Appl. Anal. 40 (1991), no. 1, 21-39.

Symmetrization in Parabolic Neumann Problems

MARCO BRAMANTI
Dipartimento di Matematica, Politecnico di Milano
via Bonardi 9, 20133 Milano, Italy

Communicated by: Carlo Pucci.

Abstract. We consider the Cauchy-Neumann problem for parabolic operators of the kind:

$$Lu = u_t - (a_{ij}(t,x)u_{x_i})_{x_j}$$

on a smooth cylinder $[0,T] \times \Omega$. By symmetrization techniques we establish for the solution u of this problem an $\mathcal{L}^p$ estimate of the kind:

$$\| u(t,\cdot) - u(t)^*(\tfrac{1}{2} \mid \Omega \mid) \|_p \leq \| U(t,\cdot) \|_p \qquad 1 \leq p \leq \infty$$

where U is the solution of a symmetrized problem and $u(t)^*(\cdot)$ is the decreasing rearrangement of $u(t,\cdot)$. We also obtain an "energy inequality" comparing $\mathcal{L}^2$ norms of the gradients of u and U.

INTRODUCTION

In this paper we consider parabolic operators of the kind:

$$Lu \equiv \frac{\partial u}{\partial t} - \frac{\partial}{\partial x_j}\left(a_{ij}(t,x)\frac{\partial}{\partial x_i}\right) \equiv u_t - Au \tag{0.1}$$

with the matrix (a_{ij}) uniformly elliptic, defined on a smooth cylinder $(0,T) \times \Omega$. We consider the Cauchy-Neumann problem for this operator, with homogeneous boundary data:

$$\begin{aligned} Lu &= f && \text{in } (0,T) \times \Omega \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } (0,T) \times \partial\Omega \\ u(0,\cdot) &= u_0 && \text{in } \Omega. \end{aligned} \tag{0.2}$$

(Here $\frac{\partial}{\partial \nu}$ stands for the conormal derivative).
Our purpose is to compare the solution u of (0.2) with the solution U of a "symmetrized" Cauchy-Dirichlet problem for the heat equation:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \gamma\Delta\right) U &= F && \text{in } (0,T) \times \Omega' \\ U &= 0 && \text{on } (0,T) \times \partial\Omega' \\ U(0,\cdot) &= U_0 && \text{in } \Omega'. \end{aligned}$$

Here F, U_0, Ω' are obtained from f, u_0, Ω by symmetrization, while γ is a number related to the relative isoperimetric constant of Ω. (See section 1 for precise definitions and statements).

The connection between Neumann and Dirichlet problems appears already in Maderna-Salsa[4] where the elliptic case is treated. The two typical results that we obtain here are the following estimates:

$$\| u(t,\cdot) - u(t)^*(\tfrac{|\Omega|}{2}) \|_{\mathcal{L}^p(\Omega)} \leq \| U(t,\cdot) \|_{\mathcal{L}^p(\Omega')} \tag{0.3}$$

for a.e. $t \in (0,T)$, any $p \in [1,\infty]$, where $u(t)^*(\cdot)$ is the decreasing rearrangement of $u(t,\cdot)$;
"energy estimate": (0.4)

$$\int_{\Omega\times(0,T)} | Du |^2 \, dx\, dt + \tfrac{1}{2} \int_\Omega u^2(T)\, dx \leq \gamma \int_{\Omega'\times(0,T)} | DU |^2 \, dx\, dt +$$

$$+\ \tfrac{1}{2}\int_{\Omega'} U^2(T)\,dx\ +\ \int_0^T h(t)\int_\Omega f(t,x)\,dx\,dt \qquad\qquad \text{with } h(t) = u(t)^*(\tfrac{|\Omega|}{2}).$$

As usual for this kind of result the main strategy is to get a differential inequality for the distribution function of the solution. In our case the key point used to carry out this procedure is a measure theory lemma (lemma 3.2) which could also be interesting in itself.

I wish to thank prof. G. Talenti and prof. S. Salsa for their kind attention and useful suggestions.

1. ASSUMPTIONS AND MAIN RESULTS

Let us consider a linear parabolic operator of the form (0.1), defined on $D = (0,T) \times \Omega$, with Ω a bounded domain in $\Re^n$, where the matrix $a_{ij}(t,x)$ is symmetric and uniformly elliptic, with normalized lowest eigenvalue:

$$a_{ij}(t,x)\xi_i\xi_j \ \geq\ |\xi|^2 \ \text{ for all } \xi \in \Re^n. \tag{1.1}$$

We consider the Cauchy-Neumann problem (0.2), and we make the following assumptions about coefficients and data:

$$a_{ij} \in \mathcal{L}^\infty(D);\ \ f \in \mathcal{L}^2(D);\ \ u_0 \in \mathcal{L}^2(\Omega). \tag{1.2}$$

Under these assumptions problem (0.2) is uniquely solvable in some weak sense. More precisely, let:

$$\Phi = \left\{\ w \in \mathcal{L}^2(0,T;H^1(\Omega)) : \tfrac{\partial w}{\partial t} \in \mathcal{L}^2(0,T;H^1(\Omega)^*)\ \right\}. \tag{1.3}$$

(Here $H^1(\Omega)^*$ denotes the dual space of $H^1(\Omega)$). Then for solution of problem (0.2) we mean a function $u \in \Phi$ such that:

$u(0,\cdot) = u_0$ (as an element of $H^1(\Omega)^*$) and:

$$\langle u_t,\phi\rangle + \int_\Omega a_{ij}\, u_{x_i}\,\phi_{x_j}\,dx = \int_\Omega f\,\phi\,dx \qquad \text{for all } \phi \in H^1(\Omega), \text{ a.e. } t \in (0,T)$$

(where $\langle\ ,\ \rangle$ denotes duality on $H^1(\Omega)$). In this sense problem (0.2) is well posed. (See for instance Treves[7], chap. IV).

Moreover, if we strengthen our assumptions asking that:

$$f \in \mathcal{L}^2(D);\ \ \tfrac{\partial a_{ij}}{\partial t} \in \mathcal{L}^1(0,T;\mathcal{L}^\infty(\Omega));\ \ u_0 \in H^1(\Omega) \tag{1.4}$$

then the solution u belongs to $H^1(D)$.

For reader's convenience we recall a few definitions and results about rearrangements of functions.

For any measurable function v defined on Ω, we define the *decreasing rearrangement* of v as:

$$v^*(s) = \inf\left\{\theta \in \Re : |\{x \in \Omega : v(x) > \theta\}| \leq s\right\} \qquad \text{for } s \in \Omega^* \equiv [0, |\Omega|].$$

($|\cdot|$ denotes Lebesgue measure). The *spherical rearrangement* of v is:

$$\tilde{v}(x) = v^*(c_n |x|^n) \quad \text{for } x \in \tilde{\Omega}$$

where c_n is the measure of the unit ball in $\Re^n$, and $\tilde{\Omega}$ is the sphere centred at the origin with measure $|\Omega|$. If v is defined on $(0,T) \times \Omega$, we write $v(t)^*$ for the decreasing rearangement of the function $v(t,\cdot)$ for t fixed. We put also:

$$\tilde{v}(t,x) = v(t)^*(c_n |x|^n).$$

Some basic facts we need about rearrangements are the following.

(*i*) The map $v \to v^*$ is non-expansive in $\mathcal{L}^p$, that is:

$$\| u^* - v^* \|_{\mathcal{L}^p(\Omega^*)} \leq \| u - v \|_{\mathcal{L}^p(\Omega)} \tag{1.5}$$

for any u, v $\in \mathcal{L}^p(\Omega)$, p $\in [1,\infty]$. (See Chiti[2] for the proof).

(*ii*) If $u \in H^1(0,T,\mathcal{L}^2(\Omega))$ then $u^* \in H^1(0,T,\mathcal{L}^2(\Omega))$ and:

$$\int_{\{u>\theta\}} \frac{\partial u}{\partial t}\, dx = \frac{\partial}{\partial t} \int_0^{\mu(\theta)} u^*(t,s)\, ds \quad \text{for a.e. } \theta. \tag{1.6}$$

(This is a basic result contained in Mossino-Rakotoson[5]).
Here we indicate with $\{u > \theta\}$ the set, depending on t: $\{x \in \Omega\colon u(t,x) > \theta\}$, and with μ the *distribution function* of u:

$$\mu(\theta) = |\{u > \theta\}|\,.$$

(Note that μ is a function of t and θ).

Let us turn now to problem (0.2). If u is the solution, we define the function:

$$h(t) = u(t)^*\left(\tfrac{|\Omega|}{2}\right).$$

We shall indicate with u_1, u_2, respectively, the positive and negative parts of $(u - h)$, and with μ_1, μ_2, respectively, the distribution functions of u_1, u_2. Note that u_1, u_2 are nonnegative functions whose supports in Ω, for every fixed t, have measure less than or equal to $|\Omega|/2$.

Let f, u_0 be the data in problem (0.2), f_1, f_2 the positive and negative parts of f and $u_1(0)$, $u_2(0)$ the positive and negative parts of $[u_0 - h(0)]$; put:

$$f_3(t) \equiv [f_1(t)^* + f_2(t)^*](s)$$

$$v_0^*(s) \equiv [u_1(0)^* + u_2(0)^*](s).$$

We define also the relative isoperimetric constant of Ω as the smaller number Q for which the following holds:

$$\left\{\min(|E|, |\Omega\backslash E|)\right\}^{1-1/n} \leq Q\cdot P_\Omega(E) \tag{1.7}$$

for any measurable subset E of Ω. Here $P_\Omega(E)$ stands for the perimeter of E relative to Ω, in the sense of De Giorgi.

Our main results are the following ones:

Theorem 1.1. Suppose that Ω is a bounded $\mathcal{C}^{2,\alpha}$ domain in $\Re^n$, and that (1.1), (1.2) hold. Let u be the solution of problem (0.2), and U the solution of:

$$\begin{aligned} &\left(\tfrac{\partial}{\partial t} - \gamma\cdot\Delta\right)U = \widetilde{f}_3 && \text{in } (0,T)\times\widetilde{\Omega}_{1/2} \\ &U = 0 && \text{on } (0,T)\times\partial\widetilde{\Omega}_{1/2} \\ &U(0,\cdot) = \widetilde{v}_0 && \text{in } \widetilde{\Omega}_{1/2} \end{aligned} \tag{1.8}$$

where $\gamma = [Q\cdot n\cdot c_n^{1/n}]^{-2}$ and $\widetilde{\Omega}_{1/2}$ is the sphere centred at the origin with measure $|\Omega|/2$. Then:

$$\|(u - h)(t)\|_{\mathcal{L}^p(\Omega)} \leq \|U(t)\|_{\mathcal{L}^p(\widetilde{\Omega}_{1/2})} \tag{1.9.a}$$

for a.e. $t \in (0,T)$, any $p \in [1,\infty]$. Moreover:

$$\sup_\Omega u(t,\cdot) - \inf_\Omega u(t,\cdot) \leq \|U(t)\|_{\mathcal{L}^\infty(\widetilde{\Omega}_{1/2})}. \tag{1.9.b}$$

Theorem 1.2. ("Energy estimate"). Suppose all the assumptions of theorem 1.1 hold. Moreover, suppose (1.4) holds, and take $h(0) = 0$. Then:

$$\int_D |Du|^2\, dx\, dt + \tfrac{1}{2}\int_\Omega u^2(T)\, dx \leq$$

$$\leq \gamma\int_{\widetilde{D}} |DU|^2\, dx\, dt + \tfrac{1}{2}\int_{\widetilde{\Omega}_{1/2}} U^2(T)\, dx + \int_0^T h(t)\int_\Omega f(t,x)\, dx\, dt. \tag{1.10}$$

Here $\widetilde{D} = (0,T) \times \widetilde{\Omega}_{1/2})$.

One can obtain from the proof of (1.9) also a rough estimate on $\mathcal{L}^p$norms of $u(t,\cdot)$ in terms of the data. Namely, the following holds:

Proposition 1.3. If $f \in \mathcal{L}^1(0,T;\mathcal{L}^p(\Omega))$ and $u_0 \in \mathcal{L}^p(\Omega)$ $(1 \leq p \leq \infty)$ then for a.e. $t \in (0,T)$ one has:

$$\| u(t,\cdot) \|_p \leq (2^{1-1/p} + 2) \cdot \int_0^T \| f(\tau) \|_p d\tau + 2 \| u_0 \|_p + 2^{1-1/p} \cdot \| u_0 - h(0) \|_p \quad (1.11)$$

Note that at the right hand side of (1.11) the relative isoperimetric constant does not appear at all. In the elliptic case such an inequality would be false, as shown in Maderna-Salsa[4].

The proof of theorems 1.1-1.3 will proceed in two main steps. First we shall derive estimates (1.9), (1.10), (1.11) under stronger assumptions (section 2). Then (section 3) we will reduce the general case to the previous one. The additional assumptions are the following:

(1.12.a) we suppose that the coefficients a_{ij} and the data u_0, f are smooth; as a consequence (see Ladyzenskaya-Solonnikov-Uraltseva[3], chap.IV, thm. 5.3) we have that: u, u_{x_i}, $u_{x_i x_j}$, u_t belong to $\mathcal{C}^\alpha(D)$;

(1.12.b) we suppose that the solution u has the following property: for a.e. t, the distribution function μ of u is strictly decreasing; this means that for a.e. t the function $u(t,\cdot)$ is not constant on any subset of Ω of positive measure.

From (1.12.a) we have the following:

Lemma 1.4. The function h is Lipschitz on [0,T]. (In particular, the functions u_1, u_2 belong to $H^1(D)$).

Proof. $| h(t+\tau) - h(t) | = | u(t+\tau)^* (\frac{|\Omega|}{2}) - u(t)^* (\frac{|\Omega|}{2}) | \leq$

$\leq \| u(t+\tau)^* - u(t)^* \|_{\mathcal{L}^\infty(\Omega^*)} \leq$ (by (1.5))

$\leq \| u(t+\tau) - u(t) \|_{\mathcal{L}^\infty(\Omega)} \leq |\tau| \cdot \sup_{\overline{D}} | \frac{\partial u}{\partial t}(t,x) |$. □

2. PROOF OF THEOREMS 1.1-1.3 IN A SPECIAL CASE

We are now in position to begin the proof of theorem 1.1. We shall keep the notations and assumptions of the previous section. Fix a number $\theta \geq 0$, and $t \in (0,T)$. Let us consider the functions u_i $(i = 1, 2)$. The Fleming-Rishel formula gives (see Maderna-Salsa[4]):

$$-\frac{\partial}{\partial\theta} \int_{\{u_i > \theta\}} | Du_i | \, dx = P_\Omega\{u_i > \theta\}. \quad (2.1)$$

By our definition of u_i and the relative isoperimetric inequality we have:

$$\mu_i(\theta)^{1-1/n} \leq Q \cdot P_\Omega\{u_i > \theta\}. \quad (2.2)$$

On the other hand, using Cauchy-Schwartz inequality and ellipticity we get:

$$-\frac{\partial}{\partial\theta} \int_{\{u_i > \theta\}} | Du_i | \, dx \leq [-\frac{\partial\mu_i}{\partial\theta}]^{\frac{1}{2}} \cdot \Big(-\frac{\partial}{\partial\theta} \int_{\{u_i > \theta\}} a_{jk} \cdot (u_i)_{x_j} (u_i)_{x_k} \, dx \Big)^{\frac{1}{2}}. \quad (2.3)$$

From (2.1), (2.2), (2.3) it follows:

$$1 \leq Q^2 \mu_i(\theta)^{-2+2/n} \cdot [-\frac{\partial\mu_i}{\partial\theta}] \cdot \Big(-\frac{\partial}{\partial\theta} \int_{\{u_i > \theta\}} a_{jk} \cdot (u_i)_{x_j} (u_i)_{x_k} \, dx \Big). \quad (2.4)$$

Since u is the solution of problem (0.2), we have that:

$$\int_\Omega \ a_{jk}\, u_{x_j}\, \phi_{x_k}\, dx \ = \ \int_\Omega \ (f - \tfrac{\partial u}{\partial t})\cdot\phi\, dx \ \text{ for a.e. t, every } \phi \in H^1(\Omega). \tag{2.5}$$

Choosing as test functions in (2.5) $\phi_i = \max(0, u_i(t,\cdot) - \theta)$ $(i = 1,2)$ we obtain:

$$\int_{\{u_i>\theta\}} \quad a_{jk}\cdot(u_i)_{x_j}(u_i)_{x_k}\, dx \ = \ (-)^i\cdot\int_{\{u_i>\theta\}} \quad (u_i - \theta)(\tfrac{\partial u}{\partial t} - f)\, dx. \tag{2.6}$$

Taking derivative in both sides of (2.6) we have, from (2.4):

$$1 \ \le \ Q^2\mu_i(\theta)^{-2+2/n}\cdot[-\tfrac{\partial\mu_i}{\partial\theta}]\cdot(-)^i\cdot\int_{\{u_i>\theta\}} \quad (\tfrac{\partial u}{\partial t} - f)\, dx. \tag{2.7}$$

Now, put:

$$F_i(t,s) \ = \ \int_0^s f_i(t)^* \ (\sigma) d\sigma \ \ i = 1,2.$$

Then Hardy-Littlewood inequality (see Maderna-Salsa[4]) implies:

$$(-)^i\int_{\{u_i>\theta\}} \quad (-f)\, dx \ \le \ \int_{\{u_i>\theta\}} \quad f_i\, dx \ \le \ F_i(t,\mu_i(\theta)). \tag{2.8}$$

Now we handle the term $(-)^i\int_{\{u_i>\theta\}} \quad \frac{\partial u}{\partial t}\, dx$ in (2.7).

Note that, by the continuity of u_i, one has:

$$(-)^i\int_{\{u_i>\theta\}} \quad \tfrac{\partial u}{\partial t}\, dx \ = \ -\int_{\{u_i>\theta\}} \quad \tfrac{\partial u_i}{\partial t}\, dx \ + \ (-)^i\mu_i(\theta)\cdot h'(t). \tag{2.9}$$

Set: $k_i(t,s) = \int_0^s u_i(t)^* \ (\sigma) d\sigma$.

Then applying (1.6) to u_i we have, by (2.9):

$$(-)^i\int_{\{u_i>\theta\}} \quad \tfrac{\partial u}{\partial t}\, dx \ = \ -\tfrac{\partial k_i}{\partial t}(t,\mu_i(\theta)) \ + \ (-)^i\mu_i(\theta)\cdot h'(t). \tag{2.10}$$

From (2.7), (2.8), (2.10) it follows:

$$1 \ \le \ Q^2\mu_i(\theta)^{-2+2/n}\cdot[-\tfrac{\partial\mu_i}{\partial\theta}]\cdot\left\{F_i(t,\mu_i(\theta)) - \tfrac{\partial k_i}{\partial t}(t,\mu_i(\theta)) \ + \right.$$

$$\left. + \ (-)^i\mu_i(\theta)\cdot h'(t)\right\}. \tag{2.11}$$

(2.11) holds for a.e. $t \in (0,T)$, every $\theta \ge 0$, $i = 1,2$.

Since u_i satisfies (1.12.b), μ_i and $u_i(t)^*$ are inverse functions. Put $s = \mu_i(\theta)$. Then (2.11) rewrites as:

$$-\tfrac{\partial^2 k_i}{\partial s^2}(t,s) \ = \ -\tfrac{\partial}{\partial s}u_i(t)^* \ (s) \ \le \tag{2.12}$$

$$\le \ Q^2\cdot s^{-2+2/n}\cdot\left\{F_i(t,s) - \tfrac{\partial k_i}{\partial t}(t,s) \ + \ (-)^i\cdot s\cdot h'(t)\right\}.$$

(2.12) are true for a.e. $t \in (0,T)$ and for all $s = \mu_i(\theta)$ with $\theta \ge 0$, i.e., since (1.12.b) holds, for all $s \in (0,\frac{1}{2} \mid \Omega \mid)$.

Moreover, k_i satisfies the following conditions:

$$k_i(t,0) \ = \ 0$$

$$\frac{\partial k_i}{\partial s}(t,\tfrac{1}{2}\mid\Omega\mid) = 0$$

$$k_i(0,s) = \int_0^s u_i(0)^*(\sigma)d\sigma$$

Now, put $k_3 = k_1 + k_2$ and $F_3 = F_1 + F_2$. Adding the two inequalities (2.12) we obtain:

$$\frac{\partial k_3}{\partial t}(t,s) - Q^{-2}\cdot s^{2-2/n}\cdot\frac{\partial^2 k_3}{\partial s^2}(t,s) \leq F_3(t,s) \tag{2.13.a}$$

with:

$$k_3(t,0) = 0 \tag{2.13.b}$$

$$\frac{\partial k_3}{\partial s}(t,\tfrac{1}{2}\mid\Omega\mid) = 0$$

$$k_3(0,s) = \int_0^s v_0^*(\sigma)d\sigma. \qquad \text{for all } s \in (0,\tfrac{1}{2}\mid\Omega\mid), \text{ a.e. } t \in (0,T).$$

Now one can see that system (2.13) is the same that is found in Mossino-Rakotoson[5] studying Dirichlet's problem for parabolic equations. So we can draw the same conclusions:

(i). For system (2.13) a maximum principle holds, that is:

$$k_3(t,s) \leq K(t,s) \text{ for } (t,s) \in (0,T)\times(0,\tfrac{1}{2}\mid\Omega\mid) \tag{2.14}$$

where K satisfies (2.13) with the equality sign in (2.13.a). Moreover:

(ii). $K(t,s) = \int_0^s U^*(t,\sigma)d\sigma$ where U is the solution of (1.8).

A lemma of Bandle (see Bandle[1], p.174) states that:

if f, g are two nonnegative measurable functions such that

$$\int_0^s f^*(\sigma)d\sigma \leq \int_0^s g^*(\sigma)d\sigma \quad \text{for all } s \geq 0, \text{ then:}$$

$$\| f \|_p \leq \| g \|_p \text{ for all } p \in [1,\infty].$$

We also note that for any function $v \in \mathcal{L}^p(\Omega)$ we have:

$$2^{-1+1/p}\cdot \| (v^+)^* + (v^-)^* \|_{\mathcal{L}^p(\Omega^*)} \leq \| v \|_{\mathcal{L}^p(\Omega)} \leq \| (v^+)^* + (v^-)^* \|_{\mathcal{L}^p(\Omega^*)}. \tag{2.15}$$

Collecting (i), (ii) and these remarks we have:

$$\| (u-h)(t) \|_{\mathcal{L}^p(\Omega)} \leq \| u_1(t)^* + u_2(t)^* \|_{\mathcal{L}^p(\Omega^*)} \leq \| U(t) \|_{\mathcal{L}^p(\widetilde{\Omega}_{1/2})} \tag{2.16}$$

for a.e. $t \in (0,T)$, for any $p \in [1,\infty]$. This is (1.9.a). Note also that:

$$\| u_1(t)^* + u_2(t)^* \|_\infty = \sup_\Omega (u-h)(t) - \inf_\Omega (u-h)(t) =$$

$$= \sup_\Omega u(t,\cdot) - \inf_\Omega u(t,\cdot).$$

Hence (1.9.b) follows. □

Remark 2.1. If one knows that $h(t)$ is constant, then inequalities (2.12) can be discussed separately (without adding them): if we call U_i ($i = 1, 2$) the solutions of a problem (1.8) with data $\widetilde{f}_i$ and $\widetilde{u(0)}_i$, then one has:

$$\| u_i(t)^* \|_{\mathcal{L}^p(\Omega)} \leq \| U_i(t) \|_{\mathcal{L}^p(\widetilde{\Omega}_{1/2})}.$$

Hence:

$$\| (u-h)(t) \|_p \leq (\| U_1(t) \|_p^p + \| U_2(t) \|_p^p)^{1/p} \text{ for } p \in [1,\infty)$$

and:

$\| (u-h)(t) \|_\infty \le \max(\| U_1(t) \|_\infty , \| U_2(t) \|_\infty).$

For instance, if $f_1^* = f_2^*$ and $u(0)_1^* = u(0)_2^*$, then $U_1 = U_2$, $U = 2U_1$, so that (if $h =$ constant, say $h \equiv 0$):

$$\| u(t) \|_p \le 2^{-1+1/p} \| U(t) \|_p \quad \text{for } p \in [1,\infty].$$

This improves the estimate of theorem 1.1 for $p > 1$.

Proof of proposition 1.3.

From (2.13.a), since $\frac{\partial^2 K}{\partial s^2} = \frac{\partial U^*}{\partial s} \le 0$, we have:

$$F_3 - \frac{\partial K}{\partial t} \ge 0 \quad \text{for all } s \in (0,\tfrac{1}{2} \mid \Omega \mid), \text{ a.e. } t \in (0,T).$$

Integrating in t this inequality we have:

$$\int_0^s U^* (t,\sigma)d\sigma \le \int_0^s g(t,\sigma)d\sigma$$

with: $g(t,s) = v_0^*(s) + \int_0^t f_3(\tau)^* (s)\, d\tau.$

So, by Bandle's lemma and (2.15), we have:

$$\| U(t) \|_{\mathcal{L}^p(\widetilde{\Omega}_{1/2})} \le \| g(t) \|_{\mathcal{L}^p(\Omega^*)} \le$$

$$\le 2^{1-1/p} \left\{ \| u_0 - h(0) \|_{\mathcal{L}^p(\Omega)} + \int_0^t \| f(\tau) \|_{\mathcal{L}^p(\Omega)} d\tau \right\}. \tag{2.17}$$

Now we want to estimate $\mid h(t) \mid$ in terms of the data. Observe that:

$$\mid h(t) \mid \cdot \frac{|\Omega|}{2} \le \int_0^{\frac{1}{2}|\Omega|} u(t)^* (\sigma)d\sigma \quad \text{if } h(t) \ge 0$$

$$\mid h(t) \mid \cdot \frac{|\Omega|}{2} \le -\int_{\frac{1}{2}|\Omega|}^{|\Omega|} u(t)^* (\sigma)d\sigma \quad \text{if } h(t) \le 0. \tag{2.18}$$

Let us turn back to (2.12). It gives in particular:

$$0 \le F_i(t,s) - \frac{\partial k_i}{\partial t}(t,s) + (-)^i \cdot s \cdot h'(t) \quad \text{for } i = 1,2.$$

Integrating in t this inequality one can obtain the following:

$$\int_0^{\frac{1}{2}|\Omega|} u(t)^* (\sigma)d\sigma \le \int_0^{\frac{1}{2}|\Omega|} u_0^* (\sigma)d\sigma + \int_0^t d\tau \int_0^{\frac{1}{2}|\Omega|} f_1(\tau)^* (\sigma)d\sigma \tag{2.19.a}$$

$$-\int_{\frac{1}{2}|\Omega|}^{|\Omega|} u(t)^* (\sigma)d\sigma \le -\int_{\frac{1}{2}|\Omega|}^{|\Omega|} u_0^*(\sigma)d\sigma + \int_0^t d\tau \int_0^{\frac{1}{2}|\Omega|} f_2(\tau)^* (\sigma)d\sigma. \tag{2.19.b}$$

(To get (2.19.b), remember that $[-u(t)]^* (s) = -u(t)^* (\mid \Omega \mid - s))$.
From (2.18)-(2.19) we get:

$$\mid h(t) \mid \cdot \frac{|\Omega|}{2} \le \| u_0 \|_{\mathcal{L}^1(\Omega)} + \int_0^t \| f(\tau) \|_{\mathcal{L}^1(\Omega)} d\tau \tag{2.20}$$

Finally, from (2.16), (2.17) and (2.20) it follows proposition 1.3. □

Proof of theorem 1.2. (Energy estimate).
From (2.5), with test function u, one has:

$$\int_\Omega f\,u\,dx - \int_\Omega u\frac{\partial u}{\partial t}\,dx = \int_\Omega a_{ij}u_{x_i}u_{x_j}\,dx \ \geq \ \text{(by ellipticity)}$$

$$\geq \int_\Omega |Du|^2 dx. \quad \text{Integrating in } t:$$

$$\int_D |Du|^2 dx\,dt + \tfrac{1}{2}\int_\Omega u^2(T)\,dx \leq \tfrac{1}{2}\int_\Omega u_0^2\,dx + \int_D f\,u\,dx\,dt.$$

Similarly one obtains:

$$\gamma\cdot\int_{\widetilde{D}} |DU|^2 dx\,dt + \tfrac{1}{2}\int_{\widetilde{\Omega}_{1/2}} U^2(T)\,dx = \tfrac{1}{2}\int_{\widetilde{\Omega}_{1/2}} \widetilde{v}_0^2\,dx + \int_{\widetilde{D}} \widetilde{f}_3\,U\,dx\,dt.$$

By (2.15), and since $h(0) = 0$:

$$\int_\Omega u_0^2\,dx \leq \int_{\widetilde{\Omega}_{1/2}} \widetilde{v}_0^2\,dx.$$

Hence:

$$\int_D |Du|^2 dx\,dt + \tfrac{1}{2}\int_\Omega u^2(T)\,dx \leq \gamma\cdot\int_{\widetilde{D}} |DU|^2 dx\,dt +$$

$$+ \tfrac{1}{2}\int_{\widetilde{\Omega}_{1/2}} U^2(T)\,dx + \int_D f\,u\,dx\,dt - \int_{\widetilde{D}} \widetilde{f}_3\,U\,dx\,dt. \tag{2.21}$$

Now:

$$\int_\Omega (u-h)f\,dx \leq \int_\Omega (u_1 f_1 + u_2 f_2)\,dx \leq \text{(by Hardy-Littlewood)}$$

$$\leq \int_0^{|\Omega|}(u_1^* f_1^* + u_2^* f_2^*)(s)\,ds \leq \text{(let us call } u_3^* = u_1^* + u_2^*)$$

$$\leq \int_0^{|\Omega|} u_3^* f_3^*\,ds = \text{(by parts)} = [f_3^* k_3]_0^{|\Omega|} - \int_0^{|\Omega|} k_3 \frac{\partial f_3^*}{\partial s}\,ds \leq$$

(since $f_3^*(t, |\Omega|) = 0$, $k_3(t,0) = 0$, and by (2.14))

$$\leq -\int_0^{|\Omega|} K\frac{\partial f_3^*}{\partial s}\,ds = \text{(by parts...)} = \int_0^{|\Omega|} U^* f_3^*\,ds = \int_{\widetilde{\Omega}_{1/2}} U\widetilde{f}_3\,dx.$$

So we have:

$$\int_D f\,u\,dx\,dt - \int_{\widetilde{D}} \widetilde{f}_3\,U\,dx\,dt \leq \int_0^T h(t)\int_\Omega f(t,x)\,dx\,dt. \tag{2.22}$$

From (2.21) and (2.22) it follows (1.10). So theorem 1.2 is proved. □

Remark 2.2. In this proof we have used, beside the assumptions of theorem 1.2, only the fact that $f_3(t)^*(\cdot)$ is absolutely continuous. This is true, for instance, when $f(t,\cdot) \in H^1(\Omega)$. However, this last assumption is unessential: in the next section we will show that theorem 1.2 holds whenever f is an $\mathcal{L}^2(D)$ function.

3. PROOF OF THEOREMS 1.1-1.3 IN THE GENERAL CASE

The aim of this section is to relax the two extra-assumptions we have done till now in proving theorems 1.1, 1.2, 1.3. Let us recall them here:

(1.12.a) the coefficients a_{ij} and the data u_0, f are smooth;

(1.12.b) the solution u has the following property: for a.e. t, the function $u(t,\cdot)$ is not constant on any subset of Ω of positive measure.

While to bypass (1.12.a) standard approximation techniques are sufficient, (1.12.b) requires a more delicate costruction. We start from this, showing how we can construct a problem of type (0.2) whose data are as close as we want to the original ones, and whose solution satisfies (1.12.b).

Let us consider, for $t \in [0.T]$ fixed, the solution $v(t,\cdot)$ of the elliptic problem:

$$-\left(a_{ij}(t,\cdot)v_{x_i}\right)_{x_j} = g \quad \text{in } \Omega \tag{3.1}$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \text{ with the condition } \int_\Omega v\,dx = 0$$

where $g(x) = |x|^2 - \fint_\Omega |y|^2 dy$.

The datum g satisfies the compatibility condition $\int g = 0$, and the solution $v(t,\cdot)$ is not constant on any set of positive measure, as we read from the equation, since g is not zero on any set of positive measure. Moreover, since the coefficients a_{ij} are smooth in D, v is smooth in D, too. Now, if we set:

$h(t,x) = t\cdot v(t,x)$

then h is the solution of the problem:

$$\frac{\partial h}{\partial t} - \left(a_{ij}h_{x_i}\right)_{x_j} = v + t\cdot\frac{\partial v}{\partial t} + t\cdot g \equiv g_1 \quad \text{in } (0,T)\times\Omega \tag{3.2}$$

$$\frac{\partial h}{\partial \nu} = 0 \quad \text{on } (0,T)\times\partial\Omega$$

$$h(0,t) = 0 \quad \text{in } \Omega$$

and for *all* $t \in [0,T]$ h is not constant on any set of positive measure. Note that the datum g_1 in (3.2) is bounded by a constant which depends on $\| \partial_t a_{ij} \|_\infty$. The next theorem clarifies the role of h.

Theorem 3.1. Let h(t,x) be a function on D such that for any fixed $t \in [0,T]$ $h(t,\cdot)$ is not constant on any subset of Ω of positive measure, and let u(t,x) be a measurable function. Then for any $\epsilon_0 > 0$ there exists $\epsilon \in (0,\epsilon_0)$ such that the function $u_\epsilon = u + \epsilon\cdot h$ for a.e. fixed $t \in (0,T)$ is not constant on any subset of Ω of positive measure.

The reason why this theorem is true is a matter of measure theory, namely the following lemma:

Lemma 3.2. Let ϵ_0 be a positive number. Suppose we have a family $\{I_\epsilon\}_{\epsilon\in(0,\epsilon_0)}$ of subsets of [0,T] and, for each $\epsilon \in (0,\epsilon_0)$, a family $\{E_{\epsilon,t}\}_{t\in I_\epsilon}$ of subsets of Ω. Then if every I_ϵ has positive measure in $\Re$ and every $E_{\epsilon,t}$ has positive measure in $\Re^n$, there exist two different numbers $\epsilon_1, \epsilon_2 \in (0,\epsilon_0)$ and a number $t_0 \in [0,T]$ such that E_{ϵ_1,t_0} and E_{ϵ_2,t_0} have intersection of positive measure.

Proof of theorem 3.1 from lemma 3.2.
By contradiction. Suppose there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0,\epsilon_0)$ there exists $I_\epsilon \subset [0,T]$, $|I_\epsilon| > 0$, such that for all $t \in I_\epsilon$ the function $u_\epsilon(t,\cdot)$ is constant on some set of positive measure, i.e. there exist $k_{\epsilon,t} \in \Re$ and $E_{\epsilon,t} \subset \Omega$, $|E_{\epsilon,t}| > 0$, such that $u_\epsilon(t,x) = k_{\epsilon,t}$ for all $t \in I_\epsilon$, $x \in E_{\epsilon,t}$. Then by lemma 3.2 there exist $\epsilon_1, \epsilon_2 \in (0,\epsilon_0)$ $(\epsilon_1 \neq \epsilon_2)$ and $t_0 \in [0,T]$ such that $|E_{\epsilon_1,t_0} \cap E_{\epsilon_2,t_0}| > 0$. Hence for all $x \in E_{\epsilon_1,t_0} \cap E_{\epsilon_2,t_0}$ we have:

$(u + \epsilon_i\cdot h)(t_0,x) = k_{\epsilon_i,t_0}$ $(i = 1,2)$.

By subtraction we get: $h(t_0,x) = \text{constant}$ for all $x \in E_{\epsilon_1,t_0} \cap E_{\epsilon_2,t_0}$, which contradicts our assumption on h. □

Proof of lemma 3.2.
Put $E_0 = (0,\epsilon_0)$ and fix $\epsilon \in E_0$.

Claim (a). We can choose an uncountable subset F of E_0 and, for any $\epsilon \in F$, a measurable subset J_ϵ of I_ϵ such that:

$|J_\epsilon| > \frac{1}{n_1}$ for all $\epsilon \in F$ and $|E_{\epsilon,t}| > \frac{1}{n_0}$ for all $\epsilon \in F$, $t \in J_\epsilon$, for some integers n_0, n_1.

In fact: for $n \in \mathcal{N}$, put:

$$I_\epsilon^n = \left\{ t \in I_\epsilon : |E_{\epsilon,t}| > \frac{1}{n} \right\}.$$

The set I_ϵ^n may not be measurable. But we always can choose a measurable set $J_\epsilon^n \subset I_\epsilon^n$ such that the *outer measure* of $I_\epsilon^n \setminus J_\epsilon^n$ is zero. Then we have:

$$\bigcup_{n=1}^{\infty} J_\epsilon^n \subset I_\epsilon \quad \text{and} \quad |I_\epsilon \setminus \bigcup_{n=1}^{\infty} J_\epsilon^n| = 0.$$

Now, since $|I_\epsilon| > 0$, there must be an integer n_ϵ such that $|J_\epsilon^{n_\epsilon}| > 0$.
For any $\epsilon \in E_0$ we choose such n_ϵ, and we call $J_\epsilon = J_\epsilon^{n_\epsilon}$. For any $m \in \mathcal{N}$, put:

$$E_m = \left\{ \epsilon \in E_0 : n_\epsilon = m \right\}.$$

Since $E_0 = \cup E_m$ and E_0 is uncountable, there exists n_0 such that E_{n_0} is uncountable. Moreover, since $|J_\epsilon| > 0$ for all $\epsilon \in E_{n_0}$, there exists n_1 such that $|J_\epsilon| > 1/n_1$ for an uncountable infinity of J_ϵ's. Let us call $\left\{ J_\epsilon \right\}_{\epsilon \in F}$ this uncountable family. We have proved (a).

Now:

Claim (b). For any fixed positive integer k, there exist k sets $J_{\epsilon_1},...,J_{\epsilon_k}$ $(\epsilon_i \in F)$ such that:

$$\bigcap_{i=1}^{k} J_{\epsilon_i} \neq \emptyset.$$

By contradiction.

There exists k such that for any $J_{\epsilon_1},...,J_{\epsilon_k}$ $(\epsilon_i \in F)$ it is:

$$\bigcap_{i=1}^{k} J_{\epsilon_i} = \emptyset.$$

Then for any finite family $J_{\epsilon_1},...,J_{\epsilon_r}$ $(\epsilon_i \in F)$, since every $t \in [0,T]$ does not belong to more than $k-1$ sets among these, one has:

$$\frac{r}{n_1} < \sum_{i=1}^{r} |J_{\epsilon_i}| \leq (k-1)\cdot |\bigcup_{i=1}^{r} J_{\epsilon_i}| \leq (k-1)\cdot T$$

and this is false for large r. Hence we have (b).

Now, choose $k > n_0 \cdot |\Omega|$, $J_{\epsilon_1},...,J_{\epsilon_k}$ satisfying claim (b), and:

$$t_0 \in \bigcap_{i=1}^{k} J_{\epsilon_i}.$$

Consider $E_{\epsilon_1,t_0},...,E_{\epsilon_k,t_0} \subset \Omega$.

We say that there are two of these sets with intersection of positive measure. If not, we should have:

$$|\Omega| \geq |\bigcup_{i=1}^{k} E_{\epsilon_i,t_0}| = \sum_{i=1}^{k} |E_{\epsilon_i,t_0}| > \frac{k}{n_0} > |\Omega|,$$

contradiction. So the lemma is proved. □

Now we show how using theorem 3.1 we can obtain theorems 1.1, 1.2, 1.3 in the general case. Let us consider problem (0.2). First, we suppose that coefficients and data are smooth. So if u does not satisfy (1.12.b) we can make the construction previously seen in this section: if h, g_1 are the functions appearing in (3.2), by theorem 3.1 we can choose a sequence $\epsilon_m \to 0$ such that the function $u_m = u + \epsilon_m \cdot h$ is the solution of:

$$\left(\frac{\partial}{\partial t} - A\right)u_m = f + \epsilon_m \cdot g_1 \equiv F_m \quad \text{in } (0,T) \times \Omega \tag{3.3}$$
$$\frac{\partial u_m}{\partial \nu} = 0 \quad \text{on } (0,T) \times \partial\Omega$$
$$u_m(0,\cdot) = u_0 \quad \text{in } \Omega$$

and this problem satisfies (1.12.a,b). So the estimates of theorems 1.1, 1.2, 1.3 hold for u_m. The "symmetrized problem" of (3.3) (see theorem 1.1) is:

$$\left(\frac{\partial}{\partial t} - \gamma \cdot \Delta\right)U_m = (\widetilde{F_m})_3 \quad \text{in } (0,T) \times \widetilde{\Omega}_{1/2} \tag{3.4}$$
$$U_m = 0 \quad \text{on } (0,T) \times \partial\widetilde{\Omega}_{1/2}$$
$$U_m(0,\cdot) = \widetilde{v}_0 \quad \text{in } \widetilde{\Omega}_{1/2}$$

with the obvious meaning of $(\widetilde{F_m})_3$.

Since g_1 is bounded in D, $F_m \to f$ uniformly in D; then, by (1.5):

$(\widetilde{F_m})_3 \to \widetilde{f}_3$ uniformly in D.

So, by continuous dependence of the solution of (3.4) from the data, we have that:

$$\| U_m(t) \|_{\mathcal{L}^p(\widetilde{\Omega}_{1/2})} \to \| U(t) \|_{\mathcal{L}^p(\widetilde{\Omega}_{1/2})} \text{ for all } t \in [0,T]. \tag{3.5}$$

As to u_m we use now the continuous dependence in the Hilbert space Φ introduced in (1.3). Since u_m is bounded in Φ, there exists $w \in \Phi$ such that, passing to a subsequence, $u_m \to w$ weakly in Φ. Then w solves problem (0.2), so that $w \equiv u$. Hence we know that for a subsequence u_m one has:

$$u_m(t) \to u(t) \text{ in } \mathcal{L}^2(\Omega) \text{ and a.e.}; \quad \| Du_m(t) \|_{\mathcal{L}^2(\Omega)} \le c, \text{ for a.e. t.} \tag{3.6}$$

Now we need the following lemma:

Lemma 3.3. Let $h_m(t) = u_m(t)^*(\frac{|\Omega|}{2})$.

If (3.6) holds, then $h_m(t) \to h(t)$ for a.e. t. If, moreover, one knows that:

$$u_m \to u \text{ in } \mathcal{L}^2(D) \text{ and a.e. in D}; \quad \| Du_m \|_{\mathcal{L}^2(D)} \le c \tag{3.7}$$

then $h_m \to h$ in $\mathcal{L}^2(0,T)$.

To prove this lemma we need further results about rearrangements of functions in $H^{1,2}(\Omega)$, namely:

Lemma 3.4. Let $u \in H^{1,2}(\Omega)$, where Ω is a bounded Lipschitz domain of $\Re^n$ $(n \ge 2)$. Then $\widetilde{u} \in H^{1,2}_{loc}(\widetilde{\Omega})$, and, if $\widetilde{\Omega}_\epsilon$ denotes the sphere centred at the origin with measure $|\Omega| - \epsilon$, one has:

$$\int_{\widetilde{\Omega}_\epsilon} |D\widetilde{u}|^2 dx \le c(\epsilon) \int_\Omega |Du|^2 dx$$

for any positive ϵ, with $c(\epsilon)$ depending on ϵ, n, $|\Omega|$, Q. Moreover, u^* is absolutely continuous in $(\epsilon, |\Omega| - \epsilon)$.

Remark 3.5. A well known theorem by Polya-Szegö states that, if v is a nonnegative function in $H^{1,2}_0(\Omega)$, then $\widetilde{v} \in H^{1,2}_0(\widetilde{\Omega})$, and:

$\| D\widetilde{v} \|_2 \le \| Dv \|_2$.

To get lemma 3.4 one has to consider a function $u \in H^{1,2}(\Omega)$ and apply to the positive and negative parts of $u - u^*(\frac{|\Omega|}{2})$ the same reasoning used in the proof of Polya-Szegö's theorem given in Talenti[6]. The *relative* isoperimetric inequality in Ω is in this case the suitable tool. We do not give account here of the details.

Proof of lemma 3.3. Let $\phi(s) = u_m(t)^*(s) - u(t)^*\ (s)$. ϕ is absolutely continuous in $(\epsilon, |\Omega| - \epsilon)$ for any $\epsilon > 0$. Then one has:

$$\phi(s)^2 \leq \frac{1}{|\Omega|-2\epsilon}\cdot\int_\epsilon^{|\Omega|-\epsilon}\phi(\sigma)^2\, d\sigma + 2\int_\epsilon^{|\Omega|-\epsilon} |\phi\cdot\phi'(\sigma)|\, d\sigma$$

for any $s \in (\epsilon, |\Omega| - \epsilon)$. So we have, for $s = \frac{1}{2}|\Omega|$:

$$|h_m(t) - h(t)|^2 \leq \frac{1}{|\Omega|-2\epsilon}\cdot\int_\epsilon^{|\Omega|-\epsilon} |u_m(t)^* - u(t)^*|^2\, d\sigma + \tag{3.8}$$

$$+ 2\int_\epsilon^{|\Omega|-\epsilon} |u_m(t)^* - u(t)^*| \cdot \left|\frac{\partial u_m}{\partial\sigma}(t)^* - \frac{\partial u}{\partial\sigma}(t)^*\right| d\sigma \equiv A_m + B_m.$$

Now, for $m \to \infty$, $A_m \to 0$ by (1.5) since $u_m(t) \to u(t)$ in $\mathcal{L}^2(\Omega)$. By Holder's inequality:

$$B_m \leq 2\left(\int_\epsilon^{|\Omega|-\epsilon}\left(\sigma^{1-1/n}\left(\frac{\partial u_m}{\partial\sigma}(t)^* - \frac{\partial u}{\partial\sigma}(t)^*\right)(\sigma)\right)^2 d\sigma\right)^{1/2}\cdot \tag{3.9}$$

$$\cdot\left(\int_\epsilon^{|\Omega|-\epsilon}\left(\sigma^{-1+1/n}(u_m(t)^* - u(t)^*)(\sigma)\right)^2 d\sigma\right)^{1/2} \equiv 2C_m\cdot D_m.$$

$D_m \leq c(\epsilon)\ \| u_m(t) - u(t) \|_2$, by (1.5).

$$C_m \leq \left(\int_0^{|\Omega|-\epsilon}\left(\sigma^{1-1/n}\left(\frac{\partial u_m}{\partial\sigma}(t)^*\ (\sigma)\right)\right)^2 d\sigma\right)^{1/2} +$$

$$+ \left(\int_0^{|\Omega|-\epsilon}\left(\sigma^{1-1/n}\left(\frac{\partial u}{\partial\sigma}(t)^*\ (\sigma)\right)\right)^2 d\sigma\right)^{1/2} =$$

$$= \left(\int_{\widetilde{\Omega}_\epsilon} |D\widetilde{u}_m(t)|^2\, dx\right)^{1/2} + \left(\int_{\widetilde{\Omega}_\epsilon} |D\widetilde{u}(t)|^2\, dx\right)^{1/2} \leq \text{(by lemma 3.4)}$$

$$\leq c(\epsilon)\left(\left(\int_\Omega |Du_m(t)|^2\, dx\right)^{1/2} + \left(\int_\Omega |Du(t)|^2\, dx\right)^{1/2}\right) \leq \text{const.}$$

So:

$$B_m \leq \text{const.}\cdot \| u_m(t) - u(t) \|_2 \tag{3.10}$$

and $h_m(t) \to h(t)$ pointwise. Moreover, if (3.7) holds, integrating in t inequalities (3.8), (3.9) one gets convergence in $\mathcal{L}^2(0,T)$. Then the proof is complete. □

Now from (3.5), (3.6) and lemma 3.3 we can pass to the limit in (1.9) (by Fatou's theorem) and obtain theorem 1.1 in tha case of smooth coefficients and data. In the same way it follows (1.11), i.e. proposition 1.3, since $F_m \to f$ in $\mathcal{L}^p(D)$. As to theorem 1.2, we already know, by remark 2.2, that it holds when coefficients and data are smooth. So what we have now to do is to relax the assumption of smoothness in deriving theorems 1.1, 1.2, 1.3..

However, this fact requires no new idea, so a detailed proof is omitted: we can regularize coefficients and data; then by continuous dependence of the solution in suitable spaces and applying lemma 3.3 we get theorems 1.1-1.3 in the general case. Note that the full statement of lemma 3.3 is needed in order to pass to the limit in the energy estimate.

Remark 3.6. Let us suppose that Ω is a ball, A is the laplacian, f, u_0 are nonnegative, radially symmetric and decreasing functions vanishing outside a ball of measure $|\Omega|/2$. Then, by reviewing the proof of theorem 1.1, one can see that:

$$u_1(t)^* + u_2(t)^* = U(t)^*$$

holds, provided that U is defined as the solution of a problem (1.8) with $\gamma = [n \cdot c_n^{1/n}]^{-1}$ (i.e. we replace Q with the isoperimetric constant of $\Re^n$). In consequence one has:

$$\| (u - h)(t) \|_1 = \| U(t) \|_1$$

and:

$$\sup_{\Omega} u(t,\cdot) - \inf_{\Omega} u(t,\cdot) = \sup_{\tilde{\Omega}_{1/2}} U(t).$$

It is an open question to find an example in which all the level sets of the functions u_i are optimal with regards to the relative isoperimetric inequality, so that the above identities may hold for the function U defined via the "right" constant Q. This example is still lacking even in the elliptic case.

REFERENCES

[1] C. Bandle, Isoperimetric inequalities and application, Pitman Advanced Publishing Program, Boston-London-Melbourne 1980.

[2] G. Chiti, Applicable Analysis, 9 (1979).

[3] O.A. Ladyzenskaya-V.A. Solonnikov-N.N. Uraltseva, Linear and quasilinear equations of parabolic type, Amer. Math. Soc., Providence, 1968.

[4] C. Maderna-S. Salsa, Applicable Analysis, 9 (1979)

[5] J. Mossino-J.M. Rakotoson, Annali Sc. Norm. Sup. Pisa, 13 (1986)

[6] G. Talenti, Annali Mat. Pura e Appl., 110 (1976)

[7] F. Treves, Basic linear partial differential equations, Academic Press, New York-San Francisco-London, 1975.