.14)

and (1.5) holds with $\gamma = Q^{-1}$.

(iii) More generally, suppose that:

$$|\left\{\mathbf{x}\in\Omega:\mathbf{u}(\mathbf{x})=0\right\}| = \epsilon$$

with $0 < \epsilon < \frac{|\Omega|}{2}$. Fix t > 0. If $\mu(t) \le \frac{|\Omega|}{2}$, (1.14) still holds. Otherwise, from (1.13) we get:

$$\mathbf{Q} \cdot \mathbf{P}_{\Omega} \Big\{ \mathbf{x} \in \Omega : \mathbf{u}(\mathbf{x}) > \mathbf{t} \Big\} \geq \left(\mid \Omega \mid -\mu(\mathbf{t}) \right)^{1-1/n} \geq \left[\alpha \, \mu(\mathbf{t}) \right]^{1-1/n}$$

with $\alpha = \frac{\epsilon}{|\Omega| - \epsilon}$. Hence (1.5) holds with:

$$\gamma = \mathbf{Q}^{-1} \alpha^{1-1/n} = \mathbf{Q}^{-1} \cdot \left(\frac{\epsilon}{|\Omega|-\epsilon}\right)^{1-1/n}$$

and (1.6) holds with constant:

$$\left(\frac{\operatorname{Qn} \operatorname{c}_n^{1/n}}{\alpha^{1-1/n}}\right)^2.$$

(*iv*) Now, suppose that:

$$\mathbf{H}_{n-1}\Big\{\mathbf{x}\in\partial\Omega:\mathbf{u}(\mathbf{x})=0\Big\} = \epsilon > 0.$$

We also suppose that Ω satisfies the following geometric property (this already appears in [10]):

$$\mathbf{H}_{n-1}\Big(\partial \mathbf{E} \cap \partial \Omega\Big) \leq \mathcal{C} \cdot \mathbf{P}_{\Omega}(\mathbf{E}) \tag{1.15}$$

for some positive constant C, for any measurable $E \subseteq \Omega$ such that $|E| \leq \frac{|\Omega|}{2}$. (If Ω is Lipschitz, (1.15) actually holds).

Fix t > 0. Again, we consider the case $\mu(t) > \frac{|\Omega|}{2}$; then, by (1.15):

$$H_{n-1}\Big(\partial\Big\{x\in\Omega\colon u(x)\leq t\Big\}\,\cap\,\partial\Omega\Big)\,\,\leq\,\,\mathcal{C}\cdot P_\Omega\Big\{x\in\Omega\colon u(x)\leq t\Big\}.$$

Hence:

$$\begin{split} & P_{\Omega}\Big\{x\in\Omega:u(x)>t\Big\} \ = \ P_{\Omega}\Big\{x\in\Omega:u(x)\leq t\Big\} \ \ge \\ & \geq \ \frac{1}{\mathcal{C}}\,H_{n-1}\Big(\partial\Big\{x\in\Omega:u(x)\leq t\Big\}\,\cap\,\partial\Omega\Big) \ \ge \ \frac{1}{\mathcal{C}}\,H_{n-1}\Big\{x\in\partial\Omega:u(x)=0\Big\} \ = \\ & = \ \frac{\epsilon}{\mathcal{C}} \ \ge \ \frac{\epsilon}{\mathcal{C}}\Big(\frac{\mu(t)}{|\Omega|}\Big)^{1-1/n}. \end{split}$$

So (1.5) holds with:

$$\gamma = \min \Big(\mathrm{Q}^{-1}, \, rac{\epsilon}{\mathcal{C} |\Omega|^{1-1/n}} \Big)$$

and (1.6) holds with constant:

$$\max\left\{\left(\operatorname{Qn} \operatorname{c}_n^{1/n}\right), \left(\frac{\operatorname{Cn} \operatorname{c}_n^{1/n} |\Omega|^{1-1/n}}{\epsilon}\right)\right\}^2.$$

Note that: $\frac{nc_n^{1/n}|\Omega|^{1-1/n}}{\epsilon} \leq \frac{H_{n-1}(\partial\Omega)}{\epsilon}$, which is a more expressive ratio. (*v*) Suppose that $E = \left\{ x \in \Omega : u(x) = 0 \right\}$ is such that its projection on at least one hyperplane has positive (n-1)-dimensional Hausdorff measure, in symbols:

$$H_{n-1}(\Pi(E)) = \epsilon > 0$$
 for some projection Π .

For any t > 0, the set $A = \{u \le t\}$ contains E, so:

$$\mathbf{P}(\mathbf{A}) = \mathbf{H}_{n-1}(\partial \mathbf{A}) \geq \mathbf{H}_{n-1}(\mathbf{\Pi}(\mathbf{A})) \geq \mathbf{H}_{n-1}(\mathbf{\Pi}(\mathbf{E})) = \epsilon$$

Now, if $\mu(t) > \frac{|\Omega|}{2}$, one has:

$$\mathcal{C}{\cdot}P_{\Omega}\{u>t\} \ \geq \ H_{n-1}\Big(\partial\Big\{u\leq t\Big\} \ \cap \ \partial\Omega\Big).$$

Hence:

$$\mathbf{P}_{\Omega}\{\mathbf{u} > \mathbf{t}\} \geq \frac{1}{\mathcal{C}+1} \mathbf{P}\{\mathbf{u} \leq \mathbf{t}\} \geq \frac{\epsilon}{\mathcal{C}+1} \geq \frac{\epsilon}{\mathcal{C}+1} \frac{\mu(\mathbf{t})^{1-1/n}}{|\Omega|^{1-1/n}}.$$

So (1.5) holds with:

$$\gamma = \min\left(\mathbf{Q}^{-1}, \frac{\epsilon}{(\mathcal{C}+1)|\Omega|^{1-1/n}}\right).$$

Now we state separately the results obtained from (*iii*)-(*iv*)-(*v*).

Theorem 1.2. Let Ω be a bounded Lipschitz domain in \Re^n , $n \ge 2$; let $u \in H^{1,2}(\Omega)$, $u \ge 0$ in Ω , and suppose that | support of $u | = |\Omega| - \epsilon$ for some $\epsilon > 0$. Then $\tilde{u} \in H_0^{1,2}(\tilde{\Omega})$ and:

$$\int_{\widetilde{\Omega}} | D\widetilde{u} |^2 dx \leq L^2 \cdot \int_{\Omega} | Du |^2 dx \qquad (1.16)$$

with $L = \left(\frac{Q n c_n^{1/n}}{\alpha^{1-1/n}}\right)$, where Q is as in (1.13) and: $\alpha = \frac{\epsilon}{|\Omega| - \epsilon}$ if $\epsilon \le \frac{|\Omega|}{2}$; $\alpha = 1$ otherwise.

Theorem 1.3. Let Ω be as above, let $u \ge 0$, $u \in \mathcal{H}$, where \mathcal{H} is the closure in $H^{1,2}$ -norm of the space:

$$\mathcal{H} \mathcal{I} \equiv \left\{ \varphi \in \operatorname{Lip}(\Omega) \colon \operatorname{supp} \varphi \cap \mathbf{F} = \emptyset \right\}$$

and F is a fixed closed subset of $\partial\Omega$ with $H_{n-1}(F) = \epsilon > 0$. Then $\tilde{u} \in H_0^{1,2}(\tilde{\Omega})$ and (1.16) holds with:

$$\mathbf{L} = \max\left\{ \left(\mathbf{Q} \, \mathbf{n} \, \mathbf{c}_n^{1/n} \right), \left(\frac{\mathcal{C} \, \mathbf{n} \, \mathbf{c}_n^{1/n} |\Omega|^{1-1/n}}{\epsilon} \right) \right\}$$
(1.17)

and C as in (1.15).

Theorem 1.4. Let Ω be as above, let $u \ge 0$, $u \in \mathcal{H}$, where \mathcal{H} is the closure in $H^{1,2}$ -norm of the space:

$$\mathcal{H} \boldsymbol{\prime} \equiv \left\{ \varphi \in \operatorname{Lip}(\Omega) \colon \operatorname{supp} \varphi \subseteq \overline{\Omega} \setminus \mathbf{F} \right\}$$

where F is a closed subset of Ω with the property stated in (v). Then $\tilde{u} \in H_0^{1,2}(\tilde{\Omega})$ and (1.16) holds with:

$$\mathbf{L} = \max\left\{ \left(\mathbf{Q} \, \mathbf{n} \, \mathbf{c}_n^{1/n} \right), \left(\frac{(\mathcal{C}+1) \, \mathbf{n} \, \mathbf{c}_n^{1/n} |\Omega|^{1-1/n}}{\epsilon} \right) \right\}.$$

Remark 1.5. We note that the spaces \mathcal{H} defined in thms. 1.3-1.4 are properly contained in $H^{1,2}(\Omega)$ whenever F has positive capacity. This is the case, in particular, if F has positive (n - 1)-measure. Moreover, if F has (positive and) finite (n - 1)-measure and is a *regular* set in the sense of geometric measure theory (that is a. e. (H_{n-1}) point of F is a density point in sense H_{n-1}) then property (ν) is certainly satisfied. (See [5], p.87).

Proof of theorem 1.2. If $u \in H^{1,2}(\Omega)$, $u \ge 0$ and Ω is Lipschitz, u may be approximated in $H^{1,2}$ -norm with smooth functions u_m in $\overline{\Omega}$. (See [1], thm. 3.18). Moreover, if the support of u has measure $|\Omega| - \epsilon$, then for any $\epsilon_1 \in (0, \epsilon) \{u_m\}$ can be choosen such that:

$$\left|\left\{\mathbf{x}\in\Omega:\mathbf{u}_m(\mathbf{x})=\mathbf{0}\right\}\right|\ \geq\ \epsilon_1.$$

Hence, for every m, u_m satisfies (1.16) (with ϵ replaced by ϵ_1), so that $\{\tilde{u}_m\}$ is a bounded sequence in $H_0^{1,2}(\tilde{\Omega})$. Let \tilde{u}_m be a subsequence converging to some $v \in H_0^{1,2}(\tilde{\Omega})$ weakly in $H^{1,2}$ and strongly in \mathcal{L}^2 . By [3], $u_m \to u$ in $\mathcal{L}^2(\Omega)$ implies $\tilde{u}_m \to \tilde{u}$ in $\mathcal{L}^2(\tilde{\Omega})$, so $v \equiv \tilde{u}$ and $\tilde{u} \in H_0^{1,2}(\tilde{\Omega})$. Then from weak convergence it follows that u satisfies (1.16) for any $\epsilon_1 < \epsilon$, and hence for ϵ , too.

Proof of theorem 1.3. If $u_m \in \mathcal{H}$, $u_m \to u$ in $H^{1,2}(\Omega)$, then u_m satisfies (1.16)-(1.17). Hence arguing as above, it follows that these hold for u. Note that the condition $\operatorname{supp} u_m \cap F = \emptyset$ implies that $| \operatorname{supp} u_m | < |\Omega|$; hence $\tilde{u}_m \in H^{1,2}_0(\tilde{\Omega})$, and so does u.

In a similar way it follows theorem 1.4. Incidentally, we note that a Sobolev embedding theorem for functions vanishing on part of the boundary can be derived from thm. 1.3:

Corollary 1.6. Let $u \in H$, where H is as in theorem 1.3 or 1.4. Then the following estimate holds:

$$\| u \|_{\mathcal{L}^{2^*}(\Omega)} \leq \text{const.} \| Du \|_{\mathcal{L}^{2}(\Omega)}.$$
(1.18)

Proof. It is sufficient to prove (1.18) for $u \ge 0$. Then $\tilde{u} \in H_0^{1,2}(\tilde{\Omega})$, so by Sobolev's embedding theorem and theorem 1.3 (or 1.4) one has:

$$\| u \|_{\mathcal{L}^{2^{*}}(\Omega)} = \| \widetilde{u} \|_{\mathcal{L}^{2^{*}}(\widetilde{\Omega})} \leq C \cdot \| D\widetilde{u} \|_{\mathcal{L}^{2}(\widetilde{\Omega})} \leq C \cdot \| Du \|_{\mathcal{L}^{2}(\Omega)} \qquad \Box$$

2. Local integrability of $| D\tilde{u} |^2$ for $u \in H^{1,2}(\Omega)$

Theorem 1.2 allows us to prove the following result, which holds for any function $u \in H^{1,2}(\Omega)$ (even assuming negative values):

Theorem 2.1. Let Ω be as in theorem 1.2, $u \in H^{1,2}(\Omega)$. Then $\tilde{u} \in H^{1,2}_{loc}(\tilde{\Omega})$ and, for any $\epsilon > 0$, one has:

$$\int_{\widetilde{\Omega}_{\epsilon}} | \mathbf{D}\widetilde{\mathbf{u}} |^{2} d\mathbf{x} \leq \mathbf{c}(\epsilon) \cdot \left(\mathbf{Q} \, \mathbf{n} \, \mathbf{c}_{n}^{1/n} \right)^{2} \cdot \int_{\Omega} | \mathbf{D}\mathbf{u} |^{2} d\mathbf{x}$$

where $\tilde{\Omega}_{\epsilon}$ is the sphere centred at the origin with measure $\mid \Omega \mid -\epsilon \;$ and:

$$c(\epsilon) = \left(\frac{|\Omega| - \epsilon}{\epsilon}\right)^{2-2/n}$$
 if $\epsilon \le \frac{|\Omega|}{2}$, $c(\epsilon) = 1$ otherwise.

Moreover, $\mathbf{u}^* \in \mathrm{AC}(\epsilon, |\Omega| - \epsilon)$.

Proof. Put $h = u^*(\frac{|\Omega|}{2})$, and let u_1, u_2 be the positive and negative parts of (u - h). Then $u_i \in H^{1,2}(\Omega)$, | supp $u_i | \leq \frac{|\Omega|}{2}$ (i = 1, 2). So by theorem 1.2 $\tilde{u}_i \in H^{1,2}_0(\tilde{\Omega})$ and:

$$\int_{\widetilde{\Omega}} | \mathrm{D}\widetilde{\mathfrak{u}}_i |^2 \, \mathrm{d} x \leq \left(\mathrm{Q} \, \mathrm{n} \, \mathrm{c}_n^{1/n} \right)^2 \cdot \int_{\Omega} | \mathrm{D} \mathfrak{u}_i |^2 \, \mathrm{d} x.$$

In particular, $\mathbf{u}_i^* \in \mathrm{AC}(\epsilon, |\Omega|)$ for any $\epsilon > 0$. Now, noting that:

$$(v^{+})^{*}(s) = (v^{*})^{+}(s)$$
 (2.1)

$$(v^{-})^{*}(s) = (v^{*})^{-}(|\Omega| - s)$$
(2.2)

one has:

$$(\mathbf{u}^* - \mathbf{h})^+ \in \mathrm{AC}(\epsilon, |\Omega|), \ (\mathbf{u}^* - \mathbf{h})^- \in \mathrm{AC}(0, |\Omega| - \epsilon), \text{ so that:}$$
$$\mathbf{u}^* \in \mathrm{AC}(\epsilon, |\Omega| - \epsilon) \text{ for any } \epsilon > 0.$$
(2.3)

Note also that:

$$(u - h)^{+} = (u - h)^{+}$$
 (2.4)

whereas the same is *not* true for the *negative* part. To handle the gradient of $(u - h)^{-1}$, let us observe that, for any $\epsilon > 0$, one has, by (0.4):

$$\int_{\tilde{\Omega}_{\epsilon}} | \mathbf{D} (\mathbf{u} - \mathbf{h})^{-} |^{2} d\mathbf{x} = \left(\mathbf{n} \, \mathbf{c}_{n}^{1/n} \right)^{2} \int_{\frac{1}{2} |\Omega|}^{|\Omega| - \epsilon} \mathbf{s}^{2-2/n} | \mathbf{D} \mathbf{u}^{*}(\mathbf{s}) |^{2} d\mathbf{s}$$
(2.5)

while, by (0.4) and (2.2):

$$\int_{\tilde{\Omega}_{\epsilon}} |D(\mathbf{u} - \mathbf{h})^{-} |^{2} d\mathbf{x} = \left(n c_{n}^{1/n} \right)^{2} \int_{\epsilon}^{\frac{1}{2} |\Omega|} s^{2-2/n} |D\mathbf{u}^{*}(|\Omega| - s)|^{2} ds = (2.6)$$
$$= \left(n c_{n}^{1/n} \right)^{2} \int_{\frac{1}{2} |\Omega|}^{|\Omega|-\epsilon} \left(|\Omega| - s \right)^{2-2/n} |D\mathbf{u}^{*}(s)|^{2} ds.$$

Comparing (2.5) and (2.6) we can write:

$$\int_{\widetilde{\Omega}_{\epsilon}} |D(\mathbf{u}-\mathbf{h})^{-}|^{2} d\mathbf{x} \leq \left(\frac{|\Omega|-\epsilon}{\epsilon}\right)^{2-2/n} \int_{\widetilde{\Omega}_{\epsilon}} |D(\mathbf{u}-\mathbf{h})^{-}|^{2} d\mathbf{x}.$$
(2.7)

Finally, we can estimate:

$$\begin{split} \int_{\widetilde{\Omega}_{\epsilon}} & \mid D\widetilde{u}(x) \mid^{2} dx = \int_{\widetilde{\Omega}_{\epsilon}} & \mid D\left(u-h\right)^{+} \mid^{2} dx + \int_{\widetilde{\Omega}_{\epsilon}} & \mid D\left(u-h\right)^{-} \mid^{2} dx \leq (by \ (2.4), (2.7)) \\ & \leq \int_{\widetilde{\Omega}} & \mid D\widetilde{u}_{1} \mid^{2} dx + \left(\frac{\mid \Omega \mid -\epsilon}{\epsilon}\right)^{2-2/n} \int_{\widetilde{\Omega}} & \mid D\widetilde{u}_{2} \mid^{2} dx \leq (by \ (2.1)) \\ & \leq \left(Q n c_{n}^{1/n}\right)^{2} \cdot max \left(1, \left(\frac{\mid \Omega \mid -\epsilon}{\epsilon}\right)^{2-2/n}\right) \cdot \left\{\int_{\Omega} & \mid Du_{1} \mid^{2} dx + \int_{\Omega} & \mid Du_{2} \mid^{2} dx\right\} = \\ & = \left(Q n c_{n}^{1/n}\right)^{2} \cdot max \left(1, \left(\frac{\mid \Omega \mid -\epsilon}{\epsilon}\right)^{2-2/n}\right) \cdot \int_{\Omega} & \mid Du \mid^{2} dx. \end{split}$$

So the theorem is completely proved.

From the previous theorem it follows the next estimate, giving an approximation result for rearrangements:

Corollary 2.2. Let Ω be as above, $u, v \in H^{1,2}(\Omega)$. Then for any $\epsilon > 0$ one has:

$$\begin{split} \sup_{s \in (\epsilon, |\Omega| - \epsilon)} &| (u^* - v^*)(s) | \leq c_1(n, Q, |\Omega|) || u - v ||_2 + \\ &+ c_2(\epsilon, n, Q, |\Omega|) || u - v ||_2^{1/2} \cdot \Big\{ || Du ||_2 + || Dv ||_2 \Big\}^{1/2}. \end{split}$$

In particular, if u_m is a sequence of $H^{1,2}$ functions converging to u in $H^{1,2}(\Omega)$, then u_m^* converges to u^* uniformly in $(\epsilon, |\Omega| - \epsilon)$ for any $\epsilon > 0$.

Proof. We start by noting that if φ is an absolutely continuous function on [a,b], then:

$$\varphi(\mathbf{s}) \leq \frac{1}{\mathbf{b}-\mathbf{a}} \int_{\mathbf{a}}^{\mathbf{b}} \varphi(\sigma) \, \mathrm{d}\sigma + \int_{\mathbf{a}}^{\mathbf{b}} | \varphi'(\sigma) | \, \mathrm{d}\sigma$$

for every $s \in [a,b]$. Applying this formula to the function:

$$\varphi(\mathbf{s}) = \left(\mathbf{u}_m^*(\mathbf{s}) - \mathbf{u}^*(\mathbf{s})\right)^2$$
 on the interval $(\epsilon, |\Omega| - \epsilon)$

we have, by Hölder's inequality:

$$\left(\mathbf{u}_{m}^{*}(\mathbf{s}) - \mathbf{u}^{*}(\mathbf{s}) \right)^{2} \leq \frac{1}{|\Omega| - 2\epsilon} \int_{\epsilon}^{|\Omega| - \epsilon} | \mathbf{u}_{m}^{*}(\sigma) - \mathbf{u}^{*}(\sigma) | ^{2} d\sigma + + 2 \left(\int_{\epsilon}^{|\Omega| - \epsilon} [\sigma^{-1 + 1/n} | \mathbf{u}_{m}^{*}(\sigma) - \mathbf{u}^{*}(\sigma) |]^{2} d\sigma \right)^{\frac{1}{2}} \cdot \left(\int_{\epsilon}^{|\Omega| - \epsilon} [\sigma^{1 - 1/n} | \frac{d\mathbf{u}_{m}^{*}}{d\sigma}(\sigma) - \frac{d\mathbf{u}^{*}}{d\sigma}(\sigma) |]^{2} d\sigma \right)^{\frac{1}{2}} \equiv \equiv \mathbf{A}_{m} + 2\mathbf{B}_{m} \cdot \mathbf{C}_{m}.$$

$$(2.8)$$

Now:

$$A_m \leq \frac{1}{|\Omega| - 2\epsilon} \| \mathbf{u}_m - \mathbf{u} \|_2^2 \text{ and:}$$

$$(2.9)$$

$$\mathbf{B}_{m} \leq \epsilon^{-1+1/n} \| \mathbf{u}_{m} - \mathbf{u} \|_{2}$$
, while: (2.10)

$$C_{m} \leq \left(\int_{0}^{|\Omega|-\epsilon} [\sigma^{1-1/n} \mid \frac{du_{m}^{*}}{d\sigma}(\sigma) \mid]^{2} d\sigma \right)^{\frac{1}{2}} + \left(\int_{0}^{|\Omega|-\epsilon} [\sigma^{1-1/n} \mid \frac{du^{*}}{d\sigma}(\sigma) \mid]^{2} d\sigma \right)^{\frac{1}{2}} = \\ = \left(\int_{\tilde{\Omega}_{\epsilon}} \mid D\tilde{u}_{m} \mid {}^{2} dx \right)^{\frac{1}{2}} + \left(\int_{\tilde{\Omega}_{\epsilon}} \mid D\tilde{u} \mid {}^{2} dx \right)^{\frac{1}{2}} \leq \text{ (by thm. 2.1)} \\ \leq c(n,Q, \mid \Omega \mid) \cdot \Big\{ \parallel Du_{m} \parallel_{2} + \parallel Du \parallel_{2} \Big\}.$$

$$(2.11)$$

Collecting (2.8), (2.9), (2.10), (2.11) one gets the result.

3. Extension to Orlicz-Sobolev spaces

Let A: $[0, +\infty) \rightarrow [0, +\infty)$ be an "N-function" (see [7]), that is A is an increasing continuous convex function, such that:

$$\lim_{t \to 0} \frac{A(t)}{t} = 0; \lim_{t \to +\infty} \frac{A(t)}{t} = +\infty$$

By Jensen's inequality, we can repeat the proof of theorem 1.1 and obtain, under the same assumptions:

$$\int_{\Omega} A(|\operatorname{Du}(\mathbf{x})|) d\mathbf{x} \geq \int_{0}^{+\infty} A\left(\frac{\gamma \mu(t)^{1-1/n}}{-\mu(t)}\right) (-\mu(t)) dt$$
(3.1)

$$\int_{\tilde{\Omega}} A(|D\tilde{u}(x)|) dx = \int_{0}^{+\infty} A\left(\frac{nc_{n}^{1/n} \mu(t)^{1-1/n}}{-\mu'(t)}\right) (-\mu'(t)) dt$$
(3.2)

We can rewrite (3.1)-(3.2) replacing A(r) with A($\frac{r}{\lambda}$) for any fixed $\lambda > 0$.

Then, choosing $\lambda_0 = \frac{nc_n^{1/n}}{\gamma}$ we get:

$$\int_{\tilde{\Omega}} A(\frac{|D\tilde{u}|}{\lambda_0}) dx \leq \int_{\Omega} A(|Du|) dx.$$
(3.3)

Now, recall that the natural norm in the Orlicz space:

$$\mathcal{L}_{A}(\Omega) \equiv \left\{ u: \Omega \to \Omega, u \text{ measurable such that } \int_{\Omega} A(\frac{|u|}{\lambda}) \, dx < +\infty \quad \text{for some } \lambda > 0 \right\}$$

is:
$$\| \mathbf{u} \|_{\mathbf{A}} \equiv \inf \left\{ \lambda > 0 : \int_{\Omega} \mathbf{A}(\frac{|\mathbf{u}|}{\lambda}) \, \mathrm{d}\mathbf{x} \le 1 \right\}.$$

Rewriting again (3.3) with A(r) replaced by A($\frac{r}{\lambda}$), and choosing $\lambda = ||$ Du $||_A$ we get:

$$\int_{\widetilde{\Omega}} A\left(\frac{|D\widetilde{u}(x)|}{\lambda_{0}||Du||_{A}}\right) dx \leq 1. \text{ Hence:}$$

$$|D\widetilde{u} \|_{\mathcal{L}_{A}(\widetilde{\Omega})} \leq \left(\frac{nc_{\alpha}^{1/n}}{\gamma}\right) \| Du \|_{\mathcal{L}_{A}(\Omega)}. \tag{3.4}$$

So we have proved the following:

Theorem 3.1. Let Ω and A be as above, let u be a nonnegative Lipschitz function in Ω , such that one of the following holds:

(*i*) $\mathbf{u} = 0$ in $\mathbf{E} \subseteq \Omega$ with $|\mathbf{E}| = \epsilon > 0$

(*ii*)
$$\mathbf{u} = 0$$
 in $\mathbf{F} \subseteq \partial \Omega$ with $\mathbf{H}_{n-1}(\mathbf{F}) = \epsilon > 0$

(iii) $\mathbf{u} = 0$ in $\mathbf{G} \subseteq \Omega$ with $\mathbf{H}_{n-1}(\Pi(\mathbf{G})) = \epsilon > 0$ for some projection Π (see section 1).

Then $\tilde{u} \in \operatorname{Lip}(\tilde{\Omega})$ and (3.4) holds, with γ possibly depending on n, \mathcal{C} , Q, $|\Omega|$, ϵ .

Remark 3.2. We did not state the previous theorem for $u \in H^1\mathcal{L}_A(\Omega)$ because to apply a limit process as in the proof of theorems (1.2)-(1.3)-(1.4) we have to know that a bounded sequence in $H^1\mathcal{L}_A(\Omega)$ has a weakly converging subsequence. This cannot be assured without further assumptions on A. To discuss this fact, we recall some results from the theory of Orlicz-Sobolev spaces. (See [1]).

We say that A satisfies a "global Δ_2 -condition" if:

$$A(2t) \le \delta A(t) \text{ for some } \delta > 0, \text{ any } t > 0.$$
(3.5)

We say that A satisfies a " Δ_2 -condition near infinity" if (3.5) holds only for any $t \ge t_0$, for some $t_0 > 0$. We say that (A,Ω) is Δ -regular if: A satisfies a global Δ_2 -condition, or: A satisfies a Δ_2 -condition near infinity and $|\Omega| < +\infty$. If (A,Ω) is Δ -regular, then $\mathcal{L}_A(\Omega)$ and $H^1\mathcal{L}_A(\Omega)$ are reflexive spaces; if Ω is Lipschitz then $\mathcal{C}^{\infty}(\overline{\Omega})$ is dense in $H^1\mathcal{L}_A(\Omega)$; if $|\Omega| < +\infty$ then $\mathcal{L}_A(\Omega)$ is continuously embedded in $\mathcal{L}^1(\Omega)$. Using these facts one can repeat the proofs of theorems (1.2)-(1.3)-(1.4) to get the following:

Theorem 3.3. Let Ω , A be as above. Suppose that A satisfies a Δ_2 -condition near infinity, and let u satisfy the assumptions of one of theorems 1.2, 1.3, 1.4, with $H^{1,2}(\Omega)$ replaced by $H^1\mathcal{L}_A(\Omega)$. Then $\tilde{u} \in H^1_0\mathcal{L}_A(\Omega)$ and (3.4) holds, with γ possibly depending on n, \mathcal{C} , Q, $|\Omega|$, ϵ .

Example. An example of Orlicz-Sobolev space which does not reduce to a standard Sobolev space and satisfies the previous theorem is the one defined by $A(r) = r^p \log (1 + r)$ with $p \ge 1$.

Now we are interested in stating an analogue of theorem 2.1 for Orlicz-Sobolev spaces. We first consider the case of a Lipschitz function u. The analogue of formula (0.4) is:

$$\int_{\widetilde{\Omega}} A(\mid D\widetilde{u}(x)\mid) \, dx = \int_{0}^{\mid \Omega \mid} A\left(n \, c_n^{1/n} \mid Du^*(s) \mid s^{1-1/n}\right) ds.$$

Arguing as in section 2 one gets:

$$\int_{\tilde{\Omega}_{\epsilon}} A\left(\mid D\left(u-h\right)^{-} \mid \right) dx = \int_{\frac{1}{2}|\Omega|}^{|\Omega|-\epsilon} A\left(n c_n^{1/n} \mid Du^*(s) \mid s^{1-1/n}\right) ds$$
(3.6)

$$\int_{\tilde{\Omega}_{\epsilon}} A\left(\mid D\left(u-h\right)^{-} \mid \right) dx = \int_{\frac{1}{2}|\Omega|}^{|\Omega|-\epsilon} A\left(n c_{n}^{1/n} \mid Du^{*}(s) \mid (\mid \Omega \mid -s)^{1-1/n}\right) ds.$$
(3.7)

Comparing (3.6)-(3.7) one can write:

$$\int_{\widetilde{\Omega}_{\epsilon}} A\left(\frac{|D(u-h)^{-}|}{\lambda}\right) dx \leq \int_{\widetilde{\Omega}_{\epsilon}} A\left(|D(u-h)^{-}|\right) dx$$
(3.8)

with $\lambda = \left(\frac{|\Omega| - \epsilon}{\epsilon}\right)^{1 - 1/n}$ (we take $\epsilon < \frac{|\Omega|}{2}$, so $\lambda > 1$).

Applying (3.3) to the positive and negative parts of (u - h) we get, by (3.8):

$$\begin{split} &\int_{\bar{\Omega}} \ A(\frac{|D\tilde{u}|}{\lambda}) \, dx \ = \ \int_{\bar{\Omega}_{\epsilon}} \ \left\{ A\Big(\frac{|D(u-h)^{^+}|}{\lambda}\Big) \ + \ A\Big(\frac{|D(u-h)^{^-}|}{\lambda}\Big) \right\} dx \ \leq \\ &\leq \ \int_{\bar{\Omega}_{\epsilon}} \ \left\{ A\Big(\mid D(u-h)^{+}_{\sim} \mid \Big) \ + \ A\Big(\mid D(u-h)^{-}_{\sim} \mid \Big) \right\} dx \ \leq \\ &\leq \ \int_{\Omega} \ \left\{ A\Big(\lambda_{0} \mid D(u-h)^{+} \mid \Big) \ + \ A\Big(\lambda_{0} \mid D(u-h)^{-} \mid \Big) \right\} dx \ = \\ &= \ \int_{\Omega} \ A\Big(\lambda_{0} \mid Du \mid \Big) \, dx \quad \text{with} \ \lambda_{0} = Q \, n \, c_{n}^{1/n}. \end{split}$$

Again, rewriting the previous inequality for $A(\frac{r}{\rho})$ instead of A(r) and choosing $\rho = \lambda_0 \parallel Du \parallel_A$ we find:

$$\| \operatorname{D} \widetilde{u} \|_{\mathcal{L}_{A}(\widetilde{\Omega}_{\epsilon})} \leq \left(\operatorname{Qn} c_{n}^{1/n} \right) \left(\frac{|\Omega| - \epsilon}{\epsilon} \right)^{1 - 1/n} \| \operatorname{Du} \|_{\mathcal{L}_{A}(\Omega)}$$
(3.9)

for every $\epsilon \in (0, \frac{|\Omega|}{2})$.

This holds for every Lipschitz function u defined in Ω . From this fact we get, by approximation with smooth functions:

Theorem 3.4. Let Ω , A be as in theorem 3.3. If $u \in H^1\mathcal{L}_A(\Omega)$ then $\tilde{u} \in H^1_{loc}\mathcal{L}_A(\Omega)$ and (3.9) holds. Moreover, $u^* \in AC_{loc}(\epsilon, |\Omega| - \epsilon)$ for every $\epsilon > 0$.

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